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On Estimates of Solutions of the Split Problems for Some Multi-Dimensional Partial Differential Equations

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We consider multidimensional second order parabolic equations and the first order partial differential equations. We consider various splittings of the Cauchy problem in the case when the coefficients of the equation depending on time and all space variables and have a special form. The uniform correctness of the split problems, that is, a sufficient condition for the split problems solutions convergence to the solutions to the original problems and the uniform correctness of this problem is proved in classes of smooth functions.

Keywords: differential equation, Cauchy problem, split, stability, convergence.

The question of convergence of the weak approximation method — WAM (a method of fractional steps in the differential form) where the problem of the split problem solution convergence to the solution to the original problem with the Cauchy data is investigated in [1–3]. The uniform correctness of the split problem and its differential extensions is supposed in the specified works.

We consider multi-dimensional second order parabolic equations and the first order partial differential equations. We consider various splittings of the Cauchy problem in the case when the coefficients of the equation depend on time and all space variables and have a special form. The uniform correctness of the split problems, that is, a sufficient condition for the split problems solutions convergence to the solution to the original problem and the uniform correctness of this problem is proved in classes of smooth functions under sufficient smoothness condition of the initial data. The uniform correctness of the original problem is not supposed.

The obtained results can be applied to nonlinear equations and systems by the splitting method on the linear equations of the above mentioned type and equations containing nonlinear members [4-6].

For splitting of various equations and systems of the equations of mathematical physics at a differential level see [7–9]. The convergence of WAM for the differential evolutionary equations in Banach spaces is treated in [7].

The multi-dimensional problems of the coefficients identification of the second order partial differential equations in a case when known leading coefficients depend only on the time variable are investigated in [4–6]. This is connected by application of WAM for the proof of solvability of input problems and the requirement in doing so stability of the split problems in some functional classes.

We shall prove exponential stability of the split problems for the multi-dimensional equations that allows to investigate, in particular, problems of identification of unknown coefficients and in

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a case when known coefficients depend on all independent variables and the differential operator has a special structure.

We consider in $\Pi_{[0,T]} = \{(t,x)|\, 0 \leqslant t \leqslant T, x \in E_n\}$, where E_n is the n-dimensional Euclidean space, $n \geqslant 1$, the Cauchy problem for the equations

$$u_t = \sum_{i=1}^{n} b_i(t,x) u_{x_i}(t,x),$$

$$u_t = \sum_{i=1}^{n} a_i(t,x_i) u_{x_i x_i}(t,x),$$

$$u_t = \sum_{i=1}^{n} a_i(t,x_i) u_{x_i x_i}(t,x) + \sum_{i=1}^{n} b_i(t,x) u_{x_i}(t,x) + c(t,x)u(t,x) + f(t,x),$$

and we investigate the stability of the corresponding splittings of these problems.

1. Equation of the First Order

1.1. Formulation of Theorems 1.1, 1.2

Consider in $\Pi_{[0,T]} = \{(t,x)|0 \leqslant t \leqslant T,\, x \in E_n\}$, the problem

$$u_t = \sum_{i=1}^{n} b_i(t,x) \frac{\partial u(t,x)}{\partial x_i}, \tag{1.1.1}$$

$$u(0,x) = u_0(x), \quad x \in E_n. \tag{1.1.2}$$

Here, $b_i(t,x)$ and $u_0(x)$ are functions given in $\Pi_{[0,T]}$ and E_n, respectively.

We split the problem (1.1.1), (1.1.2) into one-dimensional problems [3, 7]

$$\left.\begin{array}{lll}
u_t^\tau = n b_1 u_{x_1}^\tau, & k\tau < t \leqslant (k+\frac{1}{n})\tau, & (1.1.3_1)\\
u_t^\tau = n b_2 u_{x_2}^\tau, & (k+\frac{1}{n})\tau < t \leqslant (k+\frac{2}{n})\tau, & (1.1.3_2)\\
\dots\dots\dots & & \\
u_t^\tau = n b_i u_{x_i}^\tau, & (k+\frac{i-1}{n})\tau < t \leqslant (k+\frac{i}{n})\tau, & (1.1.3_i)\\
\dots\dots\dots & & \\
u_t^\tau = n b_n u_{x_n}^\tau, & (k+\frac{n-1}{n})\tau < t \leqslant (k+1)\tau, & (1.1.3_n)
\end{array}\right\} \tag{1.1.3}$$

$$u^\tau(0,x) = u_0(x), \quad x \in E_n, \tag{1.1.4}$$

where $\tau = \dfrac{T}{N}$, $k = 0, 1, \dots, N-1$, N is an integer.

We take the initial data $u^\tau|_{t=(k+\frac{i-1}{n})\tau}$ on the fractional step i of the step k from the previous fractional step:

$$u^\tau|_{t=(k+\frac{i-1}{n})\tau} = \lim_{\substack{t\to(k+\frac{i-1}{n})\tau \\ t<(k+\frac{i-1}{n})\tau}} u^\tau(t,x), \quad i = 1, \dots, n. \tag{1.1.5}$$

Assume that the input data of problems (1.1.1), (1.1.2) satisfy

$$|D^\alpha b_i(t,x)| \leqslant M(p), \quad |\alpha| \leqslant p, \quad i = 1, \dots, n; \tag{1.1.6}$$

$$|D^\alpha u_0(x)| \leqslant C(p), \quad |\alpha| \leqslant p, \tag{1.1.7}$$

in $\Pi_{[0,T]}$ and E_n, respectively. All derivatives in (1.1.6), (1.1.7) are continuous. In (1.1.6), (1.1.7) $\alpha = (\alpha_1, \dots, \alpha_n)$ is a multi-index, $D^\alpha = \dfrac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}}$, $|\alpha| = \sum\limits_{i=1}^{n} \alpha_i$, $p > 1$ is an integer.

The following theorems are valid.

