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Classification of Hyperfunctions of Rank 2 with Respect to Membership in the Maximal Partial Ultraclones

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In this paper, we consider the set of hyperfunctions, which is a subset of the full partial ultraclone of rank 2. For hyperfunctions, the problem of their classification with respect to membership in the the maximal partial ultraclones is solved. The relation of membership in the maximal partial ultraclones is an equivalence relation and generates the corresponding partition into equivalence classes. A complete description of all equivalence classes, the total number of which is 28, is obtained.

Keywords: multifunction, hyperfunction, clone, ultraclone, maximal partial ultraclone, classification of functions.
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Introduction

A sets of multifunctions are considered. A multifunction on a finite set A is a function defined on the set A and taking as its values its subsets. Obviously, superposition in the usual sense does not work when working with multifunctions. Therefore, they need to give a new definition of superposition. Two ways of defining superposition are usually considered: the first is based on the union of subsets of the set A, and in this case the closed sets containing all the projections are called multiclones, and the second is the intersection of the subsets of A, and the closed sets containing all projections are called partial ultraclones. The set of multifunctions on A on the one hand contains all the functions of $|A|$-valued logic and on the other, is a subset of functions of $2^{|A|}$-valued logic with superposition that preserves these subset.

In the theory of functions the problem of classification is interesting. One of the known variants of the classification of functions of k-valued logic is one in which functions in a closed subset B of a closed set M can be divided according to their membership in the classes that are pre-complete in M. In this paper, the subset of B is the set of all Boolean functions, and the set of M is the set of all multifunctions on the two-element set, and the partial maximal ultraclones are pre-complete classes.

Note that the quality and type of functions are limited to the function The other k-logic logic is used, for example, in the process [1–8].

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1. Basic concepts and definitions

Let $E = \{0,1\}$ и $F = \{\varnothing, \{0\}, \{1\}, \{0,1\}\}$. We define the following sets of functions:
$P_{2,n}^{\overline{*}} = \{f | f : E^n \to F\}$, $P_2^{\overline{*}} = \bigcup\limits_n P_{2,n}^{\overline{*}}$;
$P_{2,n} = \{f | f \in P_{2,n}^{\overline{*}}$ и $|f(\tilde{\alpha})| = 1$ for any $\tilde{\alpha} \in E^n\}$, $P_2 = \bigcup\limits_n P_{2,n}$;
$P_{2,n}^{-} = \{f | f : E^n \to F \setminus \{\varnothing\}\}$, $P_2^{-} = \bigcup\limits_n P_{2,n}^{-}$;
$P_{2,n}^{*} = \{f | f \in P_{2,n}^{\overline{*}}$ and $|f(\tilde{\alpha})| \leqslant 1$ for any $\tilde{\alpha} \in E^n\}$, $P_2^{*} = \bigcup\limits_n P_{2,n}^{*}$.

Functions from P_2 are called Boolean functions, functions from P_2^* are called partial functions on E, functions from P_2^- are called hyperfunctions on E, functions from $P_2^{\overline{*}}$ are called multifunctions on E.

We believe that the superposition

$$f(f_1(x_1, \ldots, x_m), \ldots, f_n(x_1, \ldots, x_m)),$$

where $f, f_1, \ldots, f_n \in P_2^{\overline{*}}$, represents some multifunction $g(x_1, \ldots, x_m)$ on a tuple with elements from the set F, if for any $(\alpha_1, \ldots, \alpha_m) \in E^m$

$$g(\alpha_1, \ldots, \alpha_m) = \begin{cases} \bigcap\limits_{\beta_i \in f_i(\alpha_1, \ldots, \alpha_m)} f(\beta_1, \ldots, \beta_n) & \text{if the intersection is not empty;} \\ \bigcup\limits_{\beta_i \in f_i(\alpha_1, \ldots, \alpha_m)} f(\beta_1, \ldots, \beta_n) & \text{otherwise.} \end{cases}$$

On tuples containing $\varnothing$, the multifunction takes the value $\varnothing$.

This definition allows you to find the value $f(x_1, \ldots, x_n)$ for every $(\sigma_1, \ldots, \sigma_n) \in F^n$.

For simplicity we use the following code:

$$\varnothing \leftrightarrow *, \{0\} \leftrightarrow 0, \{1\} \leftrightarrow 1, \{0,1\} \leftrightarrow -.$$

We note that in this paper we will adhere to the terminology adopted in [9], which will allow us not to introduce additional definitions here.

