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Distribution of Small Values of Bohr Almost Periodic Functions with Bounded Spectrum

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For f a nonzero Bohr almost periodic function on $\mathbb{R}$ with a bounded spectrum we proved there exist $C_f > 0$ and integer $n > 0$ such that for every $u > 0$ the mean measure of the set $\{ x : |f(x)| < u \}$ is less than $C_f\, u^{1/n}$. For trigonometric polynomials with $\leq n+1$ frequencies we showed that C_f can be chosen to depend only on n and the modulus of the largest coefficient of f. We showed this bound implies that the Mahler measure $M(h)$, of the lift h of f to a compactification G of $\mathbb{R}$, is positive and discussed the relationship of Mahler measure to the Riemann Hypothesis.

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1. Distribution of small values

$\mathbb{N} := \{1, 2, \dots\}, \mathbb{Z}, \mathbb{R}, \mathbb{C}, \mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$ are the natural, integer, real, complex and circle group numbers, $C_b(\mathbb{R})$ is the C*-algebra of bounded continuous functions and $\chi_\omega : \mathbb{R} \to \mathbb{T},\ \omega \in \mathbb{R}$ the homomorphisms $\chi_\omega(x) := e^{i\omega x},\ \omega \in \mathbb{R}$. A finite sum $f = \sum_\omega a_\omega\, \chi_\omega$ with distinct ω is called a trigonometric polynomial with height $H_f := \max_\omega |a_\omega|$ and they comprise the algebra $T(\mathbb{R})$ of trigonometric polynomials. Bohr [9] defined the C*-algebra $U(\mathbb{R})$ of uniformly almost periodic functions to be the closure of $T(\mathbb{R})$ in $C_b(\mathbb{R})$ and proved that their means $m(f) := \lim_{L\to\infty}(2L)^{-1}\int_{-L}^{L} f(t)dt$ exist. The Fourier transform $\widehat{f} : \mathbb{R} \to \mathbb{C}$ of $f \in U(\mathbb{R})$ is $\widehat{f}(\omega) := m(f\,\overline{\chi}_\omega)$ and its spectrum $\Omega(f) :=$ support $\widehat{f}$. If f is nonzero then $\Omega(f)$ is nonempty and countable and we say f has bounded spectrum if its bandwidth $b(f) \in [0, \infty]$, defined by $b(f) := \sup \Omega(f) - \inf \Omega(f)$, is finite. We observe that if $S \subseteq \mathbb{R}$ is defined by a finite number of inequalities involving functions in $U(\mathbb{R})$ then $m(S) := \lim_{L\to\infty}(2L)^{-1}$measure $[-L, L] \cap S$ exists and define $J_f : (0, \infty) \to [0, 1]$ by

$$J_f(u) := m\left(\{ x \in \mathbb{R} : |f(x)| < u \}\right) \tag{1}$$

Theorem 1.1. *If $f \in U(\mathbb{R})$ is nonzero and has a bounded spectrum then there exist $C_f > 0$ and $n \in \mathbb{N}$ such that:*

$$J_f(u) \leqslant C_f\, u^{\frac{1}{n}}, \quad u > 0. \tag{2}$$

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There exists a sequence C_n such that if $f \in T(\mathbb{R})$ has $n+1$ frequencies then

$$J_f(u) \leqslant C_n\, H_f^{-\frac{1}{n}}\, u^{\frac{1}{n}}, \quad u > 0. \tag{3}$$

Proof. For $f \in \mathrm{U}(\mathbb{R}), \omega \in \mathbb{R}, k \in \mathbb{N}, u > 0$ define $\Xi_{f,\omega,k,u}\,, K_f \,:\, (0,\infty) \to [0,1]$ by

$$\Xi_{f,\omega,k,u}(v) := m\{\, x \in \mathbb{R} \,:\, |f(x)| < u,\, |(\,\chi_\omega f)^{(j)}(x)\,| < v^j, j = 1, \ldots, k\,\}, \tag{4}$$

$$K_f(u) := \inf_{\omega\in\mathbb{R}}\, \inf_{k\in\mathbb{N}}\, \inf_{v>0}\, \Big[3\,\sqrt{2}\,\pi^{-1}\, b(f)\, k\, v^{-1}\, u^{\frac{1}{k}} \;+\; \Xi_{f,\omega,k,u}(v)\Big]\,. \tag{5}$$

We first prove Theorem 1 assuming the following result which we prove latter.