Theorem 1.1. *Let the conditions (1.1.6), (1.1.7) hold. Then problem (1.1.3), (1.1.4) has a unique solution $u^\tau(t,x)$, and that solution satisfies the inequalitys*

$$|D^\alpha u^\tau(t,x)| \leqslant C_{k,j}(p) e^{l\tau/n},\ (t,x) \in \Pi_{[(k+\frac{j-1}{n})\tau,(k+\frac{j}{n})\tau]},\ j = 1, \dots, n; \tag{1.1.8}$$

$$|D^\alpha u^\tau(t,x)| \leqslant C_k(p) e^{l\tau},\ (t,x) \in \Pi_{[k\tau,(k+1)\tau]},\ k = 0, 1, \dots, N-1;\ |\alpha| \leqslant p. \tag{1.1.9}$$

Here $C_{k,j}(p) = \max\limits_{|\alpha|\leqslant p} \sup\limits_{E_n} \left| D^\alpha u^\tau \left(\left(k + \dfrac{j-1}{n} \right) \tau, x \right) \right|$, $C_k(p) = \max\limits_{|\alpha|\leqslant p} \sup\limits_{E_n} |D^\alpha u^\tau(k\tau, x)|$, the constant l depends only on p and does not depend on τ, $C_0(p) = C_{0,1}(p) = \max\limits_{|\alpha|\leqslant p} \sup\limits_{E_n} |D^\alpha u_0(x)| = C(p)$.

Theorem 1.2. *Let the conditions of Theorem 1.1 be fulfilled. Then the solutions $u^\tau(t,x)$ of the problem (1.1.3), (1.1.4) converge to the solution $u(t,x)$ of the problem (1.1.1), (1.1.2) in $\Pi_{[0,T]}$ and uniformly in $\Pi^R_{[0,T]}$ for all fixed R, $R > 0$, together with the derivatives $D^\alpha u^\tau(t,x)$ with respect to space variables up to the order $p-1$ as $\tau \to 0$:*

$$\lim_{\tau \to 0} D^\alpha u^\tau(t,x) = D^\alpha u(t,x), \quad (t,x) \in \Pi_{[0,T]}, \tag{1.1.10}$$

$$\lim_{\tau \to 0} \|D^\alpha u^\tau - D^\alpha u\|_{C(\Pi^R_{[0,T]})} = 0,\ |\alpha| \leqslant p-1. \tag{1.1.11}$$

Under $p \geqslant 2$ the function $u(t,x)$ is the solution to problem (1.1.1), (1.1.2) of the class $C^{1,p-1}(\Pi_{[0,T]})$.

In (1.1.11) $\Pi^R_{[0,T]} = \{(t,x) | 0 \leqslant t \leqslant T,\ x \in E_n,\ |x| \leqslant R\}$, $R > 0$ – const.

Consider the linear partial differential equation of the first order

$$z_t + \sum_{i=1}^{n} f_i(t,x,\lambda) z_{x_i} + f_0(t,x,\lambda) z = f(t,x,\lambda). \tag{1.1.12}$$

Here $x = (x_1, \dots, x_n)$, and $\lambda = (\lambda_1, \dots \lambda_m)$ is a parameter.

Assumption 1.1. *We make the assumption that in the domain $\Pi_{[0,T]} = \{(t,x) | t_0 \leqslant t \leqslant t_1, x \in E_n\}$ the functions f_i, $i \geqslant 1$, are bounded for every fixed λ. The functions f_i, f are continuous, and partial derivatives f_{ix_j}, $f_{i\lambda_r}$, $i = 0, 1, \dots, n$, and f_{x_j}, f_{λ_r} exist, they are continuous and $k-1$ times continuously differentiable with respect to all $n+m+1$ arguments ($k \geqslant 1$). The function $\omega(x,\lambda)$ is k times continuously differentiable according to all $n+m$ arguments in the domain $-\infty < x_1, \dots, x_n < +\infty$, $-\infty < \lambda_1, \dots, \lambda_m < +\infty$.*

The result of section 4.3 in [8] is formulated as the following theorem.

Theorem 1.3. *If the assumption 1.1 for all θ, λ from intervals $t_0 \leqslant \theta \leqslant t_1$, $-\infty < \lambda_1, \dots, \lambda_m < +\infty$ is satisfied then the equation (1.1.12) has in $\Pi_{[t_0,t_1]}$ the unique integral $z = \psi(t,x,\theta,\lambda)$ with the initial value $\psi(\theta,x;\theta,\lambda) = \omega(x,\lambda)$. This integral is k times continuously differentiable with respect to all $m+n+2$ arguments.*

If $x_i = \varphi_i(t, \theta, \eta_1, \ldots, \eta_n, \lambda)$ *are characteristic functions of the system*

$$x_i'(t) = f_i(t, x, \lambda), \quad i = 1, \ldots, n, \tag{1.1.13}$$

(i.e., integral curves of the system (1.1.13), which go through the point $(\theta, \eta_1, \ldots, \eta_n)$*), then the parametric representation of the integral takes the following form:*

$$x_i = \varphi_i(t, \theta, \eta_1, \ldots, \eta_n, \lambda), \quad i = 1, \ldots, n,$$

$$z = \exp\{-F_0\}\{\omega(\eta_1, \ldots, \eta_n, \lambda) + \int\limits_\theta^t f(t, \varphi_1, \ldots, \varphi_n, \lambda) \exp\{F_0\} dt\}, \tag{1.1.14}$$

where $F_0 = F_0(t, \theta, \eta_1, \ldots, \eta_n, \lambda) = \int\limits_\theta^t f_0(t, \varphi_1, \ldots, \varphi_n, \lambda) dt.$

Denote $\theta = t_0$, $\omega(x, \lambda) = z(t_0, x, \lambda)$ and assume that the following conditions are fulfilled:

$$|f_0(t, x, \lambda)| \leqslant M, \quad (t, x) \in \Pi_{[t_0, t_1]}, \quad \lambda \in E_m, \tag{1.1.15}$$

$$|\omega(x, \lambda)| \leqslant q, \quad x \in E_n, \quad \lambda \in E_m, \tag{1.1.16}$$

$$|f(t, x, \lambda)| \leqslant N, \quad (t, x) \in \Pi_{[t_0, t_1]}, \quad \lambda \in E_m, \tag{1.1.17}$$

where M, q, N are constants.

