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Characterizations of Layer-Finite Groups and Their Extensions

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In the paper we review the results characterizing layer-finite groups and almost layer-finite groups in the class of all groups. We also characterize the groups with layer-finite periodic parts, the groups with almost layer-finite periodic parts and the groups with generalized Chernikov periodic parts. Theorems describing the properties of quasi-layer-finite groups and simple quasi-layer-finite groups are provided.

Keywords: infinite groups, involutions, finiteness conditions.

Introduction

One of the perspective directions in the group theory is to find characterizations of a few well-known classes of groups within other groups classes, i.e., determining the weakest sufficient conditions under which the groups of the wide class are included into more narrow classes of groups (the direction has been studied since the 60's). In the paper the characterizations of layer-finite groups, almost layer-finite groups and generalized Chernikov groups in other classes of groups are provided. The review begins with the theorem of S.N.Chernikov describing layer-finite groups in the class of locally normal groups, thise theorem being the first in that direction. Also characterized are the groups with layer-finite periodic parts, the groups with almost layer-finite periodic parts and the groups with generalized Chernikov periodic parts.

The theorems describing properties of quasi-layer-finite groups and simple quasi-layer-finite groups are provided.

The results of S.N.Chernikov, V.P.Shunkov, A.I.Sozutov, V.O.Gomer, M.N.Ivko, E.I.Sedova, S.I.Shahova and V.I.Senashov are also included.

Because of restrictions on the volume of the paper in survey [1] by V.I.Senashov, A.I.Sozutov, V.P.Shunkov only one page is devote to given direction. This survey aims at providing a detailed explanation of the subject.

1. Characterizations of Layer-Finite Groups and Groups with Layer-Finite Periodic Parts

Definition 1. *A group is called layer-finite if there are no more than finite number of elements of each order in it.*

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This concept was first introduced by S.N.Chernikov in the work [2]. The theory of such groups in the developed form is introduced in monographs [3, 4].

Layer-finite groups have been studied in details by S.N.Chernikov, R.Baer, Ch.Ch.Muchamedjan, Ia.D.Polovitsky and others (see, for example, [5, 3, 6, 4]).

The first characterization of layer-finite groups is a theorem of S.N.Chernikov:

Theorem 1 (S.N.Chernikov, [3]). *The class of layer-finite groups coincides with the class of locally normal groups with Chernikov Sylow subgroups.*

Definition 2. *A group is called locally normal if any finite set of its elements is contained in a finite subgroup which is normal in the group.*

Definition 3. *A group is called Chernikov group if it is either finite or finite extension of the direct product of a finite number of quasi-cyclic groups.*

When proving the following results one can further develop the technique of considering groups satisfying finiteness conditions imposed on systems of subgroups. In particular, these conditions are imposed on certain two-generated subgroups and on layers of elements in periodic locally solvable subgroups.

In the following theorems layer-finite groups and the groups with layer-finite periodic parts (i.e., groups in which the set of elements of finite orders is a layer-finite group) are characterized.

At first groups with layer-finite periodic parts under the condition of solvability of all finite subgroups [7, 8], and then without this restriction are characterized:

Theorem 2 (V.I.Senashov, [9, 10]). *A group has a layer-finite periodic part if and only if it is conjugately biprimitively finite and its any periodic locally solvable subgroup is layer-finite.*

Let's remind, that a group G is called *conjugately biprimitively finite* if for any finite subgroup H in the quotient-group $N_G(H)/H$ any two conjugate elements of a prime order generate a finite subgroup. (Such a group is otherwise called a *Shunkov's group*.)

Further finiteness condition is imposed on a system of subgroups connected with one fixed element:

Theorem 3 (M.N.Ivko, V.P.Shunkov, [11]). *A group G containing an element a of a prime order $p \neq 2$, has a layer-finite periodic part in the only case when the following conditions are satisfied:*

1) the normalizer of any non-trivial $\langle a\rangle$-invariant locally finite subgroup of the group G has a layer-finite periodic part;

2) subgroups of the form $\langle a, a^g\rangle$, $g \in G$, are finite and almost all all of them are solvable.

Theorem 4 (V.I.Senashov, [12]). *Let G be a group, let a be its involution satisfying following conditions:*

1) All subgroups of the form $\langle a, a^g\rangle, g \in G$, are finite;

2) The normalizer of any non-trivial $\langle a\rangle$-invariant finite subgroup has a layer-finite periodic part.

Then, either set of all elements of finite orders of the group G generates a layer-finite group or G is a Φ-group.