In [10] it is proved that the maximal partial ultraclones of rank 2 are only the following 12 sets:

1) K_1 is the set consisting of all multifunctions f such that $f(\tilde{0}) \in \{0, *\}$.

2) K_2 is the set consisting of all multifunctions f such that $f(\tilde{1}) \in \{1, *\}$.

3) K_3 is the set consisting of all multifunctions f for which one of the two conditions is fulfilled:

- $f(\tilde{0}) = *$ or $f(\tilde{1}) = *$,
- $f(\tilde{0}) = 0$ and $f(\tilde{1}) = 1$.

4) K_4 is the set consisting of all multifunctions f such that on any binary tuple $\tilde{\alpha}$ one of three conditions is fulfilled:

- $f(\tilde{\alpha}) = f(\bar{\tilde{\alpha}}) = -$,
- $f(\tilde{\alpha}) = f(\bar{\tilde{\alpha}}) = *$,
- $f(\tilde{\alpha}) = \overline{f(\bar{\tilde{\alpha}})}$, where $f(\tilde{\alpha}) \in \{0,1\}$.

5) K_5 is the set consisting of all multifunctions f such that on any binary tuple $\tilde{\alpha}$ one of two conditions is fulfilled:

- $f(\tilde{\alpha}) = *$ or $f(\bar{\tilde{\alpha}}) = *$,
- $f(\tilde{\alpha}) = \overline{f(\bar{\tilde{\alpha}})}$, where $f(\tilde{\alpha}) \in \{0, 1\}$.

6) $K_6 = P_2^- \cup \{*\}$.
7) $K_7 = P_2^*$.
8) K_8 is the set of all multifunctions f that simultaneously satisfy three conditions:

- if $f(\tilde{\alpha})$, $f(\tilde{\beta})$, $f(\tilde{\gamma}) \in \{0, 1\}$, then

$$f\begin{pmatrix}\tilde{\alpha}\\ \tilde{\beta}\\ \tilde{\gamma}\end{pmatrix} \in \left\{\begin{pmatrix}0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}1\\1\\1\end{pmatrix}\right\},$$

  where $\tilde{\alpha} = (\alpha_1, \ldots, \alpha_n)$, $\tilde{\beta} = (\beta_1, \ldots, \beta_n)$, $\tilde{\gamma} = (\gamma_1, \ldots, \gamma_n)$ are binary tuples such that $(\alpha_i\beta_i\gamma_i) \in \{(000), (001), (010), (111)\}$ for any $i \in \{1, \ldots, n\}$;
- if there is a binary tuple $\tilde{\alpha}$ such that $f(\tilde{\alpha}) = -$, then for any binary tuple $\tilde{\beta}$ true $f(\tilde{\beta}) \neq 1$;
- let binary tuples $\tilde{\alpha} = (\alpha_1, \ldots, \alpha_n)$, $\tilde{\beta} = (\beta_1, \ldots, \beta_n)$ such that $\alpha_i \leqslant \beta_i$ for any $i \in \{1, \ldots, n\}$ then, if $f(\tilde{\alpha}) = *$, then $f(\tilde{\beta}) = *$.

9) K_9 is the set of all multifunctions f that simultaneously satisfy three conditions:

- if $f(\tilde{\alpha}), f(\tilde{\beta}), f(\tilde{\gamma}) \in \{0, 1\}$, then

$$f\begin{pmatrix}\tilde{\alpha}\\ \tilde{\beta}\\ \tilde{\gamma}\end{pmatrix} \in \left\{\begin{pmatrix}0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}1\\1\\1\end{pmatrix}\right\},$$

  where $\tilde{\alpha} = (\alpha_1, \ldots, \alpha_n)$, $\tilde{\beta} = (\beta_1, \ldots, \beta_n)$, $\tilde{\gamma} = (\gamma_1, \ldots, \gamma_n)$ are binary tuples such that $(\alpha_i\beta_i\gamma_i) \in \{(000), (011), (101), (111)\}$ for any $i \in \{1, \ldots, n\}$;
- if there is a binary tuple $\tilde{\alpha}$ such that $f(\tilde{\alpha}) = -$, then for any binary tuple $\tilde{\beta}$ true $f(\tilde{\beta}) \neq 0$;
- let binary tuples $\tilde{\alpha} = (\alpha_1, \ldots, \alpha_n)$, $\tilde{\beta} = (\beta_1, \ldots, \beta_n)$ such that $\alpha_i \leqslant \beta_i$ for any $i \in \{1, \ldots, n\}$ then, if $f(\tilde{\beta}) = *$, then $f(\tilde{\alpha}) = *$.