By virtue of (1.1.14) we can prove that the estimate

$$|z(t, x)| \leqslant e^{M(t - t_0)}(q + N(t - t_0)), \quad (t, x) \in \Pi_{[t_0, t_1]}. \tag{1.1.18}$$

holds due to (1.1.15)–(1.1.17).

1.2. Proof of Theorems 1.1, 1.2

In order to simplify the proof of Theorems 1.1, 1.2, we shall give it in the case $n = 2$. The general case can be treated similarly.

Consider in $\Pi_{[0,T]} = \{(t, x, y) | 0 \leqslant t \leqslant T, (x, y) \in E_2\}$ the Cauchy problem

$$\frac{\partial u}{\partial t} + a \frac{\partial u}{\partial x} + b \frac{\partial u}{\partial y} = 0, \tag{1.2.1}$$

$$u(0, x, y) = u_0(x, y). \tag{1.2.2}$$

Here $a = a(t, x, y)$, $b = b(t, x, y)$, $u_0(x, y)$ are functions given in $\Pi_{[0,T]}$, $\Pi_{[0,T]}$, E_2 respectively.

We replace the problem (1.2.1), (1.2.2) by the problem:

$$\frac{\partial u^\tau}{\partial t} + 2a \frac{\partial u^\tau}{\partial x} = 0, \; k\tau < t \leqslant \left(k + \frac{1}{2}\right)\tau, \tag{1.2.3}$$

$$\frac{\partial u^\tau}{\partial t} + 2b \frac{\partial u^\tau}{\partial y} = 0, \; (k + \frac{1}{2})\tau < t \leqslant (k + 1)\tau, \tag{1.2.4}$$

$$u^\tau(0, x, y) = u_0(x, y), \tag{1.2.5}$$

$k = 0, 1, \ldots, N-1$; $\tau N = T$. The initial data on each fractional step is taken from the previous fractional step:

$$u^\tau\left(\left(k + \frac{j-1}{2}\right)\tau, x, y\right) = \lim_{\substack{t \to (k+\frac{j-1}{2})\tau \\ t<(k+\frac{j-1}{2})\tau}} u^\tau(t, x, y), \quad j = 1, 2;\ k = 0, \ldots, N-1.$$

The conditions (1.1.6), (1.1.7) become

$$|D^\alpha a(t, x, y)| \leqslant M(p), \quad |D^\alpha b(t, x, y)| \leqslant M(p), \quad (t, x, y) \in \Pi_{[0,T]}, \tag{1.2.6}$$

$$|D^\alpha u_0(x, y)| \leqslant C(p), \quad (x, y) \in E_2, \tag{1.2.7}$$

$\alpha = (\alpha_1, \alpha_2)$, $|\alpha| = \alpha_1 + \alpha_2$, $D^\alpha g = \dfrac{\partial^{|\alpha|} g}{\partial x^{\alpha_1} \partial y^{\alpha_2}}$.

By virtue of Theorem 1.3 we can prove that in the view of (1.2.6), (1.2.7) the solution $u^\tau(t, x, y)$ to the problem (1.2.3)–(1.2.5) exists for all $\tau > 0$ in $\Pi_{[0,T]}$ (has continuous derivatives $D^\alpha u^\tau$ in $\Pi_{[0,T]}$ for $|\alpha| \leqslant p$ and continuous derivatives u^τ_t in $\Pi_{[0,T]}$, but u^τ_t has discontinuities on hyperplanes $t_{k,j} = \left(k + \frac{j}{2}\right)\tau$, $k = 0, \ldots, N-1$; $j = 0, 1, 2$).

Consider the zeroth whole step ($k = 0$). At the first fractional step we solve the problem

$$u^\tau_t + 2a u^\tau_x = 0, \quad 0 < t \leqslant \frac{\tau}{2}, \tag{1.2.8}$$

$$u^\tau(0, x, y) = u_0(x, y). \tag{1.2.9}$$

From (1.1.18), (1.2.6), (1.2.7) (here $\omega(x, y) = u_0(x, y)$, $f = 0$, $M = 0$) it follows that

$$|u^\tau(t, x, y)| \leqslant C(p), \quad 0 \leqslant t \leqslant \frac{\tau}{2}. \tag{1.2.10}$$

By differentiating the problem (1.2.8), (1.2.9) with respect to x, we get for $v^\tau = u^\tau_x$ the problem

$$\begin{aligned} &v^\tau_t + 2a v^\tau_x + 2a_x v^\tau = 0, \quad 0 < t \leqslant \frac{\tau}{2}, \\ &v^\tau(0, x, y) = u_{0x}(x, y). \end{aligned} \tag{1.2.11}$$

By virtue of (1.1.18), (1.2.6), (1.2.7) from (1.2.11) we get the inequality

$$|u^\tau_x(t, x, y)| = |v^\tau(t, x, y)| \leqslant C(p) e^{M(p)\tau}, \quad 0 \leqslant t \leqslant \frac{\tau}{2}. \tag{1.2.12}$$

By differentiating the problem (1.2.11) with respect to x (differentiating twice the problem (1.2.8) with respect to x), for $\omega^\tau = u^\tau_{xx}$ we get the problem

$$\begin{aligned} &\omega^\tau_t + 2a\omega^\tau_x + 4a_x \omega^\tau + 2a_{xx} u^\tau_x = 0, \quad 0 < t \leqslant \frac{\tau}{2}, \\ &\omega^\tau(0, x, y) = u_{0xx}(x, y). \end{aligned} \tag{1.2.13}$$

Taking into account (1.1.18), (1.2.6), (1.2.7), (1.2.12) from (1.2.13) we obtain the inequality

$$\begin{aligned} |u^\tau_{xx}(t, x, y)| &= |\omega^\tau(t, x, y)| \leqslant \\ &\leqslant e^{2M(p)\tau}(C(p) + M(p)C(p)e^{M(p)\tau}\tau) \leqslant C(p) e^{4M(p)\tau}. \end{aligned} \tag{1.2.14}$$

By differentiating j times, $j = 0, 1, \dots, p$, with respect to x the problem (1.2.8), (1.2.9), we obtain the inequality

$$\left| \frac{\partial^j}{\partial x^j} u^\tau(t,x,y) \right| \leqslant C(p) e^{C_1(p)M(p)\tau}, \quad j = 0,1,\dots,p, \ (t,x,y) \in \Pi_{[0,\frac{\tau}{2}]}, \tag{1.2.15}$$

where the constant $C_1(p)$ depends on p only and does not depend on τ.

