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Domains of Convergence for A-hypergeometric Series and Integrals

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We prove two theorems on the domains of convergence for A-hypergeometric series and for associated Mellin-Barnes type integrals. The exact convergence domains are described in terms of amoebas and coamoebas of the corresponding principal A-determinants.

Keywords: A-hypergeometric series, Mellin-Barnes integral, Γ*-integral, Principal A-determinant.*
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1. Introduction and preliminaries

The hypergeometric functions constitute a substantial section within the realm of special functions of Mathematics and Physics. It may therefore appear quite surprising that this wide class of hypergeometric functions in essence is obtained from a single "exponential" series by restriction of the summation to a suitable sublattice.

We rewrite the exponential Taylor series as a Laurent series

$$\exp(a) = \sum_{l\in\mathbb{N}^N} \frac{a_1^{l_1}}{l_1!}\cdots\frac{a_N^{l_N}}{l_N!} = \sum_{k\in\mathbb{Z}^N} \frac{a_1^{l_1}}{\Gamma(l_1+1)}\cdots\frac{a_N^{l_N}}{\Gamma(l_N+1)},$$

using the fact that the Euler gamma-function Γ has poles in negative numbers. Now consider the shifted exponential series

$$\exp_{\boldsymbol{\gamma}}(a) = \sum_{k\in\mathbb{Z}^N} \frac{a^{\boldsymbol{\gamma}+l}}{\Gamma(\gamma_1+l_1+1)\cdots\Gamma(\gamma_N+l_N+1)},$$

with $\boldsymbol{\gamma}=(\gamma_1,\ldots,\gamma_N)\in\mathbb{C}^N$, and introduce the notion of Γ-series.

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Definition 1. *Let L be an integer sublattice of $\mathbb{Z}^N$. The diagonal subseries*

$$\phi(a) = \Phi_{\boldsymbol{\gamma}}(a) = \sum_{l\in L} \frac{a^{\boldsymbol{\gamma}+l}}{\Gamma(\gamma_1 + l_1 + 1)\cdots\Gamma(\gamma_N + l_N + 1)} \tag{1}$$

is called a Γ*-series.*

This definition was introduced by Gelfand, Kapranov and Zelevinsky [1]. They observed that choosing bases of L, one can rewrite any Γ-series as a product of a monomial $a^{\boldsymbol{\gamma}}$ with a formal Laurent series which is a hypergeometric series as defined by Horn ([2]) in m variables, where $m = \operatorname{rank}(L)$. Recall that a Laurent series is hypergeometric in the sense of Horn if the quotients of neighbor coefficients are rational functions in the variables of summation. Usually such coefficients are represented by ratio of Γ-factors, i.e. of factors of gamma-function Γ composed with affine function in the variables of summation.

In the mentioned paper [1] a new fruitful approach to the general theory of hypergeometric functions was developed. It has connections to toric geometry, combinatorics of polytopes and a number of other fields. The basic idea of the GKZ-approach is to cleverly introduce extra variables a_j, one for each Γ-factor in the hypergeometric series in the sence of Horn. Their main observation was that the new function (of many variables) thus obtained will satisfy a very simple (binomial) system of differential equations with constant coefficients.

The corresponding hypergeometric system of differential equations, see Definition 2 in the Section 2, may be coded by an integer matrix A of size $n \times N$, where $n = N - m$, whose kernel sublattice is L (recall that $m = rank\, L$).

Note that a $(N \times m)$-matrix B as an annihilator of A is called the *Gale transform* of the configuration A, see Section 5.4 of [3]. We are interesting in the Gale transforms B of two types. The first type consist of integer matrices whose columns generate the lattice L. The second one contains rational matrices B which have a unit matrix E_m on some m rows of B.