10) K_{10} is the set of all multifunctions preserve the predicate
$R_{10} = \begin{pmatrix}0&0&0&0&1&1&1&1&-&\alpha\\0&0&1&1&1&1&0&0&-&\beta\\0&1&0&1&1&0&1&0&-&\gamma\\0&1&1&0&1&0&0&1&-&\delta\end{pmatrix}$, where $(\alpha, \beta, \gamma, \delta)^t$ are all sorts of columns in which $\alpha, \beta, \gamma, \delta \in F$ are simultaneously satisfy two conditions:

- in every column $(\alpha, \beta, \gamma, \delta)^t$ among $\alpha, \beta, \gamma, \delta$ least two assume the value $*$;
- in every column $(\alpha, \beta, \gamma, \delta)^t$, if 0 or 1 are found among $\alpha, \beta, \gamma, \delta$ then all of them are not equal to $-$.

11) K_{11} is the set of all multifunctions preserve the predicate

$$R_{11} = \begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 0 & 0 & - & - & 0 & 1 & - & * & * & * & * & * & * & * \\ 0 & 0 & 1 & 0 & 1 & 0 & - & 0 & - & * & * & * & 0 & 1 & - & * & * & * & * \\ 0 & 1 & 0 & 0 & 1 & - & 0 & 0 & - & * & * & * & * & * & * & 0 & 1 & - & * \end{pmatrix}.$$

12) K_{12} is the set of all multifunctions preserve the predicate

$$R_{12} = \begin{pmatrix} 0 & 0 & 1 & 1 & 1 & 1 & 1 & - & - & 0 & 1 & - & * & * & * & * & * & * & * \\ 0 & 1 & 0 & 1 & 1 & 1 & - & 1 & - & * & * & * & 0 & 1 & - & * & * & * & * \\ 0 & 1 & 1 & 0 & 1 & - & 1 & 1 & - & * & * & * & * & * & * & 0 & 1 & - & * \end{pmatrix}.$$

For any multifunction f we uniquely define vector $\tau(f) = (\tau_1, \dots, \tau_{12})$. This vector $\tau(f)$ is a vector of membership in the sets $K_1 - K_{12}$ and for every $i \in \{1, \dots, 12\}$ $\tau_i = \begin{cases} 0 & \text{if } f \in K_i \\ 1 & \text{if } f \notin K_i \end{cases}$.

The membership relation in the sets $K_1 - K_{12}$ is an equivalence relation and generates a partition of $P_2^{\bar{*}}$ into equivalence classes. For multifunctions from one class, the membership vectors in the sets $K_1 - K_{12}$ are the same. Since there are 12 maximal partial ultraclones, the largest possible number of equivalence classes is $2^{12} = 4096$.

In this paper, we find the number of equivalence classes that consist only of hyperfunctions.

2. The main result

In [1] it is shown that the set of Boolean functions is divided into 15 equivalence classes with respect to membership in the maximal partial ultraclones. Therefore, throughout the paper, we consider only hyperfunctions from the set $P_2^- \setminus P_2$. Obviously, any hyperfunction belongs to the K_6 and does not belong to the K_7.

Lemma 1. *For any $f \in P_2^- \setminus P_2$ the following statements are true:*
1) $f \notin K_5$,
2) if f is not the constant hyperfunction $-$, then $f \notin K_{10}$,
3) $f \in K_1 \bigcap K_2$ if and only if $f \in K_3$.

Proof. 1) Let f be an arbitrary hyperfunction from the set $P_2^- \setminus P_2$. There must be a tuple $\tilde{\alpha}$ such that $f(\tilde{\alpha}) = -$. Moreover, it is obvious that $f(\bar{\tilde{\alpha}}) \in \{0, 1, -\}$. Therefore, $f \notin K_5$.