Below we estimate the derivatives containing differentiation with respect to the variable y. We notice that in the problem (1.2.8), (1.2.9) the variable y is a parameter.

Let us estimate u_y^τ. We differentiate (1.2.8), (1.2.9) with respect to y.
The function $\alpha^\tau(t,x,y) = u_y^\tau(t,x,y)$ is the solution to problem

$$\begin{aligned} \alpha_t^\tau + 2a\alpha_x^\tau + 2a_y u_x^\tau = 0, \quad 0 < t \leqslant \frac{\tau}{2}, \\ \alpha^\tau(0,x,y) = u_{0y}(x,y), \quad (x,y) \in E_2. \end{aligned} \tag{1.2.16}$$

By virtue of (1.1.18), (1.2.6), (1.2.7), (1.2.12) from (1.2.16) we get the inequality

$$\begin{aligned} |u_y^\tau(t,x,y)| = |\alpha^\tau(t,x,y)| \leqslant C(p) + M(p)C(p)e^{M(p)\tau}\tau \leqslant \\ \leqslant C(p)e^{M(p)\tau}(1 + M(p)\tau) \leqslant C(p)e^{M(p)2\tau}. \end{aligned} \tag{1.2.17}$$

We differentiate the problem (1.2.11) with respect to y. For the function $\beta^\tau = \dfrac{\partial^2 u^\tau}{\partial x \partial y}$ we get the problem

$$\begin{aligned} \beta_t^\tau + 2a\beta_x^\tau + 2a_x\beta^\tau + 2a_y u_{xx}^\tau + 2a_{xy} u_x^\tau = 0, \\ \beta^\tau(0,x,y) = u_{0xy}(x,y). \end{aligned} \tag{1.2.18}$$

By virtue of (1.1.18), (1.2.6), (1.2.7), (1.2.12), (1.2.14) from (1.2.18) we have

$$\begin{aligned} |u_{xy}^\tau(t,x,y)| &= |\beta^\tau(t,x,y)| \leqslant \\ &\leqslant e^{M(p)\tau}(C(p) + M(p)C(p)e^{4M(p)\tau}\tau + M(p)C(p)e^{M(p)\tau}\tau) \leqslant \\ &\leqslant C(p)e^{M(p)\tau}e^{4M(p)\tau}(1 + 2M(p)\tau) \leqslant C(p)e^{7M(p)\tau}. \end{aligned} \tag{1.2.19}$$

We differentiate the problem (1.2.16) with respect to y. For the function $\gamma^\tau = u_{yy}^\tau$ we get the problem

$$\begin{aligned} \gamma_t^\tau + 2a\gamma_x^\tau + 4a_y u_{xy}^\tau + 2a_{yy} u_x^\tau = 0, \\ \gamma^\tau(0,x,y) = u_{0yy}(x,y). \end{aligned} \tag{1.2.20}$$

By virtue of (1.1.18), (1.2.6), (1.2.7), (1.2.12), (1.2.19) from (1.2.20) we obtain the following inequality

$$\begin{aligned} |u_{yy}^\tau| = |\gamma^\tau| &\leqslant C(p) + 2M(p)C(p)e^{7M(p)\tau}\tau + M(p)C(p)e^{M(p)\tau}\tau \leqslant \\ &\leqslant C(p)e^{7M(p)\tau}(1 + 2M(p)\tau + M(p)\tau) \leqslant \\ &\leqslant C(p)e^{10M(p)\tau}, \ (t,x,y) \in \Pi_{[0,\frac{\tau}{2}]}. \end{aligned} \tag{1.2.21}$$

From (1.2.10), (1.2.12), (1.2.17), (1.2.19), (1.2.21) it follows that

$$|D^\alpha u^\tau(t,x,y)| \leqslant C(p)e^{10M(p)\tau}, \ |\alpha| \leqslant 2, \ (t,x,y) \in \Pi_{[0,\frac{\tau}{2}]}. \tag{1.2.22}$$

Let the integer k belong to the segment $[2, p-1]$. Assume that the estimate

$$|D^\alpha u^\tau(t,x,y)| \leqslant C(p)e^{l_1(k)M(p)\tau},\ |\alpha| \leqslant k,\ (t,x,y) \in \Pi_{[0,\frac{\tau}{2}]}, \tag{1.2.23}$$

holds.

Let us prove that

$$|D^\alpha u^\tau(t,x,y)| \leqslant C(p)e^{l_1(k+1)M(p)\tau},\ |\alpha| \leqslant k+1,\ (t,x,y) \in \Pi_{[0,\frac{\tau}{2}]}. \tag{1.2.24}$$

In (1.2.23), (1.2.24) the constants $l_1(p)$, $l_1(p+1)$ depend on p and $p+1$, respectively.

The derivative $D^\beta u^\tau = \omega^\tau$ for $|\beta| = k+1$ is a solution to problem

$$\begin{aligned} &\omega^\tau_t + 2a\omega^\tau_x + \sum_{|\alpha|\leqslant k} \Phi_\alpha D^\alpha u^\tau + 2D^\gamma a \frac{\partial^{k+1}u^\tau}{\partial x^{k+1}} = 0, \\ &\omega^\tau|_{t=0} = D^\beta u_0. \end{aligned} \tag{1.2.25}$$

In (1.2.25) the multi-index γ satisfies the condition $|\gamma| = 1$ and the coefficients Φ_α are functions that linearly depend on the derivatives $D^\alpha a$, $|\alpha| \leqslant k$.