Lemma 1.1. *Every nonzero $f \in U(\mathbb{R})$ with bounded spectrum satisfies $J_f \leqslant K_f$.*

We observe that for every $\omega \in \mathbb{R}$ and every $a \in \mathbb{R}\backslash\{0\}$, if $h(x) = \chi_\omega(x)\, f(ax)$ then $J_h = J_f$ and $K_h = K_f$. Without loss of generality we can assume that $\Omega(f) \subset \Big[-\dfrac{b(f)}{2}, \dfrac{b(f)}{2}\Big]$. If $b(f) = 0$ then $f = c$ and $J_f(u) \leqslant |c|^{-1}u$. If $b(f) > 0$ then Bohr [10] proved that f extends to an entire function F of exponential type $\frac{b(f)}{2}$, and Boas [6], ([7], p. 11, Equation 2.2.12) proved that

$$\limsup_{k\to\infty} |f^{(k)}(x)|^{\frac{1}{k}} = \frac{b(f)}{2} \tag{6}$$

uniformly in x. Therefore for any $v_0 > \dfrac{b(f)}{2}$ there exists $k \in \mathbb{N}$ such that $\Xi_{f,0,k,u}(v_0) = 0$ so Lemma 1.1 implies J_f satisfies (2) with $C_f = 3\,\sqrt{2}\,\pi^{-1}\, b(f)\, k\, v_0^{-1}$ and $n = k$. This proves the first assertion. To prove the second we assume, without loss of generality, that $b(f) = 1$, $\Omega(f) \subset [0,1]$ and

$$f(x) = \sum_{j=1}^{n+1} a_j\, e^{i\,\omega_j\, x}, \quad 0 = \omega_1 < \cdots < \omega_{n+1} = 1, \quad H_f = \max\{|a_j| \,:\, j = 2, \ldots, n+1\}.$$

Define $C_1 := \dfrac{1}{2}$. If $n = 1$ and f has $n+1 = 2$ terms and $f = a_0 + a_1\chi_1$ with $|a_1| = H_f$ and $h = H_f(1-\chi_1)$, then $J_f(u) \leqslant J_h(u) = (2/\pi)\sin^{-1}(u/(2H_f)) \leqslant C_1 H_f^{-1}u$ therefore (3) holds for $n = 1$. For $n \geqslant 2$ we assume by induction that (3) holds for $n-1$ and therefore, since $f^{(1)}$ has n terms and $H_{f^{(1)}} = H_f$, it follows that for all $v > 0$,

$$J_{f^{(1)}}(v) \leqslant C_{n-1}\, H_f^{\frac{1}{n-1}}\, v^{\frac{1}{n-1}}, \tag{7}$$

$$\Xi_{f,0,1,u}(v) \leqslant C_{n-1}\, H_f^{\frac{1}{n-1}}\, v^{\frac{1}{n-1}}. \tag{8}$$

Therefore Lemma 1 with $\omega = 0,\, b(f) = k = 1$ gives

$$J_f(u) \leqslant \inf_{v>0} \left[3\,\sqrt{2}\,\pi^{-1}\; v^{-1}\, u + C_{n-1} H_f^{\frac{1}{n-1}}\, v^{\frac{1}{n-1}}\right] = C_n\, H_f^{\frac{1}{n}}\, u^{\frac{1}{n}} \tag{9}$$

$$\text{where } C_n := C_{n-1}^{1-\frac{1}{n}}\, [3\sqrt{2}\,\pi^{-1}\,(n-1)]^{\frac{1}{n}}\, n(n-1)^{-1}. \tag{10}$$

Remark 1.1. *Computation of 200 million terms shows that $n^{-1}C_n \to 0.900316322$*

Conjecture 1.1. *In (3) C_n can be replaced by a bounded sequence.*

Lemma 1.2. *If $\phi : [a,b] \to \mathbb{C}$ is differentiable and $\phi'([a,b])$ is contained in a quadrant then*

$$b - a \leqslant 2\sqrt{2}\, \frac{\max|\phi|([a,b])}{\min|\phi'|([a,b])}. \tag{11}$$