We define the *hypergeometric series* associated to rational Gale transform B with a unit matrix on the last m rows as follows:

$$\phi(a) = \phi_B(a) = \sum_{k\in\mathbb{N}^m} \frac{a^{\gamma+\langle B,k\rangle}}{\prod_{j=1}^{N-m} \Gamma(\gamma_j + \langle b_j, k\rangle + 1)\, k!}\,, \tag{2}$$

where the b_j denote the first n rows in B, $k! = k_1! \cdots k_m!$, and

$$a^{\gamma+\langle B,k\rangle} = a^{\gamma}(a^{b^1})^{k_1} \cdots (a^{b^m})^{k_m}$$

with $b^1, \ldots, b^m$ being the columns of B. In fact series (2) is a sum of Γ-series (1) with suitable shifts $\boldsymbol{\gamma}^{(i)}$.

It is well known that the hypergeometric series in one variable can also be represented as a so called Mellin-Barnes integral. Given the hypergeometric series (2), its formal integral representation is the following Mellin-Barnes integral (about the multiple Mellin-Barnes integrals see [4])

$$\hat{\phi}(a) = \frac{a^{\gamma}}{(2\pi i)^m} \int_{\delta+i\mathbb{R}^m} \frac{\Gamma(s_1)\cdots\Gamma(s_m)}{\prod_{j=1}^{N-m} \Gamma(\gamma_j - \langle b_j, s\rangle + 1)} \prod_{j=1}^{m} (-a^{b^j})^{-s_j} ds,$$

where $ds = ds_1 \wedge \cdots \wedge ds_m$ and $\delta \in \mathrm{R}^m_+$ is chosen in appropriate way. This means that granted that the integral $\hat{\phi}$ converges, it will coincide with series (2), where the convergence domains of the integral and the series overlap.

In this paper we also introduce another type of integrals called Γ-*integrals*, as the continuous analogue of the Γ-series (1):

$$\hat{\phi}(a) = \hat{\Phi}_{\boldsymbol{\gamma}(a)} = \int_{h \subset L_{\mathbb{C}}} \Gamma(\xi - \gamma) a^{\boldsymbol{\gamma} - \xi} d\mu \tag{3}$$

and call it a Γ-*integral*. Here h is a cycle of dimension $m = N - n$ with a closed support, $\mathrm{d}\mu$ is an m-dimensional measure on $L_{\mathbb{C}}$, and

$$\Gamma(\xi - \gamma) a^{\boldsymbol{\gamma} - \xi} = \prod_{j=1}^{N} \Gamma(\xi_j - \gamma_j) a_j^{\gamma_j - \xi_j}.$$

Notice that we have put all Γ-factors in the numerator. This will turn out to be very beneficial since it will make the domain of convergence larger.

The aims of this paper is to specify the convergence domains for the series (8) which is in fact the dehomogenization of the series (2). We will use the theory of amoebas (see Definition 5), and the so called principal A-determinant, see Definition 3, which defines the singularities of the hypergeometric functions. Parallely, we will be studying the representation of the hypergeometric function in terms of Mellin-Barnes type integrals. We will describe the convergence domains of these integrals and their connection with the coamoeba of the principal A-determinant.

These results were largely obtained already in 2009 and included in the thesis by L. Nilsson [5], advised by M. Passare and A. Tsikh, but have not been published until now. The years since then have shown an increasing interest to them from specialists in hypergeometric and algebraic functions. Recall that already H. Mellin noticed [6] a general algebraic function $y(x) = y(x_1, \ldots, x_{n-1})$ is a hypergeometric function. The convergence domains for the series and integrals representing this function were described in the papers [7] and [8], correspondingly. Using this results, in [9] the monodromy of $y(x)$ was described. We hope that Theorems 2, 3, 4 of this paper can be applied to give a similar description of monodromy for an arbitrary hypergeometric function.

2. Basic definitions and algebraic preparations

2.1. A-hypergeometric systems

Gelfand, Kapranov and Zelevinsky constructed a holonomic system of linear differential equations satisfied by the Horn series as well as their integral representations of Euler type generalizing the integral representation for ${}_2F_1$ in [1, 10]. This is done through associating a system of hypergeometric functions to a finite subset A of the integer lattice $\mathbb{Z}^n$. These functions are called A-hypergeometric functions.