2) Let f be an arbitrary hyperfunction other than the constant hyperfunction $-$. There are tuples $\tilde{\alpha}^1$ and $\tilde{\alpha}^2$ such that $f(\tilde{\alpha}^1) = -$ and $f(\tilde{\alpha}^2) = \lambda \in \{0, 1\}$. Then $f \begin{pmatrix} \tilde{\alpha}^1 \\ \tilde{\alpha}^1 \\ \tilde{\alpha}^2 \\ \tilde{\alpha}^2 \end{pmatrix} = \begin{pmatrix} - \\ - \\ \lambda \\ \lambda \end{pmatrix} \notin R_{10}$, where column $(\alpha_i^1 \alpha_i^1 \alpha_i^2 \alpha_i^2)^t$ belongs to the predicate R_{10} for any $i \in \{1, \dots, n\}$.

3) The assertion follows directly from the definitions of the K_1, K_2, K_3. □

Lemma 2. *For any $f \in P_2^- \setminus P_2$ the following statements are true:*
1) if $f \notin K_8$ then $f \notin K_{11}$,
2) if $f \notin K_9$ then $f \notin K_{12}$.

Proof. 1) Let $f \notin K_8$. Suppose that f does not satisfy the first condition in the definition of the K_8. There are tuples $\tilde{\alpha}^i = (\alpha_1^i, \dots, \alpha_n^i)$, where $i \in \{1, 2, 3\}$, such that column $(\alpha_j^1 \alpha_j^2 \alpha_j^3)^t$ coincides with one of the columns $(000)^t$, $(001)^t$, $(010)^t$, $(111)^t$ for any j and $f \begin{pmatrix} \tilde{\alpha}^1 \\ \tilde{\alpha}^2 \\ \tilde{\alpha}^3 \end{pmatrix} \in \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\}$. If $f \begin{pmatrix} \tilde{\alpha}^1 \\ \tilde{\alpha}^2 \\ \tilde{\alpha}^3 \end{pmatrix} \in \left\{ \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\}$, then considering that $(000)^t$, $(001)^t$, $(010)^t$, $(111)^t \in R_{11}$ and $(011)^t$, $(101)^t$, $(110)^t \notin R_{11}$, we get $f \notin K_{11}$. If $f\begin{pmatrix}\tilde{\alpha}^1\\ \tilde{\alpha}^2\\ \tilde{\alpha}^3\end{pmatrix} = \begin{pmatrix}1\\0\\0\end{pmatrix}$, then $f\begin{pmatrix}\tilde{\alpha}^1\\ \tilde{\beta}\\ \tilde{\alpha}^3\end{pmatrix} = \begin{pmatrix}1\\-\\0\end{pmatrix} \notin R_{11}$, where the tuple $\tilde{\beta} = (\beta_1, \dots, \beta_n)$ is such that $\beta_k = -$ for those k for which $(\alpha_k^1\alpha_k^2\alpha_k^3)^t = (010)^t$, and $\beta_k = \alpha_k^2$ for the remaining k. Therefore, $f \notin K_{11}$.

Now suppose that f does not satisfy the second condition in the definition of the K_8. There are tuples $\tilde{\alpha}^1$ and $\tilde{\alpha}^2$ such that $f(\tilde{\alpha}^1) = -$ and $f(\tilde{\alpha}^2) = 1$. Consider the value of f on the tuple consisting only of 1. If $f(\tilde{1}) \in \{0, 1\}$, then $f\begin{pmatrix}\tilde{\alpha}^1\\ \tilde{1}\\ \tilde{\alpha}^1\end{pmatrix} \in \left\{\begin{pmatrix}-\\0\\-\end{pmatrix}, \begin{pmatrix}-\\1\\-\end{pmatrix}\right\} \notin R_{11}$. If $f(\tilde{1}) = -$, then $f\begin{pmatrix}\tilde{\alpha}^2\\ \tilde{1}\\ \tilde{\alpha}^2\end{pmatrix} = \begin{pmatrix}1\\-\\1\end{pmatrix} \notin R_{11}$. Therefore, $f \notin K_{11}$.

2) The proof is similar to the proof of the preceding item due to duality. □

Lemma 3. *For any $f \in P_2^- \setminus P_2$ the following statements are true:*
1) if $f \in K_1$, then $f \notin K_9$ and $f \notin K_{12}$;
2) if $f \in K_2$, then $f \notin K_8$ and $f \notin K_{11}$;
3) if $f \in K_1 \bigcap K_2$, then $f \notin K_8 \bigcup K_9$ and $f \notin K_{11} \bigcup K_{12}$.