Proof of Lemma 1.2. We first proved this result in ([18], Lemma 1) where we used it to give a proof, of a conjecture of Boyd [11] about monic polynomials related to Lehmer's conjecture [20], which was reviewed in ([13], Section 3.5) and extended to monic trigonometric polynomials in ([19], Lemma 2). The triangle inequality $|\phi'| \leqslant |\Re\,\phi'| + |\Im\,\phi'|$ gives

$$(b-a)\min|\phi'|([a,b]) \leqslant \int_a^b |\phi'(y)|\,dy \leqslant \int_a^b (\,|\Re\,\phi'(y)| + |\Im\,\phi'(y)|\,)\;dy.$$

Since $\phi'([a,b])$ is contained in a quadrant of $\mathbb{C}$ there exist $c,d \in \{1,-1\}$ such that $|\Re\,\phi'(y)| = c\,\Re\,\phi'(y)$ and $|\Im\,\phi'(y)| = d\,\Im\,\phi'(y)$ for all $y \in [a,b]$. Therefore

$$\int_a^b (\,|\Re\,\phi'(y)| + |\Im\,\phi'(y)|\,)\;dy = (c\,\Re\phi(b) + d\,\Im\phi(b)) - (c\,\Re\phi(a) + d\,\Im\phi(a)).$$

The result follows since the right side is bounded above by $2\sqrt{2}\,\max|\phi|([a,b])$. □

Proof of Lemma 1.1, Assume that $f \in \mathrm{U}(\mathbb{R})$ is nonzero. We may assume without loss of generality that $\Omega(f) \subset [-\frac{b(f)}{2}, \frac{b(f)}{2}]$. For $k \in \mathbb{N}, u > 0, v > 0$ we define the set

$$S_{f,k,u,v} := \{\, x \in \mathbb{R} \,:\, |f(u)| < u,\ \max_{j\in\{1,\dots,k\}} |f^{(j)}(x)|^{\frac{1}{j}} \geqslant v \,\}. \tag{12}$$

We observe that the set of functions in $\mathrm{U}(\mathbb{R})$ whose spectrums are in $[-\frac{b(f)}{2}, \frac{b(f)}{2}]$ is closed under differentiation, and define $s(f,k,u,v) := m(S_{f,k,u,v})$.

$$\text{It suffices to prove that} \quad s(f,k,u,v) \leqslant 3\,\sqrt{2}\,\pi^{-1}\,b(f)\,k\,v^{-1}\,u^{\frac{1}{k}}. \tag{13}$$

Define $\gamma_j := u^{\frac{k-j}{k}}\,v^j, j \in \{0,...,k\}$, and $\mathcal{I}$:= set of closed intervals I satisfying, for some $j \in \{0,1,...,k-1\}$, the following three properties:

1. $f^{(j+1)}(I)$ is a subset of a closed quadrant,
2. $\max|f^{(j)}|(I) \leqslant \gamma_j$ and $\min|f^{(j+1)}|(I) \geqslant \gamma_{j+1}$,
3. I is maximum with respect to properties 1 and 2.

Define $\mathcal{E}$:= set of endpoints of intervals in $\mathcal{I}$, and

$$\psi := \prod_{j=0}^{k-1} (\Re f^{(j+1)})(\Im f^{(j+1)})(|f^{(j)}(x)|^2 - \gamma_j^2)(|f^{(j+1)}(x)|^2 - \gamma_{j+1}^2). \tag{14}$$

$$\text{Lemma 1.2 implies that} \quad \text{length }(I) \leqslant 2\sqrt{2}\,\frac{\gamma_k}{\gamma_{k+1}} = 2\sqrt{2}\,v^{-1}\,u^{\frac{1}{k}}, \quad I \in \mathcal{I}, \tag{15}$$

$$\text{and (12) and Property 3 implies that} \quad S_{f,k,u,v} \subset \bigcup_{I\in\mathcal{I}} I. \tag{16}$$

Clearly $\psi = \Psi|_{\mathbb{R}}$ where Ψ is the product of $6k$ entire functions each having bandwidth $b(f)$ so a theorem of Titchmarsh [25] implies that the density of real zeros of Ψ is bounded above by $3\pi^{-1}\, b(f)\, k$. Property 3 implies that all points in $\mathcal{E}$ are zeros of Ψ so the upper density of intervals in $\mathcal{I}$ is bounded by $\frac{3}{2}\,\pi^{-1}\, b(f)\, k$. Combining these facts gives $s(f,k,u,v) \leqslant (\frac{3}{2}\pi^{-1}\, b(f)\, k)\,(2\sqrt{2}v^{-1}u^{\frac{1}{k}}) = 3\sqrt{2}\pi^{-1}\, b(k)\, k\, v^{-1}u^{\frac{1}{k}}$ which proves (13) and concludes the proof of Lemma 1. $\square$