Definition 4. *Let G be a group, i be its involution, satisfying following conditions:*

1) all subgroups of the form $\langle i, i^g\rangle, g \in G$, are finite;

2) $C_G(i)$ is infinite and has a layer-finite periodic part;

3) $C_G(i) \neq G$ and $C_G(i)$ is not contained in any other subgroup of G with a periodic part;

4) if K is a finite subgroup of G, not lying in $C_G(i)$, and $V = K \cap C_G(i) \neq 1$, then K is a Frobenius group with a complement V.

The group G with some involution i, satisfying the conditions 1—4 is called Φ-group.

This class of groups was introduced by V.P.Shunkov.

Let's present a number of characterizations of layer-finite groups.

Theorem 5 (V.O.Gomer, [13]). *Let G be a periodic binary solvable group, and let a be its element of a prime order p such that:*

1) in $C_G(a)$ any locally finite subgroup is layer-finite and has finite Sylow q-subgroups for all prime q;

2) any locally finite subgroup containing an element a is layer-finite.

Then G is a locally layer-finite (locally finite) group.

Theorem 6 (M.N.Ivko, V.I.Senashov, [12]). *A periodic locally solvable group is layer-finite if and only if for some its finite subgroup B the following condition is satisfied: the normalizer of any non-trivial B-invariant finite subgroup is layer-finite.*

Theorem 7 (M.N.Ivko, [14]). *A group G without involutions has a layer-finite periodic part if and only if for any element a of a prime order p the following conditions are satisfied:*

1) the normalizer of any non-trivial $\langle a\rangle$-invariant finite elementary Abelian subgroup has a layer-finite periodic part;

2) almost all subgroups of the form $\langle a, a^g\rangle$, $g \in G$, are finite.

Let's present a series of criteria of layer-finiteness of a group.

Theorem 8 (E.I.Sedova, [15]). *A periodic finitely approximated F^*-group in which any locally finite subgroup is layer-finite is a locally finite layer-finite group.*

Definition 5. *A group is called finitely approximated if for any its different elements there exists a homomorphism on to a finite group whose images of these elements also are different.*

Definition 6. *A group G is called a F_q-group if for any its finite subgroup K and for any elements a, b of the same simple order $q \in \pi(G)$ from $T = N_G(K)/K$ there will be an element $c \in T$, that $\langle a, b^c\rangle$ is finite. If this property is inherited by subgroups of group G then it is called F_q^*-group (see question 7.42 of [16]).*

Theorem 9 (E.I.Sedova, [15]). *A periodic group is a locally solvable layer-finite group in the only case when it is binary solvable and any its locally solvable subgroup is layer-finite.*

Theorem 10 (E.I.Sedova, V.P.Shunkov, [15]). *A periodic F^*-group without involutions with finite locally finite primary subgroups is a layer-finite group with finite Sylow p-subgroups for all p in the only case when it is a F^{**}-group and any its locally finite subgroup is layer-finite.*

Definition 7. *A group G is called F_p^{**}-group if it is an F_p^*-group and for any finite subgroup H in $N_G(H)/H$ any element of order p with almost each conjugate element generates a finitely approximated subgroup.*

Theorem 11 (M.N.Ivko, [14]). *A 2-biprimitivly finite group G of the form $G = H\lambda L$, where H is a subgroup without involutions and L is a Klein's group of order four, is layer-finite in the only case when the centralizer in G of any involution from L is layer-finite.*

Corollary (M.N.Ivko, [14]). *A periodic group H without involutions whose holomorph possesses a Klein's subgroup L of order four is layer-finite in the only case when the centralizer in H of any involution from L is layer-finite.*

2. On Quasi-Layer-Finite Groups

Definition 8. *By a Quasi-layer-finite group we mean a non-layer-finite group, all of whose own subgroups are layer-finite.*

With the help of techniqnes of fans in [17] the structure of quasi-layer-finite groups is described.

Definition 9. *Any set X of finite subgroups of the group G is called a fan with the basis T, if T is intersection of all subgroups in the set X, and $T \neq 1$ (the basis of an empty fan is considered any).*

Theorem 12 (A.I.Sozutov, S.I.Shahova, [17]). *For an infinite group G all of whose own subgroups are layer-finite, one of the following statements is valid.*

1) G is a layer-finite group;

2) $G = P\lambda\langle a\rangle$, where P is a Chernikov complete Abelian p-group not containing own infinite $\langle a\rangle$-invariant subgroups, $\langle a\rangle$ is a primary group and $|G : C_G(P)|$ is a prime number;

3) $G/Z(G)$ is a simple not locally finite group.

Simple quasi-layer-finite groups have an interesting structure. We will give some properties of simple quasi-layer-finite groups.