Proof. 1) Let $f \in K_1$. Then $f(\tilde{0}) = 0$, i.e. there is the tuple on which the value of f is equal to 0. So, taking into account the mandatory existence of a tuple on which the value of the function f is equal to $-$, we obtain that the hyperfunction f does not satisfy the second condition in the definition of the K_9. Therefore, $f \notin K_9$ and, by the point *2)* of Lemma 2, we obtain that $f \notin K_{12}$.

2) The proof is similar to the proof of the preceding item due to duality.

3) The validity of the statement follows from the items *1)* and *2)* of the present Lemma, as well as the items *1)* and *2)* of Lemma 2. □

Lemma 4. *Let $f \in P_2^- \setminus P_2$. If $f \in K_1 \setminus K_2$ or $f \in K_2 \setminus K_1$, then $f \notin K_4$.*

Proof. For definiteness, let $f \in K_1 \setminus K_2$. Then $f(\tilde{0}) = 0$ and $f(\tilde{1}) \in \{0, -\}$. We show that in each case the hyperfunction f does not satisfy the conditions in the definition of the K_4. If $f(\tilde{1}) = 0$, then $f(\bar{\tilde{1}}) = f(\tilde{0}) = 0 \neq 1 = \overline{f(\tilde{0})} = \overline{f(\bar{\tilde{1}})}$. If $f(\tilde{1}) = -$, then $f(\bar{\tilde{1}}) = f(\tilde{0}) = 0 \neq -$. In the case when the hyperfunction f belongs to the set $K_2 \setminus K_1$, the proof is similar. □

Lemma 5. *For any $f \in P_2^- \setminus P_2$ the following statements are true:*
1) if $f \in K_8 \bigcap K_9$, then f is the constant hyperfunction $-$;
2) if $f \in K_{11} \bigcap K_{12}$, then f is the constant hyperfunction $-$.

Proof. 1) Suppose f is not the constant hyperfunction $-$. Then there is a tuple $\tilde{\alpha}$ such that $f(\tilde{\alpha}) = \lambda \in \{0, 1\}$. There is necessarily a tuple in which the value of f is equal to $-$. So, if $\lambda = 0$, then f does not satisfy the second condition in the definition of the K_9, if $\lambda = 1$, then f does not satisfy the second condition in the definition of the K_8. Therefore, either $f \notin K_9$ or $f \notin K_8$, which contradicts the fact that f belongs to the set $K_8 \bigcap K_9$.

2) Suppose f is not the constant hyperfunction $-$. From the previous item, we obtain either $f \notin K_9$ or $f \notin K_8$. Further, using the assertions of Lemma 2, we obtain that either $f \notin K_{12}$ or $f \notin K_{11}$, which contradicts the fact that f belongs to the set $K_{11} \bigcap K_{12}$. □

Lemma 6. *Let $f \in P_2^- \setminus P_2$. If $f \notin K_1 \bigcup K_2$ and $f \in K_4$, then either f is the constant hyperfunction $-$ or $f \notin K_8 \bigcup K_9$ and $f \notin K_{11} \bigcup K_{12}$.*

Proof. Since $f \notin K_1 \bigcup K_2$, then $f(\tilde{0}) \in \{1, -\}$ and $f(\tilde{1}) \in \{0, -\}$. Considering that $f \in K_4$ it is enough to consider cases $f(\tilde{0}) = 1$, $f(\tilde{1}) = 0$ and $f(\tilde{0}) = f(\tilde{1}) = -$. In the case when $f(\tilde{0}) = 1$, $f(\tilde{1}) = 0$ as well as in the proof of Lemma 3, we get that $f \notin K_8 \bigcup K_9$ and $f \notin K_{11} \bigcup K_{12}$. If $f(\tilde{0}) = f(\tilde{1}) = -$, then either f is the constant hyperfunction $-$ and the statement of the Lemma holds, or there is a tuple $\tilde{\alpha}$ such that $f(\tilde{\alpha}) \in \{0, 1\}$. Without loss of generality, we can assume that $f(\tilde{\alpha}) = 0$. Since $f \in K_4$, then $f(\tilde{\bar{\alpha}}) = 1$. Thus, there are tuples on which the hyperfunction f is equal to 0, 1, and $-$. Therefore, $f \notin K_8 \bigcup K_9$ and $f \notin K_{11} \bigcup K_{12}$. □