For $p \in [1,\infty)$ Besicovitch [4] proved that the completion $\mathrm{B}^p(\mathbb{R})$ of $\mathrm{U}(\mathbb{R})$ with norm $(m(|f|^p))^{\frac{1}{p}}$ is a subset of $L^p_{loc}(\mathbb{R})$. For $x \geqslant 0$ we define $\log^+(x) := \log(\max\{1,x\}) \in [0,\infty)$, $\log^-(x) := \log(\min\{1,x\}) \in [-\infty,0]$, and $|x|_j := \max\{|x|, \frac{1}{j}\}$ for $j \in \mathbb{N}$.

Corollary 1.1. *If $f \in U(\mathbb{R})$ satisfies (2), then $\log^- \circ |f| \in B^p(\mathbb{R})$,*

$$m(\,|\log^- \circ |f|\,|^p\,) \leqslant \int_0^1 |\log(u)|^p\, d\, C_f\, u^{\frac{1}{n}} = C_f\, n^p\, \Gamma(p), \tag{17}$$

and $\log \circ |f| \in B^p(\mathbb{R})$.

Proof of Corollary 1.1. Since the means of the functions $\log^- \circ |f|_j\,|^p$ are nondecreasing and bounded by the right side of (17), the sequence $\log^- \circ |f|_j$ is a Cauchy sequence in $B^p(\mathbb{R})$ so it converges to a function $\eta \in B^p(\mathbb{R})$. Therefore $\log^- \circ |f| = \eta$ since it is the pointwise limit of $\log^- \circ |f|_j$ and $\eta \in L^p_{loc}(\mathbb{R})$. The last fact follows since $\log = \log^+ + \log^-$. $\square$

2. Compactifications and Hardy Spaces

Definition 2.1. *A compactification of $\mathbb{R}$ is a pair (G,θ) where G is a compact abelian group and $\theta : \mathbb{R} \to G$ is a continuous homomorphism with a dense image.*

$\mathrm{C}(G)$ is the set of continuous functions on G and $L^p(G), p \in [1,\infty)$ are Banach spaces. If $h \in \mathrm{C}(G)$ then $f := h \circ \theta \in \mathrm{U}(\mathbb{R})$ since by a theorem of Bochner [8] every sequence of translates of f has a subsequence that converges uniformly. We call h the lift of f to G. The Pontryagin dual [24] $\widehat{G}$ of a compact abelian group G is the discrete group of continuous homomorphisms $\chi : G \to \mathbb{T}$ under pointwise multiplication. Bohr proved the existence of a compactification $(\mathbb{B},\theta)$ such that $\mathrm{U}(\mathbb{R}) = \{\, h \circ \theta \,:\, h \in \mathrm{C}(\mathbb{B})\,\}$. The group $\mathbb{B}$ is nonseparable and $\widehat{\mathbb{B}}$ is isomorphic to $\mathbb{R}_d :=$ real numbers with the discrete topology.

Lemma 2.1. *For every $f \in U(\mathbb{R})$ there exists a compactification $(G(f),\theta)$, with $G(f)$ separable, and $h \in C(\mathbb{R})$ such that $f = h \circ \theta$.*

Proof of Lemma 2.1. If $f \in \mathrm{U}(\mathbb{R})$ is nonzero its spectrum $\Omega(f)$ is nonempty and countable so the product group $\mathbb{T}^{\Omega(f)}$ is compact and separable. The function $\theta : \mathbb{R} \to \mathbb{T}^{\Omega(f)}$ defined by $\theta(x)(\omega) := \chi_\omega(x)$ is a continuous homomorphism. Define $G(f) := \overline{\theta(\mathbb{R})}$. Then $(G(f),\theta)$ is a compactification. The function $\widetilde{h} : \theta(\mathbb{R}) \to \mathbb{C}$ defined by $\widetilde{h}(\theta(x) := f(x)$ is uniformly continuous so extends to a unique function $h : G \to \mathbb{C}$ and $f = h \circ \theta$. $\square$

Lemma 2.2. *If (G,θ) is a compactification, $h \in C(G)$, $f = h \circ \theta$, and $\log \circ |f| \in B^p(\mathbb{R})$, then $\log \circ |h| \in L^p(G)$ and $\int_G |\log \circ |h|\,|^p = m(|\log \circ |f|\,|^p)$ for all $p \in [1,\infty)$.*