By virtue of (1.2.22) from (1.2.25) we get the estimate (1.2.24), whence follows, that

$$|D^\alpha u^\tau(t,x,y)| \leqslant C(p)e^{l_1(p)M(p)\tau},\ |\alpha| \leqslant p,\ (t,x,y) \in \Pi_{[0,\frac{\tau}{2}]}. \tag{1.2.26}$$

On the second fractional step (at $k = 0$) we consider the problem

$$\begin{aligned} &\frac{\partial u^\tau}{\partial t} + 2b\frac{\partial u^\tau}{\partial y} = 0, \quad \frac{\tau}{2} < t \leqslant \tau, \\ &u^\tau\left(\frac{\tau}{2}, x, y\right) = \lim_{\substack{t\to\frac{\tau}{2} \\ t<\frac{\tau}{2}}} u^\tau(t,x,y), \end{aligned} \tag{1.2.27}$$

where $u^\tau(\frac{\tau}{2}, x, y)$ satisfies the conditions

$$|D^\alpha u^\tau(\frac{\tau}{2},x,y)| \leqslant C(p)e^{l_1(p)M(p)\tau},\ |\alpha| \leqslant p. \tag{1.2.28}$$

From the conditions (1.2.6), (1.2.28), we get the estimate

$$\begin{aligned} |D^\alpha u^\tau(t,x,y)| &\leqslant C(p)e^{l_1(p)M(p)\tau}e^{l_1(p)M(p)\tau} = \\ &= C(p)e^{l(p)M(p)\tau},\ (t,x,y) \in \Pi_{[\frac{\tau}{2},\tau]}. \end{aligned} \tag{1.2.29}$$

In the relation (1.2.29) $l(p) = 2l_1(p)$ is a constant.

From (1.2.26), (1.2.29) we have

$$|D^\alpha u^\tau(t,x,y)| \leqslant C(p)e^{l(p)M(p)\tau},\ |\alpha| \leqslant p,\ (t,x,y) \in \Pi_{[0,\tau]}. \tag{1.2.30}$$

Considering the k-th whole step ($0 \leqslant k \leqslant N-1$) and repeating the reasonings which have been lead in the proof of the relation (1.2.30), considering $C_k(p)$ instead of $C(p)$, we shall obtain the estimate (1.1.9). The estimate (1.1.8) follows from the proof of the estimate (1.1.9). □

Proof of the Theorem 1.2. On the second fractional step of the first whole step ($k = 1$), repeating the reasonings which have been lead in the deduction of the estimate (1.2.30), we shall obtain that the inequality

$$|D^\alpha u^\tau(t,x,y)| \leqslant C(p)e^{l(p)M(p)\tau}e^{l(p)M(p)\tau} = C(p)e^{l(p)M(p)2\tau},\ |\alpha| \leqslant p,$$

is valid for $(t, x, y) \in \Pi_{[\tau, 2\tau]}$.

Clearly, at k-th step

$$|D^\alpha u^\tau(t,x,y)| \leqslant C(p)e^{l(p)M(p)T},\ (t,x,y) \in \Pi_{[k\tau,(k+1)\tau]},\ |\alpha| \leqslant p,\ 0 \leqslant k \leqslant N-1,$$

and, hence,

$$|D^\alpha u^\tau(t,x,y)| \leqslant C(p)e^{l(p)M(p)T} = d_1,\ (t,x,y) \in \Pi_{[0,T]},\ |\alpha| \leqslant p. \tag{1.2.31}$$

From (1.2.31) and the system (1.1.3) it follows that

$$\left|\frac{\partial}{\partial t} D^\alpha u^\tau(t,x,y)\right| \leqslant d_2,\ (t,x,y) \in \Pi_{[0,T]},\ |\alpha| \leqslant p-1. \tag{1.2.32}$$

The estimates (1.2.31), (1.2.32) guarantee the uniform boundedness and the equicontinuity of the functions $\{D^\alpha u^\tau\}$, $|\alpha| \leqslant p-1$, in $\Pi_{[0,T]}$.

By virtue of Arzela's theorem (on compactness in C) we can choose a subsequence u^τ (without changing the notation) such that

$$D^\alpha u^\tau \longrightarrow D^\alpha u, \quad |\alpha| \leqslant p-1, \quad \tau \to 0, \tag{1.2.33}$$

in $\Pi_{[0,T]}$, and uniformly in $\Pi^R_{[0,T]}$ for all fixed $R > 0$.

By virtue of (1.2.33) and by the convergence Theorem WAM (Theorem 2.4.1 in [7]) the function $u(t,x,y)$ is the solution to the problem (1.2.1), (1.2.2) in the class $C^{1,p-1}(\Pi_{[0,T]})$. As the solution of a class $C^{1,p-1}(\Pi_{[0,T]})$ is unique, for this reason all sequence $\{u^\tau(t,x,y)\}$ converges to u as well as the subsequence chosen above (relations (1.1.10), (1.1.11) are fulfilled). □

2. Parabolic Equation

2.1. Parabolic Equation Containing the Central Derivatives

Consider in $\Pi_{[0,T]} = \{(t,x) | 0 \leqslant t \leqslant T,\ x \in E_n\}$ the problem

$$u_t = \sum_{i=1}^n a_i(t,x_i) u_{x_i x_i}, \tag{2.1.1}$$

$$u(0,x) = u_0(x), \quad x \in E_n. \tag{2.1.2}$$

Conditions on the initial data of the problem (2.1.1), (2.1.2):

$$|D^\alpha u_0(x)| \leqslant C_m, \quad |\alpha| = m, \quad m \geqslant 0, \tag{2.1.3}$$

$$\left|\frac{\partial^m a_i(t,x_i)}{\partial x_i^m}\right| \leqslant M_m, \quad i = 1,\dots,n, \quad m \geqslant 0. \tag{2.1.4}$$

All derivatives in relation (2.1.3) are continuous in E_n.