Lemma 7. *Let $f \in P_2^- \setminus P_2$. If $f \notin K_1 \bigcup K_2$ and $f \notin K_4$, then $f \notin K_{11} \bigcup K_{12}$.*

Proof. Since $f \notin K_1 \bigcup K_2$, then $f(\tilde{0}) \in \{1, -\}$ and $f(\tilde{1}) \in \{0, -\}$. For the cases when $f(\tilde{0})$ and $f(\tilde{1})$ are not equal to $-$ simultaneously, we have $f\begin{pmatrix}\tilde{0}\\ \tilde{1}\\ \tilde{0}\end{pmatrix} \in \left\{\begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}1\\-\\1\end{pmatrix}, \begin{pmatrix}-\\0\\-\end{pmatrix}\right\} \notin R_{11}$ and $f\begin{pmatrix}\tilde{1}\\ \tilde{0}\\ \tilde{1}\end{pmatrix} \in \left\{\begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\-\\0\end{pmatrix}, \begin{pmatrix}-\\1\\-\end{pmatrix}\right\} \notin R_{12}$. Suppose $f(\tilde{0}) = f(\tilde{1}) = -$. Since $f \notin K_4$, it is not the constant hyperfunction $-$ and, so, there is a tuple $\tilde{\alpha}$ such that $f(\tilde{\alpha}) = \lambda \in \{0, 1\}$. Then $f\begin{pmatrix}\tilde{0}\\ \tilde{\alpha}\\ \tilde{0}\end{pmatrix} = \begin{pmatrix}-\\ \lambda\\ -\end{pmatrix} \notin R_{11}$ and $f\begin{pmatrix}\tilde{1}\\ \tilde{\alpha}\\ \tilde{1}\end{pmatrix} = \begin{pmatrix}-\\ \lambda\\ -\end{pmatrix} \notin R_{12}$. □

Theorem 1.1. *The set of all hyperfunctions of rank 2 other than Boolean functions generates no more than 13 equivalence classes with respect to membership in the maximal partial ultraclones.*

Proof. From the first two points of Lemma 1, it follows that for any of the considered hyperfunctions f the components τ_5 and τ_{10} of the vector $\tau(f) = (\tau_1\tau_2\tau_3\tau_4\tau_5 01\tau_8\tau_9\tau_{10}\tau_{11}\tau_{12})$ are equal to 1, where $\tau(f)$ is the vector of membership in the $K_1 - K_{12}$. From the third point of the Lemma 1 we get that $(\tau_1\tau_2\tau_3) \in \{(000), (011), (101), (111)\}$. Consider all these cases.

From the third point of Lemma 3, it follows that the hyperfunctions belonging simultaneously to the K_1, K_2, K_3 are divided into no more than 2 equivalence classes, these classes correspond to the vectors (000010111111), (000110111111).

Now consider the hyperfunctions that either belong to K_1 and do not belong to K_2, K_3, or belong to K_2 and do not belong to K_1, K_3. Using Lemma 2, the first two points of Lemma 3 and Lemma 4, we obtain that the number of equivalence classes for such hyperfunction is no more than 6 and the vectors corresponding to these classes have the form (011110101101), (011110101111), (011110111111), (101110110110), (101110110111), (101110111111).

It remains to consider hyperfunctions that do not belong to any of the K_1, K_2, K_3. It is obvious that among such hyperfunctions there are those that take the value $-$ on each tuple. It is easy to verify that the vector of membership in the $K_1 - K_{12}$ for these hyperfunctions has the form (111010100000). Further we assume that hyperfunctions are not constant. By Lemma 6, we obtain that hyperfunctions belonging to the K_4 can generate at most one equivalence class, to which the membership vector (111010111111) corresponds. Further, applying Lemmas 5 and 7, we obtain that hyperfunctions that do not belong to the K_4 are divided into no more than 3 equivalence classes, which correspond to the vectors (111110101111), (111110110111), (111110111111). □

Theorem 1.2. *The set of all hyperfunctions of rank 2 generates 28 equivalence classes with respect to membership in the maximal partial ultraclones.*

Proof. Since the number of classes of Boolean functions is 15 considering the previous theorem, we obtain that all hyperfunctions are divided into no more than 28 equivalence classes.

As a result of computer calculations on hyperfunctions of three variables, 28 different vectors of membership in the $K_1 - K_{12}$ were found. The Tab. 1 shows the vectors of affiliation and the corresponding hyperfunctions. Table 1 shows membership vectors and the corresponding hyperfunctions. Note that at number 23 there is the constant hyperfunction $-$.