Proof of Lemma 2.2. The theorem of averages ([3], p. 286) implies that

$$\int_G |\log^- \circ |h|_j\,|^p = m(|\log^- \circ |f|_j\,|^p) \leqslant m(|\log^- \circ |f||^p). \tag{18}$$

The result follows from Lebesgue's monotone convergence theorem since the sequence $|\log \circ |h|_j\,|^p$ is nondecreasing, converges pointwise to $|\log \circ |h|\,|^p$ pointwise and by (18) their integrals are uniformly bounded. $\square$

Definition 2.2. *The Fourier transform* $\mathfrak{F} : L^1(G) \to \ell^\infty(\widehat{G})$ *is defined by* $\mathfrak{F}(h)(\chi) := \int_G f\,\overline{\chi}$.

We define the spectrum $\Omega(h) :=$ support $\mathfrak{F}(h)$. The Hausdorff-Young theorem [15,26] implies that the restrictions give bounded operators $\mathfrak{F} : L^p(G) \to \ell^q(\widehat{G})$ for $p \in [1,\infty)$ and $p^{-1}+q^{-1} = 1$.

Definition 2.3. *A compactification* (G,θ) *induces an injective homomorphism* $\xi : \widehat{G} \to \mathbb{R}$, $\xi(\chi) := \omega$ *where* $\chi \circ \theta = \chi_\omega$, *by which we will identity* $\widehat{G}$ *as a subset of* $\mathbb{R}$ *with the same archimedian order. Therefore if* $h \in C(G)$ *is the lift of* $f \in U(\mathbb{R})$, *then* $\Omega(h) = \Omega(f)$. *The compactification gives Hardy spaces* $H^p(G,\theta) := \{h \in L^p(G) : \Omega(h) \subset [0,\infty)\}$, $p \in [1,\infty]$.

Definition 2.4. *A function* $h \in H^p(G,\theta)$ *is outer if* $\int_G h \neq 0$, $\log \circ |h| \in L^1(G)$, *and*

$$\int_G \log \circ |h| = \log \left| \int_G h \right|. \tag{19}$$

A function $h \in H^p(G,\theta)$ *is inner if* $|h| = 1$.

A polynomial h is outer iff it has no zeros in the open unit disk since a formula of Jensen [16] gives $\int_G \log \circ |h| = \log |h(0)| - \sum_{h(\lambda)=0} \log^-(|\lambda|)$. Beurling [5] proved that a function $h \in H^2(\mathbb{T})$ admits a factorization $h = h_o\, h_i$, with h_o outer and h_i inner, iff $\log \circ |h| \in L^1(\mathbb{T})$.

Let (G,θ) be a compactification. If $h \in \mathrm{C}(G)$ has a bounded spectrum $\Omega(h) \subset [0,\infty)$ and $\int_G h\, d\sigma > 0$ then $f = h \circ \theta$ extends to an entire function F bounded in the upper half plane. We observe that if F has no zeros in the upper half plane, then $\chi_{-b(f)/2}F$ is the Ahiezer spectral factor [1] of the entire function $F(z)\overline{F(\overline{z})}$.

Conjecture 2.1. *h above is outer iff F has no zeros in the open upper half plane.*

3. Mahler Measure and the Riemann Hypothesis

Definition 3.1. *For G a compact abelian group the Mahler measure [22, 23] of* $h \in L^1(G)$ *is* $M(h) := \exp\left(\int_G \log \circ |h|\right) \in [0,\infty)$. *We also define* $M^\pm(h) := \exp\left(\int_G \log^\pm \circ |h|\right)$.

Since $M(h) = M^+(h)M^-(h)$ and $M^+(h) \in [1, \max\{1, ||h||_\infty\}]$, it follows that $M(h) > 0$ iff $\log^- \circ |h| \in L^1(G)$ and then $M^-(h) = \exp\left(-|| \log^- \circ |h|\, ||_1\right)$. Lemma 2.2 implies that this condition holds whenever $h \in \mathrm{C}(G)$ is nonzero and $\Omega(h)$ is bounded.