Remark 2.1. *It is obvious, that by virtue of (2.1.3), (2.1.4) for any integer $p \geqslant 0$ there are constants $C = C(p)$, $M = M(p)$ depending on p only such that*

$$\left|\frac{\partial^k a_i(t,x_i)}{\partial x_i^k}\right| \leqslant M(p), \quad k \leqslant p, \quad i = 1,\dots,n,$$

$$|D^\alpha u_0(x)| \leqslant C(p), \quad |\alpha| \leqslant p. \tag{2.1.5}$$

We split the problem (2.1.1), (2.1.2) into one-dimensional problems according to WAM:

$$u_t^\tau = na_1 u^\tau_{x_1x_1}, \quad k\tau < t \leqslant \left(k+\frac{1}{n}\right)\tau,$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$u_t^\tau = na_i u^\tau_{x_ix_i}, \quad \left(k+\frac{i-1}{n}\right)\tau < t \leqslant \left(k+\frac{i}{n}\right)\tau,$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$u_t^\tau = na_n u^\tau_{x_nx_n}, \quad \left(k+\frac{n-1}{n}\right)\tau < t \leqslant (k+1)\tau, \tag{2.1.6}$$

$$u^\tau(0,x) = u_0(x), \quad x \in E_n,$$

$\tau = \dfrac{T}{N}$, $k = 0, 1, \dots, N-1$, N is an integer.

We take the initial data $u^\tau|_{t=(k+\frac{i-1}{n})\tau}$ on the i-th fractional step from the previous fractional step:

$$u^\tau|_{t=(k+\frac{i-1}{n})\tau} = \lim_{\substack{t\to(k+\frac{i-1}{n})\tau \\ t<(k+\frac{i-1}{n})\tau}} u^\tau(t,x), \quad i = 1,\dots,n.$$

By virtue of conditions (2.1.3) and the Theorem 5.1 in [11] we can show that the solution $u^\tau(t,x)$ to the problem (2.1.6) exists in $\Pi_{[0,T]}$ for any τ ($\tau N = T$), $D^\alpha u^\tau \in C(\Pi_{[0,T]})$, for all $|\alpha| \geqslant 0$, $(D^\alpha u^\tau)_t$ are continuous in $\Pi_{[0,T]}$, but $(D^\alpha u^\tau)_t$ in general has discontinuities in the time variable points $t_{k,j} = \left(k + \frac{j-1}{n}\right)\tau$, $j = 1,\dots,n$; $k = 0,1,\dots,N-1$.

The following theorems hold.

Theorem 2.1. *Let the conditions (2.1.3), (2.1.4) be fulfilled. Then for a fixed* $p \geqslant 0$ *the estimates*

$$|D^\alpha u^\tau(t,x)| \leqslant C_{k,j}(p)e^{d\tau/n},\ (t,x)\in \Pi_{[(k+\frac{j-1}{n})\tau,(k+\frac{j}{n})\tau]},\ j=1,\dots,n, \tag{2.1.7}$$

$$|D^\alpha u^\tau(t,x)| \leqslant C_k(p)e^{d\tau},\ (t,x)\in\Pi_{[k\tau,(k+1)\tau]},\ k=0,1,\dots,N-1;\ |\alpha|\leqslant p, \tag{2.1.8}$$

hold, where u^τ *is the solution to the problem (2.1.6) and* $C_{k,j}(p) = \max\limits_{|\alpha|\leqslant p}\sup\limits_{E_n}|D^\alpha u^\tau((k+\frac{j-1}{n})\tau,x)|$, $C_k(p) = \max\limits_{|\alpha|\leqslant p}\sup\limits_{E_n}|D^\alpha u^\tau(k\tau,x)|$, *the constant* d *depends on* $C(p)$ *only and does not depend on* τ.

Let's note that in (2.1.7), (2.1.8) we take

$$C_0(p) = C_{0,1}(p) = \max_{|\alpha|\leqslant p}\sup_{E_n}|D^\alpha u_0(x)|.$$

Theorem 2.2. *If the conditions of Theorem 2.1 are fulfilled, then the solutions* $u^\tau(t,x)$ *of the problem (2.1.6) converge as* $\tau\to 0$ *to the solution* $u(t,x)$ *of the problem (2.1.1), (2.1.2) in* $\Pi_{[0,T]}$ *and uniformly in* $\Pi^R_{[0,T]}$ *for all fixed* R, $R > 0$, *together with the derivatives* $D^\alpha u^\tau(t,x)$, $|\alpha|\geqslant 0$:

$$\lim_{\tau\to 0} D^\alpha u^\tau(t,x) = D^\alpha u(t,x), \quad (t,x)\in\Pi_{[0,T]},$$

$$\lim_{\tau\to 0}\|D^\alpha u^\tau - D^\alpha u\|_{C(\Pi^R_{[0,T]})} = 0.$$

The function $u \in C^{1,\infty}(\Pi_{[0,T]})$ *is the solution to problem (2.1.1), (2.1.2).*

Here $C^{1,\infty}(\Pi_{[0,T]}) = \{u(t,x)|D^\alpha u,\ u_t \in C(\Pi_{[0,T]}),\ |\alpha|\geqslant 0\}$.

2.2. General Parabolic Equation

We consider in $\Pi_{[0,T]} = \{(t,x)|0 \leqslant t \leqslant T,\, x \in E_n\}$ the problem

$$u_t = L(u) + f, \tag{2.2.1}$$

$$u(0,x) = u_0(x), \tag{2.2.2}$$

where $L(u) = \sum_{i=1}^{n} a_i(t,x_i)u_{x_ix_i} + \sum_{i=1}^{n} b_i(t,x)u_{x_i} + c(t,x)u$.