The set A has to satisfy the following conditions: the $\mathbb{Z}$-span of A is $\mathbb{Z}^n$ and there exists a linear form h with the property $h(\alpha) = 1$ for all $\alpha \in A$. As mentioned in Section 1 we choose h as a coordinate function and therefore we are able to represent A as a matrix

$$A = \begin{pmatrix} 1 & 1 & \ldots & 1 \\ \alpha^{(1)} & \alpha^{(2)} & \ldots & \alpha^{(N)} \end{pmatrix}, \tag{4}$$

and sometimes we will identify A with a set of vectors

$$\mathfrak{A} = \{\alpha^{(1)}, \dots, \alpha^{(N)}\} \subset \mathbb{Z}^{n-1}.$$

Denote by $L = L(A) \subset \mathbb{Z}^A$ the lattice of affine relations among elements of A, that is, the set of integer vectors $l = (l_\alpha)_{\alpha \in A}$, such that

$$\sum_{\alpha \in A} l_\alpha \alpha = 0.$$

Let $\mathbb{C}^A$ be the space of vectors $a = (a_\alpha)_{\alpha \in A}$. Corresponding to the order of the set A we also denote the elements $a \in \mathbb{C}^A$ by $a = (a_1, \dots, a_N)$. For any $l = (l_1, \dots, l_N) \in L$ we define the differential operator $\Box_l$ on $\mathbb{C}^A$ by

$$\Box_l = \prod_{j:l_j>0} (\partial/\partial a_j)^{l_j} - \prod_{j:l_j<0} (\partial/\partial a_j)^{-l_j}.$$

We also define the differential operators

$$\mathcal{E}_i = \sum_{j=1}^{N} \alpha_i^{(j)} a_j (\partial/\partial a_j), \quad i = 1, \dots, n,$$

on $\mathbb{C}^A$, where $\alpha_i^{(j)}$ is the i:th coordinate of the j:th column of A.

Definition 2 ([1]). *Let $\boldsymbol{\beta} = (\beta_1, \dots, \beta_n)$ be a complex vector. The A-hypergeometric system with parameters $\boldsymbol{\beta}$ is the following system of linear differential equations on a function $\Phi(a)$, $a \in \mathbb{C}^A$:*

$$\begin{aligned} &\Box_l \Phi(a) = 0, \ l \in L \\ &\mathcal{E}_i \Phi - \beta_i \Phi = 0, \quad i = 1, \dots, n. \end{aligned} \tag{5}$$

Hence the A-hypergeometric system consists of firstly a binomial equation in partial differentials given by $\Box_l$ and secondly of a number of a equations given by $\mathcal{E}_i$ that should really be considered as a number of homogeneity conditions, originating from the relations among the elements of A.

Remark 1. Formally $\Box_l \Phi = 0$ with $l \in L$ in (5) represents an infinite number of equations, but in fact it is enough to consider a finite number of equations with l corresponding to a basis of the lattice L.

Remark 2. One can check immediately (see [1]) that any Γ-series (1) with $\boldsymbol{\gamma} \in A^{-1}(\boldsymbol{\beta})$ formally satisfies the A-hypergeometric system (5).

Definition 3 ([1]). *Holomorphic solutions of (5) will be called A-hypergeometric functions.*

It turns out that the solution space to the system in Definition 2 is finite dimensional, more precisely, we have the following theorem.

Theorem 1 ([1, 10]). *The number of linearly independent holomorphic solutions of the A-hypergeometric system (5) at a generic point of $\mathbb{C}^A$ is equal to the normalized volume of the convex hull $Q(A)$ of the points in A.*

The volume is normalized so that the minimal $(n-1)$-simplex in $\mathbb{R}^{n-1}$ with vertices on $\mathbb{Z}^{n-1}$ has volume 1.

In [1] it is also proved that choosing corresponding vectors $\boldsymbol{\gamma}$ one can construct a basis of solutions to (5) by means of nonformal (convergent) Γ- series (in more details see subsection 3.1).