Definition 3.2. *For* $N \in \mathbb{N}$, $\Phi_N :=$ *product of the first N cyclotomic polynomials.*

Amoroso ([2], Theorem 1.3) proved that the Riemann Hypothesis is equivalent to

$$\log M^+(\Phi_N) \ll_\epsilon N^{\frac{1}{2}+\epsilon}, \quad \epsilon > 0. \tag{20}$$

Define $f_N := \Phi_N \circ \chi_1 \in \mathrm{U}(\mathbb{R})$ and define $J_{f_N} : (0,\infty) \to [0,1]$ by (1). Jensen's formula implies that $M(\Phi_N) = 1$ therefore

$$\log M^+(\Phi_N) = -\int_0^1 \log(u)\, d\, J_{f_N}(u). \tag{21}$$

The bounds that we obtained for J_f in (2) and (3) were exceptionally crude and totally inadequate to obtain (20). When deriving (3) for general polynomials we used the bound (8) $\Xi_{f,0,1,u}(v) =$

$= m(\{ x : |f(x)| < u, |f^{(1)}(x)| < v \} \leqslant m(\{ x : |f^{(1)}(x)| < v \}$. Conjecture (1.1) was based on our intuition that a smaller upper bound holds. We suspect that much smaller upper bounds hold for specific sequences of polynomials as illustrated by the following examples. Construct sequences of height 1 polynomials

$$P_n(z) := 1 + z + \cdots + z^n \ ; \quad Q_n(z) := \binom{n}{[n/2]}^{-1} (1+z)^n \tag{22}$$

and $p_n := P_n \circ \chi_1$, $q_n := Q_n \circ \chi_1$. Both polynomials have maxima at $z = 1$, $||P_n||_\infty = n+1$, Stirling's approximation gives $||Q_n||_\infty \approx \sqrt{\pi\, n/2}$ for large n, and for $u \in (0, 1]$

$$J_{p_n}(u) \leqslant \frac{2}{\pi} \sin^{-1} (\min\{ 1, u \}) \leqslant u \Rightarrow \log(M^-(P_n)) > -1, \tag{23}$$

$$J_{q_n}(u) = \frac{2}{\pi} \sin^{-1} \left(\min \left\{ 1, \frac{1}{2} \binom{n}{[n/2]}^{\frac{1}{n}} u^{\frac{1}{n}} \right\} \right) \geqslant \frac{2}{\pi} u^{\frac{1}{n}} \Rightarrow \log(M^-(Q_n)) < -\frac{2n}{\pi}. \tag{24}$$

Differences between these polynomials arise from their root discrepancy. Those of P_n are nearly evenly spaced. Those of Q_n, all at $z = -1$, have maximally discrepancy.

Conjecture 3.1. *If R_n is a polynomial with $n+1$ terms and height $H(R_n) = 1$ then $M^-(Q_n) \leqslant M^-(R_n) \leqslant M^-(P_n)$.*

The roots of Φ_N have the form $\exp(2\pi i a_k), k = 1, \ldots, \deg \Phi_N$ where a_k are the Farey series consisting of rational numbers in $[0, 1)$ whose denominators are $\leqslant N$. Bounds on the discrepancy of the Farey series were shown by Franel [14] and by Landau [17] to imply the Riemann Hypothesis. The relationship between the discrepancy of roots of a polynomial and its coefficients, and the distributions of roots of entire functions have been extensively studied since the seminal paper by Erdös and Turán [12] and the extensive work by Levin and his school [21]. We suggest that investigation of the functions $\Xi_{f,\omega,k,v,u}$ in (4) and derived functions K_f in (5) may further elucidate how the distribution of small values of polynomials and entire functions depend on their coefficients and roots.

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Распределение малых значений почти периодических функций Бора с ограниченным спектром

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Для f ненулевой почти периодической функции Бора на $\mathbb{R}$ с ограниченным спектром мы доказали, что существуют $C_f > 0$ и целое число $n > 0$ такие что для каждого $u > 0$ средняя мера установить $\{ x : |f(x)| < u \}$ меньше $C_f\, u^{1/n}$. Для тригонометрических полиномов с частотами $\leqslant n+1$ мы показали, что C_f можно выбрать так, чтобы он зависел только от n и модуль наибольшего коэффициента f. Из этой оценки следует, что мера Малера $M(h)$, подъема h из f к компактификации G из $\mathbb{R}$ положительна и обсуждена связь меры Малера с гипотезой Римана.

Ключевые слова: почти периодическая функция, целая функция, факторизация Берлинга, мера Малера, гипотеза Римана.