Table 1

N	$\tau(f)$	$f(x_1, x_2, x_3)$	№	$\tau(f)$	$f(x_1, x_2, x_3)$
1	(000000000000)	(00001111)	15	(101110000000)	(11111111)
2	(000000011011)	(01101001)	16	(101110011011)	(10011001)
3	(000000011111)	(00010111)	17	(101110011111)	(10000001)
4	(000010111111)	(000−−111)	18	(101110110110)	(−1111111)
5	(000110001101)	(00000001)	19	(101110110111)	(111111−1)
6	(000110010110)	(00111111)	20	(101110111111)	(100000−1)
7	(000110011111)	(00000111)	21	(111000011011)	(10010110)
8	(000110111111)	(000000−1)	22	(111000011111)	(10001110)
9	(011110000000)	(00000000)	23	(111010100000)	−
10	(011110011011)	(00111100)	24	(111010111111)	(100−−110)
11	(011110011111)	(00000010)	25	(111110011111)	(10000000)
12	(011110101101)	(0000000−)	26	(111110101111)	(−0000000)
13	(011110101111)	(000000−0)	27	(111110110111)	(1111111−)
14	(011110111111)	(0000001−)	28	(111110111111)	(1000000−)

□

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References

[1] S.A.Badmaev, On the Classes of Boolean Functions Generated by Maximal Partial Ultraclones, *The Bulletin of Irkutsk State University. Series Mathematics*, **27**(2019), 3–14 (in Russian).

[2] S.V.Zamaratskaya, V.I.Panteleev, Classification and Types of Bases of All Ultrafunctions on Two-Element Set, *The Bulletin of Irkutsk State University. Series Mathematics*, **16**(2016), 58–70 (in Russian).

[3] A.S.Zinchenko, V.I.Panteleev, On Classes of Hyperfunctions of Rank 2 Generated by Maximal Multiclones, *The Bulletin of Irkutsk State University. Series Mathematics*, **21**(2017), 61–76 (in Russian).

[4] A.S.Kazimirov, V.I. Panteleyev, L.V.Tokareva, Classification and Enumeration of Bases in Clone of All Hyperfunctions on Two-Elements Set, *The Bulletin of Irkutsk State University. Series Mathematics*, **7**(2014), 61–78 (in Russian).

[5] A.S.Kazimirov, V.I.Panteleyev, On the Classes of Boolean Functions Generated by Maximal Multiclones, *The Bulletin of Buryat State University. Mathematics and Informatics*, **9**(2015), 16–22 (in Russian).

[6] S.V.Yablonskij, On the Superpositions of Logic Functions, *Mat. Sbornik*, bf 30(1952), no. 2(72), 329-348 (in Russian).

[7] M.Miyakawa, I.Stojmenović, D.Lau, I.Rosenberg, Classification and Basis Enumerations in Many-Valued Logics, Proc. 17th International Symposium on Multi-Valued logic. Boston, 1987, 151–160.

[8] M.Miyakawa, I.Stojmenović, D.Lau, I.Rosenberg, Classification and Basis Enumerations of the Algebras for Partial Functions, Proc. 19th International Symposium on Multi-Valued logic. Rostock, 1989, 8–13.

[9] S.A.Badmaev, On Some Maximal Clone of Partial Ultrafunctions on a Two- Element Set, *Journal of Siberian Federal University. Mathematics and Physics*, **10**(2017), no. 2, 140–145.

[10] S.A.Badmaev, A Completeness Criterion of Set of Multifunctions in Full Partial Ultraclone of Rank 2, *Siberian Electronic Mathematical Reports*, **15**(2018), 450–474 (in Russian).

Классификация гиперфункций ранга 2 относительно принадлежности максимальным частичным ультраклонам

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В данной работе рассматривается множество гиперфункций, которое является подмножеством полного частичного ультраклона ранга 2. Для гиперфункций решена задача их классификации относительно принадлежности максимальным частичным ультраклонам. Отношение принадлежности максимальным частичным ультраклонам является отношением эквивалентности и порождает соответствующее разбиение на классы эквивалентности. Получено полное описание всех классов эквивалентности, общее число которых равно 28.

Ключевые слова: мультифункция, гиперфункция, клон, ультраклон, максимальный частичный ультраклон, классификация функций.