Theorem 13 (A.I.Sozutov, S.I.Shahova, [17]). *In a simple infinite group, all of whose own subgroups are layer-finite, the intersection of any two infinite maximal subgroups is the unit.*

Theorem 14 (A.I.Sozutov, S.I.Shahova, [17]). *If a simple quasi-layer-finite group G contains an involution i, then $C_G(i) = H$ is an infinite maximal subgroup of the group G, involution in H is unique, all involutions in G are conjugated, the Sylow 2-subgroups in G are conjugated and are either (locally) cyclic or finite (generalized) quaternion groups.*

Theorem 15 (A.I.Sozutov, S.I.Shahova, [18]). *For any pair of non-unit elements a, b of infinite simple quasi-layer-finite group G, at least one of which is not an involution, there will be infinitely many elements b^g such, that $G = \langle a, b^g\rangle$.*

3. Characterizations of Almost Layer-Finite Groups

An almost layer-finite group is a group which is a finite extension of a layer-finite group. Almost layer-finite groups represent an essentially wider class of groups than layer-finite groups, which, in particular, contains all Chernikov's groups.

Like Schmidt's problem (whether there is a finite group in which any own subgroup is finite) it is natural to propose the following question: what properties are transferred to the whole group from some system of its subgroups? Here the question is solved for the condition of almost layer-finiteness. At first this question is solved in the class of locally finite groups:

Theorem 16 (V.P. Shunkov, [19]). *A locally finite group G is an almost layer-finite in the only case when next condition is valid in G: the normalizer of any nontrivial finite subgroup from in G is an almost layer-finite group.*

Then the same question is solved in the class of conjugately biprimitively finite groups without involutions:

Theorem 17 (V.I.Senashov, [20]). *Let G be a periodic conjugately biprimitively finite group without involutions. If in G the normalizer of any non-trivial finite subgroup is almost layer-finite, then the group G is almost layer-finite.*

It is impossible to remove the condition of conjugately biprimitively finitenesses in the theorem in view of the examples of the Novikov-Adian group [21] and of the Olshansky group [22].

The following theorem establishes the existence of not almost layer-finite subgroups in not almost layer-finite group which satisfies to the conjugately biprimitively finiteness condition and has a strongly embedded subgroup.

Theorem 18 (V.I.Senashov, [23, 24]). *Every conjugately biprimitively finite group with a strongly embedded subgroup is either almost layer-finite or contains an own not almost layer-finite subgroup.*

Definition 10. *A subgroup H of a group G i thes called strongly embedded in G if H is an own subgroup of the group G with involutions and intersection of H with $x^{-1}Hx$ does not contain involutions for $x \in G \setminus H$.*

One more criterion of almost layer-finiteness of a group was obtained by M.N. Ivko.

Theorem 19 (M.N.Ivko, [14]). *Let G be a periodic almost locally solvable group with an elementary Abelian subgroup V of order p^2. If the centralizer in G of any non-trivial element from V is layer-finite, then the group G is almost layer-finite.*

A group satisfies the *minimality condition for not almost layer-finite subgroups* if any strictly decreasing chain of its subgroups after finite number of steps reaches to an almost layer-finite subgroup.

The following theorems characterise almost layer-finite groups with the use of this condition:

Theorem 20 (V.I.Senashov, [25, 26]). *Let G be a conjugately biprimitively finite group without elements of the third order. If the group G satisfies the minimality condition for not almost layer-finite subgroups, then G is an almost layer-finite group.*

Theorem 21 (V.I.Senashov, [26]). *Let G be a conjugately biprimitively finite, binary solvable group. If the group G satisfies the minimality condition for not almost layer-finite subgroups, then G is almost layer-finite.*

Theorem 22 (V.I.Senashov, [26]). *Let G be a conjugately biprimitively finite group, such that the centralizers of every involution in it are Chernikov. If the group G satisfies the minimality condition for not almost layer-finite subgroups, then G is an almost layer-finite group.*

Further research of a question on carrying over the property of almost layer-finiteness from normalizers of non-trivial finite subgroups to the whole group has resulted in the following theorems.

Theorem 23 (V.I.Senashov, [27]). *Let a periodic Shunkov's group G contain a strongly embedded subgroup. If the normalizers of any non-trivial finite subgroups in G are almost layer-finite, then the group G is also almost layer-finite.*

Theorem 24 (V.I.Senashov, [28]). *Let G be a periodic Shunkov's group without subgroups $PSL_2(q)$. If the normalizers of any non-trivial finite subgroup in G are almost layer-finite, then the group G is also almost layer-finite.*

The following theorems provide characterizations of groups with almost layer-finite periodic parts. Let's adduce results for the mixed groups with conjugately biprimitively finiteness condition.