Assume that the functions u_0, a_i satisfy the conditions (2.1.3), (2.1.4), respectively, functions b_i, c, f satisfy the conditions

$$|D^\alpha b_i(t,x)| \leqslant M_m, \quad |\alpha| = m, \quad m \geqslant 0, \tag{2.2.3}$$

$$|D^\alpha c(t,x)| \leqslant \chi_m, \quad |\alpha| = m, \quad m \geqslant 0, \tag{2.2.4}$$

$$|D^\alpha f(t,x)| \leqslant \sigma_m, \quad |\alpha| = m, \quad m \geqslant 0. \tag{2.2.5}$$

Remark 2.2. *It follows from relations (2.2.3)-(2.2.5), that for any integer $p \geqslant 0$ there are constants $M(p)$, $\chi(p)$, $\sigma(p)$, which depend on p only, such that*

$$|D^\alpha b_i(t,x)| \leqslant M(p), \quad |\alpha| \leqslant p, \quad i = 1,\dots,n, \tag{2.2.6}$$

$$|D^\alpha c(t,x)| \leqslant \chi(p), \quad |\alpha| \leqslant p, \tag{2.2.7}$$

$$|D^\alpha f(t,x)| \leqslant \sigma(p), \quad |\alpha| \leqslant p. \tag{2.2.8}$$

We split the problem (2.2.1), (2.2.2) into the one-dimensional problems:

$$u_t^\tau = (n+1)a_1 u^\tau_{x_1x_1} + (n+1)b_1 u^\tau_{x_1}, \quad k\tau < t \leqslant (k + \frac{1}{n+1})\tau,$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$u_t^\tau = (n+1)a_n u^\tau_{x_nx_n} + (n+1)b_n u^\tau_{x_n}, \quad (k + \frac{n-1}{n+1})\tau < t \leqslant (k + \frac{n}{n+1})\tau,$$

$$u_t^\tau = (n+1)cu^\tau + (n+1)f, \quad (k + \frac{n}{n+1})\tau < t \leqslant (k+1)\tau, \tag{2.2.9}$$

$$u^\tau(0,x) = u_0(x). \tag{2.2.10}$$

The system (2.2.9) contains $(n+1)$ one-dimensional parabolic equations and one ordinary differential equation, containing the parameters. The whole step contains $(n+1)$ fractional steps.

The following theorems hold.

Theorem 2.3. *Let $C_0(p) \geqslant 1$ and conditions (2.1.3), (2.1.4), (2.2.3)-(2.2.5) be fulfilled. Then the solution $u^\tau(t,x)$ to the problem (2.2.9), (2.2.10) satisfies the estimates*

$$|D^\alpha u^\tau(t,x)| \leqslant C_{k,j}(p)e^{\theta\tau/(n+1)},\ (t,x) \in \Pi_{[(k+\frac{j-1}{n+1})\tau,(k+\frac{j}{n+1})\tau]},\ j = 1,\dots,n+1, \tag{2.2.11}$$

$$|D^\alpha u^\tau(t,x)| \leqslant C_k(p)e^{\theta\tau},\ (t,x) \in \Pi_{[k\tau,(k+1)\tau]},\ k = 0,1,\dots,N-1;\ |\alpha| \leqslant p, \tag{2.2.12}$$

where $C_{k,j}(p) = \max_{|\alpha|\leqslant p} \sup_{E_n} |D^\alpha u^\tau((k + \frac{j-1}{n+1})\tau, x)|$, $C_0(p) = C_{0,1}(p)$, the constant θ depends on p (on $M(p)$, $\chi(p)$, $\sigma(p)$ from (2.1.5), (2.2.6)–(2.2.8)) and does not depend on τ.

Theorem 2.4. *Let the conditions of Theorem 2.3 be satisfied. Then the solution $u^\tau(t,x)$ of the problem (2.2.9), (2.2.10) converge as $\tau\to 0$ to the function $u(t,x)$ in $\Pi_{[0,T]}$ and uniformly in $\Pi^R_{[0,T]}$ for any fixed R, $R>0$, together with the derivatives $D^\alpha u^\tau(t,x)$, $|\alpha|\geqslant 0$:*

$$\lim_{\tau\to 0} D^\alpha u^\tau(t,x) = D^\alpha u(t,x), \quad (t,x)\in\Pi_{[0,T]},$$

$$\lim_{\tau\to 0}\|D^\alpha u^\tau - D^\alpha u\|_{C(\Pi^R_{[0,T]})} = 0, \quad |\alpha|\leqslant p-2.$$

The function $u\in C^{1,\infty}(\Pi_{[0,T]})$ is the solution to problem (2.2.1), (2.2.2).

Example 2.1. *Inverse problem.* We consider the equation

$$u_t = \sum_{i=1}^{2} L_i(t,x,u) + u_{zz} + g(t,x)f(t,x,z), \tag{2.2.13}$$

where $x=(x_1,x_2)$, $L_i(t,x,u)=a_i(t,x_i)u_{x_ix_i}+b_i(t,x)u_{x_i}$, the functions $a_i=a_i(t,x_i)$ depend on t and x_i only, $b_i=b_i(t,x)$ depend on t, x, $f=f(t,x,z)=f(t,x_1,x_2,z)$ are given functions, the coefficient $g(t,x)$ and the function $u(t,x,z)$ are unknown.

We set the initial data

$$u(0,x,z)=u_0(x,z), \quad x,z\in E_2, \tag{2.2.14}$$

and the overdetermination condition

$$u(t,x,0)=\varphi(t,x), \quad (t,x)\in\Pi_{[0,T]}. \tag{2.2.15}$$

Let's put $z=0$ in (2.2.13), we express the function $g(t,x)$ from the obtained relation and substitute the expression for $g(t,x)=\dfrac{\Psi(t,x)-u_{zz}(t,x,0)}{f(t,x,0)}$ in (2.2.13). We arrive at the equation

$$u_t = \sum_{i=1}^{2} L_i(t,x,u) + u_{zz} + \frac{\Psi(t,x)-u_{zz}(t,x,0)}{f(t,x,0)}f(t,x,z), \tag{2.2.16}$$

In (2.2.16) the function $\Psi=\varphi_t-\sum_{i=1}^{2}L_i(t,x,\varphi)$ is a known function depending on φ and on coefficients a_i, b_i to operators L_i.