2.2. The principal A-determinant, and the notions of amoeba and coamoeba

The solutions to the A-hypergeometric system define regular functions everywhere in $\mathbb{C}^A$ except on the algebraic hypersurface, $\{E_A(a_\alpha) = 0\}$. The description of the defining polynomial is the following.

Definition 4 ([11])**.** *Let $A \subset \mathbb{Z}^{n-1}$ be a finite subset which affinely generates $\mathbb{Z}^{n-1}$. For any $f = f(x_1, \ldots x_{n-1}) \in \mathbb{C}^A$, where $\mathbb{C}^A$ is interpreted as the space of Laurent polynomials with monomials from A, the principal A-determinant is defined as a resultant*

$$E_A(f) = R_A\Big(x_1\frac{\partial f}{\partial x_1}, \ldots, x_{n-1}\frac{\partial f}{\partial x_{n-1}}, f\Big).$$

Note that E_A is clearly a polynomial function in coefficients of f. One major aim of this paper is to describe the exact convergence domains of the solutions to the A-hypergeometric system. However the solutions can be represented in various different ways, such as power series and different types of integrals. The different representations will be solutions to the same hypergeometric system, but have different convergence domains. They are in fact analytic extensions of each other. Naturally the convergence domains of the hypergeometric power series only depend on the absolute value of the variables, which leads us to the key tool for describing the situation further, that is, amoebas: For a Laurent polynomial P in m variables, we denote by $\mathcal{Z}_P$ the hypersurface determined by the equation $P = 0$, and introduce the logarithmic mapping $(\mathbb{C}^*)^m \to \mathbb{R}^m$ given by

$$\mathrm{Log} : (\zeta_1, \ldots, \zeta_m) \mapsto (\log|\zeta_1|, \ldots, \log|\zeta_m|).$$

Definition 5. *The amoeba of the polynomial P is denoted by $\mathcal{A}_P$ and*

$$\mathcal{A}_P := \mathrm{Log}(\mathcal{Z}_P).$$

The amoebas properties are discribed in the papers [12, 13] and [14]. We also consider solutions represented as so called Mellin-Barnes integrals or Γ-integrals. The convergence of the integrals depend only on the argument of the variables, and hence it is natural to use coamoebas in the study of the convergence domains of these integrals.

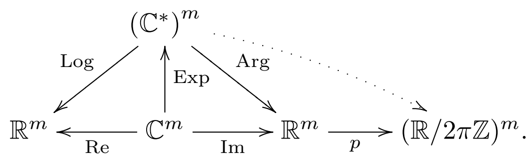
Similarity to the definition of an a amoeba above, we use the argument map for defining coamoebas. $\mathrm{Arg} : (\mathbb{C}^*)^m \to \mathbb{R}^m/[0, 2\pi]^n$ given by

$$\mathrm{Arg} : (\zeta_1, \ldots, \zeta_m) \mapsto (\arg(\zeta_1), \ldots, \arg(\zeta_m)).$$

Definition 6. *The coamoeba is denoted by $\mathcal{A}'_P$, where*

$$\mathcal{A}'_P := \mathrm{Arg}(\mathcal{Z}_P).$$

It is convenient to incorporate the maps Log and Arg into the following diagram where p is a projection on the m-dimension real torus:



2.3. Algebraic preparations

Proposition 1. *Let A be a matrix of size $n \times N$, $n \leqslant N$ with integer entries. Then the following claims are equivalent:*
(i) *the column span of A is the entire lattice $\mathbb{Z}^n$;*
(ii) *the maximal minors of A are relatively prime;*
(iii) *there is a unimodular matrix M of size $N \times N$ such that $AM = (E_n|0)$ (the unity matrix of size $n \times n$ enlarged by adding zeros to a $n \times N$-matrix);*
(iv) *there is a completion $\widetilde{A}$ which is unimodular, i.e. we can enlarge A to a $N \times N$ integer square matrix $\widetilde{A}$ with $\det \widetilde{A} = 1$.*

Proof. (i) $\Rightarrow$ (ii) Given the condition (i), there exists an integer $N{\times}n$-matrix C with the property that AC is equal to the unit matrix E_n of size $n \times n$. From this it follows that all the maximal minors of the matrix A are relatively prime, since by the well known Binet-Cauchy formula ([15]) the determinant of AC equals the sum of the maximal minors of A multiplied with the corresponding minors of C.