Theorem 25 (V.I.Senashov, [19]). *Let G be a Shunkov's group without involutions. If in G normalizers of any non-trivial finite subgroups have almost layer-finite periodic parts, then the group G also has an almost layer-finite periodic part.*

Theorem 26 (V.I.Senashov, [29]). *Let Shunkov's group G contain a strongly embedded subgroup with Chernikov almost layer-finite periodic part. If the normalizers of all non-trivial finite subgroup in G have an almost layer-finite periodic parts, then the group G has the almost layer-finite periodic part.*

4. Characterizations of Generalized Chernikov Groups

Characterizations of a few well-known classes of groups within other groups classes is determining of the weakest sufficient conditions under which the groups of the wide class are included into more narrow classes of groups.

Above we considered from this point of view the class of layer-finite groups, i.e., groups, in which every set of elements of any given order is finite, and the class of finite extensions of layer-finite groups, i.e., almost layer-finite groups. The following natural step in this direction is a description of generalized Chernikov groups which are extensions of layer-finite groups by means of of locally normal groups. We will now consider this class of groups.

Definition 11. *A group G satisfies the primary minimality condition, if for any prime p every chain $G_1 > G_2 > ... > G_n > ...$ of subgroups in G such that in any difference $G_n \setminus G_{n+1}$ there is an element g_n which in degree p^{k_n} is contained in G_{n+1} for some k_n, terminates after a finite number of steps (this definition belongs to S.N. Chernikov).*

Definition 12. *A periodic almost locally solvable group, satisfying the primary minimality condition is called a generalized Chernikov group.*

The term "generalized Chernikov group" has appeared for the first time in 1979 in the work of V.P.Shunkov, A.A.Shafiro [30]. Its use can be proved that by theorem of Ia.D. Polovitsky [31] a generalized Chernikov group G is an extension of the direct product A of quasicyclic p-groups with a finite number of multipliers for every prime number p with the help of a locally normal group B, and every element in G not permutable with elements for only finite number of primary Sylow subgroups of A. For comparison a Chernikov group is a finite extension of the direct product of quasi-cyclic groups taken in finite number.

In the research of infinite groups one of the basic methods is imposing the condition of termination of chains of subgroups on a group. Such conditions: are primary minimality condition, the minimality condition for Abelian subgroups and, at last, the minimality condition for subgroups.

A little later there were finiteness conditions which have not been connected with termination of chains of subgroups: binary finiteness, biprimitive finiteness, conjugately biprimitively finiteness. These conditions concern the finiteness of some two-generated subgroups in a group or in its sections.

The relations between conditions of finiteness of these two types were investigated by S.N.Chernikov, V.P.Shunkov, M.I.Kargapolov and by other authors. Here it is possible to connect conditions of the specified types, namely, to prove extendability of a condition of the first type a from a system of subgroups to the whole group by imposing a condition of the second type on a group.

Theorem 27 (V.I.Senashov, [32]). *Let G be a group with involutions, satisfying the conditions:*

1) any two involutions from G generate a finite subgroup;

2) the normalizer of any finite non-trivial subgroup, containing involutions, has generalized Chernikov periodic part.

Then either the group G has a generalized Chernikov periodic part or G is a T-group.

Definition 13. *Let G be a group with involutions. With every involution i from G we will associate a set of subgroups W_i, defined as follows. If the Sylow 2-subgroups from G are dihedral groups of 8th order and i is contained in a Klein's subgroup R_i of order four such that $C_G(i) < N_G(R_i)$, then suppose $V_i = N_G(R_i)$ and such a subgroup V_i is included in the set W_i. The group G with involutions is called a T-group if it satisfies the conditions:*

1) any two involutions from G generate a finite subgroup;

2) the normalizer of any non-trivial locally finite subgroup from G, containing involutions, has a locally finite periodic part;

Further, for any involution i and for any subgroup V_i from W_i

3) the set $G \setminus V_i$ possesses an involution;

4) for any element c from $G \setminus V_i$, strictly real with respect to i, i.e., $c^i = c^{-1}$, there exists an element s_c in $C_G(i)$ such that the subgroup $\langle c, c^{s_c} \rangle$ is infinite.

This class of groups was introduced by V.P.Shunkov.

Let's give one more criterion for a group to have a generalized Chernikov periodic part.

Theorem 28 (V.I.Senashov, [33, 34]). *A group has a generalized Chernikov periodic part in the only case when it is conjugately biprimitively finite and the normalizer of any its finite non-trivial subgroup has a generalized Chernikov periodic part.*

The examples of the groups constructed by S.P.Novikov, S.I.Adian [21] and by A.Yu.Olshansky [22] show, that the condition of conjugately biprimitive finiteness in the theorem is cannot be released.

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