Let

$$f(t,x,0)\geqslant\delta>0, \quad \delta-\text{const}. \tag{2.2.17}$$

We split the problem (2.2.16), (2.2.14) into the one-dimensional problems:

$$\left.\begin{array}{ll}
u^\tau_t = 4L_1(t,x,u^\tau),\ n\tau<t\leqslant(n+\frac14)\tau, & (2.2.18_1)\\
u^\tau_t = 4L_2(t,x,u^\tau),\ (n+\frac14)\tau<t\leqslant(n+\frac12)\tau, & (2.2.18_2)\\
u^\tau_t = 4u^\tau_{zz},\ (n+\frac12)\tau<t\leqslant(n+\frac34)\tau, & (2.2.18_3)\\
u^\tau_t = 4\frac{\Psi(t,x)-u^\tau_{zz}(t-\frac{\tau}{4},x,0)}{f(t,x,0)}f(t,x,z),\ (n+\frac34)\tau<t\leqslant(n+1)\tau, & (2.2.18_4)
\end{array}\right\} \tag{2.2.18}$$

$$u^\tau(0,x,z)=u_0(x,z), \quad (x,z)\in E_2. \tag{2.2.19}$$

It is assumed that conditions (2.1.4), (2.2.3) and

$$|D^\gamma f(t,x,z)|\leqslant\sigma_m, \quad |\gamma|=m, \quad m\geqslant 0, \tag{2.2.20}$$

$$|D^\alpha \varphi(t,x)| \leqslant \chi_m, \quad |\alpha| = m, \quad m \geqslant 0, \tag{2.2.21}$$

$$|D^\gamma u_0(x,z)| \leqslant \sigma_m, \quad |\gamma| = m, \quad m \geqslant 0 \tag{2.2.22}$$

are fulfilled. In (2.2.20), (2.2.22) vector $\gamma = (\gamma_1, \gamma_2, \gamma_3)$ is a multi-index of dimension 3, $D^\gamma = \dfrac{\partial^{|\gamma|}}{\partial x_1^{\gamma_1} \partial x_2^{\gamma_2} \partial z^{\gamma_3}}$, $|\gamma| = \sum\limits_{i=1}^{3} \gamma_i$.

We differentiate (2.2.18), (2.2.19) twice with respect to z.

By virtue of independence of z the coefficients of the equations $(2.2.18_1)$–$(2.2.18_3)$ and conditions (2.2.20)-(2.2.22) we conclud that the set $u^\tau_{zz}(t,x,z)$ is bounded in $\Pi_{[0,T]}$ uniformly on $\tau > 0$:

$$|u^\tau_{zz}(t,x,z)| \leqslant C, \quad (t,x,z) \in G_{[0,T]}. \tag{2.2.23}$$

In the view of (2.2.20)-(2.2.23), differentiating with respect to z the problem (2.2.18), (2.2.19), we obtain in $G_{[0,T]}$ the estimate

$$\left| \frac{\partial^j}{\partial z^j} u^\tau(t,x,z) \right| \leqslant N(p), \quad j = 1, 2, \ldots, p;\ p \geqslant 2. \tag{2.2.24}$$

Let's estimate the derivatives $D^\gamma u^\tau$. We differentiate the problem (2.2.18), (2.2.19) with respect to z. The function $\dfrac{\partial^j u^\tau}{\partial z^j}$ is a solution of system of the equations (2.2.18), in which instead of f is taken $\dfrac{\partial^j f}{\partial z^j}$ and

$$\frac{\partial^j u^\tau(0,x,z)}{\partial z^j} = \frac{\partial^j u_0(x,z)}{\partial z^j}. \tag{2.2.25}$$

The problem (2.2.18), (2.2.25) is exponentially stable on first two fractional steps (Theorem 2.3, inequality (2.2.11)). It is not difficult to show the exponential stability of the problem (2.2.18), (2.2.25) on the third and fourth fractional steps. Hence in $G_{[0,T]}$ the estimate

$$\left| \frac{\partial^j}{\partial z^j} D^\alpha u^\tau(t,x,z) \right| \leqslant N_1(p), \quad |\alpha| + j \leqslant p,\ N_1(p) - const. \tag{2.2.26}$$

holds. Due to (2.2.26) and (2.2.18) it follows that in $G_{[0,T]}$

$$\left| \frac{\partial}{\partial t} D^\gamma u^\tau(t,x,z) \right| \leqslant N_2(p), \quad |\gamma| \leqslant p - 4,\ N_2(p) - const. \tag{2.2.27}$$

By virtue of (2.2.26), (2.2.27) and Arzela's theorem we can choose a subsequence u^τ (without changing the notation) which converges at $\tau \to 0$ to the function u together with all derivatives $D^\gamma u^\tau$:

$$D^\gamma u^\tau(t,x,z) \to D^\gamma u(t,x,z), \quad (t,x,z) \in G_{[0,T]},$$

$$||D^\gamma u^\tau(t,x,z) - D^\gamma u(t,x,z)||_{C(G^K_{[0,T]})} \to 0 \tag{2.2.28}$$

for any fixed $K > 0$ and all $|\gamma| \geqslant 0$.

In view of (2.2.28) by Theorem 2.4.1 in [7] the function $u(t,x,z)$ is the solution to the problem (2.2.16), (2.2.14) in the class $C^\infty(G_{[0,T]})$. Taking into account the matching condition $u_0(x,0) = \varphi(0,x)$ by the known way [12] we can prove that $u(t,x,z)$ satisfies to the overdetermination condition (2.2.15) and the pair of functions $u(t,x,z)$, $g(t,x)$ is the solution to problem (2.2.13)-(2.2.15). □

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