(ii) $\Rightarrow$ (iii) By the invariant factor theorem ([16]) it follows that there exist unimodular integer matrices D and F of size n and N respectively, such that $DAF = (\delta|0)$ where δ is a diagonal $n \times n$-matrix with integers $\epsilon_1, \ldots, \epsilon_n$ on the diagonal, and 0 is the zero-matrix of size $n \times (N-n)$. From the representation $A = D^{-1}(\delta|0)F^{-1}$ it is now easy to see that in fact $\delta = E_n$, because some ϵ_j being different from 1 would contradict the fact that the maximal minors of A are relatively prime.

(iii) $\Rightarrow$ (iv) By (iii) we have

$$A = D^{-1}(E_n|0)F^{-1} = (D^{-1}|0)F^{-1},$$

and the desired enlargement of A may be taken to be

$$\widetilde{A} = \left(\begin{array}{c|c} D^{-1} & 0 \\ \hline 0 & E_{N-n} \end{array}\right) F^{-1}.$$

(iv) $\Rightarrow$ (i) This is obvious. □

We now introduce some notation. Given an integer $(n{\times}N)$-matrix A, denote by B the integer dual matrix of A, i.e. a $(N{\times}\,m)$-matrix such that the columns form a $\mathbb{Z}$-basis for the kernel of A. For any increasingly ordered subset $I = (i_1, \ldots, i_n)$ of $\{1, 2, \ldots, N)$, we let $J = (j_1, \ldots, j_m)$ be its complement, also increasingly ordered. We let A_I denote the $(n{\times}n)$-minor of A with columns indexed by I, and similarly, we write B_J for the $(m \times m)$-minor of B with rows having indices from the complementary subset J.

Proposition 2. *Assume that columns of A generate the entire lattice $\mathbb{Z}^n$. Then* $|\det A_I| = |\det B_J|$.

Proof. We can assume that $I = \{1, \dots, n\}$ and $J = n+1, \dots, N$. So we can write A in the block form $(A_I \,|\, A_J)$. By analogy with respect to rows we write the matrix B in block form

$$B = \left(\frac{B_I}{B_J.} \right) .$$

Since the columns of A generate $\mathbb{Z}^n$ it follows from Proposition 1 that there is a unimodular completion $\widetilde{A}$ of A, which we write in the block form

$$\widetilde{A} = \left(\begin{array}{c|c} A_I & A_J \\ \hline * & * \end{array} \right) .$$

Let us consider the corresponding block composition of the inverse matrix :

$$\widetilde{A}^{-1} = \left(\begin{array}{c|c} * & B'_I \\ \hline * & B''_J \end{array} \right) .$$

According to Jacobi's formula ([15], formula (11)) one has $\det B'_J \cdot det\widetilde{A} = \det A_I$ and hence $|\det B'_J| = |\det A_I|$. It is clear that the block column

$$\left(\frac{B'_I}{B'_J} \right)$$

in $\widetilde{A}^{-1}$ constitute a basis of L, so B'_J differs from B_J only by an unimodular factor. Finally we get $|\det B_J| = |\det B'_J| = |\det A_I|$. □

3. Domains of convergence for A-hypergeometric series

3.1. A-hypergeometric series

Let us firstly explain the GKZ-approach of the constructing basis A-hypergeometric series for the A-hypergeometric system (5). Given the data for the system(5), that is, an integer $(n \times N)$-matrix A of type (4), and a generic complex column n-vector $\boldsymbol{\beta}$, we fix a choice of a basis of the sublattice $L = KerA$. This means that we choose an integer $(N \times m)$-matrix B, where $m = N - n$, such that $AB = 0$ and the columns of B form a $\mathbb{Z}$-basis for L.

Using the row vectors b_j of the matrix B and a complex column vector $\boldsymbol{\gamma} \in A^{-1}(\boldsymbol{\beta})$, we can rewrite the Γ-series (1) on the form

$$\Phi_{\boldsymbol{\gamma}}(a) = \sum_{k \in \mathbb{Z}^m} \prod_{j=1}^{N} \frac{a_j^{\gamma_j + \langle b_j, k \rangle}}{\Gamma(1 + \gamma_j + \langle b_j, k \rangle)} \,. \tag{6}$$

Due to Remark 2 the series (6) gives a formal solution to the A-hypergeometric system provided that the vector $\boldsymbol{\gamma}$ is chosen so that $A\boldsymbol{\gamma} = \boldsymbol{\beta}$.

In order to obtain convergent Γ-series, in [1] was suggested to choose the vector $\boldsymbol{\gamma}$ so that m of its entries are integers. The point is that when $\gamma_j \in \mathbb{Z}$ the factor $\Gamma(1 + \gamma_j + \langle b_j, k \rangle)$ will be infinite for all integer k in the halfplane $1 + \gamma_j + \langle b_j, k \rangle \leqslant 0$, so the coefficients in (6) are zero for such k. With m different entries $(\gamma_{j_1}, \ldots, \gamma_{j_m}) =: \gamma_J$ of $\boldsymbol{\gamma}$ being integers, the support of summation in the series (6) will be contained in a simplicial cone, if the matrix B_J with rows $(b_{j_1}, \ldots, b_{j_m})$ is nondegenerate, and one gets a series with a non-empty domain of convergence.

According to Proposition 2 we have $|\det A_I| = |\det B_J|$, and we denote this number by δ_I. Now, the choice of I, that is, the choice of n columns of the matrix A, is of course equivalent to the choice of a subsimplex $\sigma = \sigma_I$ of the point configuration $\mathfrak{A} \subset \mathbb{Z}^{n-1} \subset \mathbb{R}^{n-1}$. One clearly has $\delta_I = (n-1)!\,\mathrm{Vol}\,(\sigma_I)$.

The objective is now, for each subset I, to first construct δ_I linearly independent convergent Γ-series, and then to determine their common domain of convergence. Of course we only have to consider those I for which $\delta_I \neq 0$, so that the minors A_I and B_J are invertible. In the linear compatibility equation $A\boldsymbol{\gamma} = \boldsymbol{\beta}$ for $\boldsymbol{\gamma} = (\gamma_I, \gamma_J)$, we can then solve γ_I in terms of $\boldsymbol{\beta}$ and γ_J. This leads to the identity

$$\boldsymbol{\gamma} = C\boldsymbol{\beta} + BB_J^{-1}\gamma_J \tag{7}$$

where C is the $(N \times m)$-matrix obtained from A_I^{-1} by inserting zero rows corresponding to the indices from the complement J. We have here made use of the relation $A_I^{-1}A_J = -B_I B_J^{-1}$, which follows from the equation $0 = AB = A_I B_I + A_J B_J$.

From equation (7) we see how the vector $\boldsymbol{\gamma}$ that defines the series (6) is computed from the shorter vector γ_J. The algorithm used by GKZ in [1] is to take all choices of integer vectors γ_J such that each entry of the vector $B_J^{-1}\gamma_J$ belongs to the half-open interval $[0, 1)$. There are precisely δ_I distinct such choices of γ_J, each giving a different vector $\boldsymbol{\gamma}$ by (7), and hence producing a different series (6).

Recall that due the Definition 7 a general hypergeometric series we defined for every rational Gale transformation R of the form $(R', E_m)^{tr}$ and for $\gamma' \in \mathbb{C}^n$, to be the following power series

$$\phi(a) = \phi_{R,\gamma'}(a) = \sum_{k \in \mathbb{N}^m} \frac{a^{\gamma' + \langle R, k\rangle}}{\prod_{j=1}^{N-m} \Gamma(\gamma_j + \langle r_j, k\rangle + 1)\, k!}\,, \tag{8}$$

where the r_j denote the rows in the matrix R', and $k! = k_1! \cdots k_m!$.

Remark that up to the factor $a^{\gamma'}$ one can consider (8) as a power series with the exponents from the lattice $R\mathbb{Z}^m$. Clearly, $R = BB_J^{-1}$ for each basis B of L, so L is a sublattice of $R\mathbb{Z}^m$, and hence (8) is a finite sum of Γ-series $\varphi_\gamma^{(i)}$ where $\gamma^{(i)}$ runs over the $R\mathbb{Z}^n/L$. In example above one has $RX = \left(\frac{1}{2}b^1, b^2\right)$ with unimodular matrix $X = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}$, and it means that $R\mathbb{Z}^2 = (RX)\mathbb{Z}^2 = L \bigsqcup \left(\frac{1}{2}b^1, b^2\right)$. So we get $\gamma^{(1)} = 0$ and $\gamma^{(2)} = \frac{1}{2}b^1$ like above in the GKZ-approach. In general we have the following statement.

Proposition 3. *For a given integer $n \times N$-matrix A of the type (4), whose columns generate $\mathbb{Z}^n$, and a chosen simplex $\sigma \subset Q$ of normalized volume Δ, there will be precisely Δ distinct A-hypergeometric series of type (8) which are linearly independent.*

Proof. Let B be a basis of $L = A^{-1}(0)$. Consider the lattice $M := R\mathbb{Z}^m = BB_J^{-1}\mathbb{Z}^m$. By the invariant factor theorem there are unimodular $m \times m$-matrices X and Y, such that the new bases for the lattices L and M given by

$$\widetilde{R} = R \cdot X\,, \quad \widetilde{B} = B \cdot Y$$

have the property that the basis $\widetilde{B}$ is expressed in the basis $\widetilde{R}$ by means of a diagonal integer matrix:

$$\widetilde{B} = \widetilde{R} \cdot \begin{pmatrix} \delta_1 & & \\ & \ddots & \\ & & \delta_m \end{pmatrix}, \qquad \delta_j \in \mathbb{Z}.$$

In other words, if $\tilde{b}^1, \ldots \tilde{b}^m$ denote the column vectors of $\widetilde{B}$, then the lattice M is Q-generated by the basis $\widetilde{B}$ of L as

$$M = \left\{ \frac{\tilde{b}^1}{\delta_1} s_1 + \ldots + \frac{\tilde{b}^m}{\delta_m} s_m \right\}_{s \in \mathbb{Z}^m}.$$

From this it is easily seen that the series (8) can be re-written in the powers

$$\left(a^{\tilde{b}^1/\delta_1}\right)^{s_1} \cdots \left(a^{\tilde{b}^m/\delta_m}\right)^{s_m}.$$

Clearly, the index $M : L$ is equal to $\Delta = |\det R''| = |\delta_1 \cdots \delta_m|$ and by choosing various radicals $(a^{\tilde{b}^j})^{1/\delta_j}$ we obtain Δ linearly independent series. □

We now introduce a dehomogenization of the series (8), where all other variables than the ones chosen by the position of the unit matrix equals 1, that is $a_1 = \ldots = a_n = 1$ and $a_{n+k} = z_k$, $k = 1 \ldots m$.

Definition 7. *For every Gale transformation $R = (R', E_m)^{tr}$, we define the dehomogenized hypergeometric series in m variables*

$$\phi(z) = \phi_B(z) = \sum_{k \in \mathbb{N}^m} \frac{z^k}{\prod_{j=1}^{N-m} \Gamma(\gamma_j + \langle r_j, k \rangle + 1)\, k!}, \tag{9}$$

where the r_j denote the rows in the matrix R', $z^k = z_1^{k_1} \ldots z_m^{k_m}$ and $k! = k_1! \cdots k_m!$.

Notice that the series in the Definition 7 are represented as Taylor series, due to the fact that we have chosen the matrix R to be on the form $(B', E_m)^{tr}$, with the unit matrix positioned in the last m rows. However by performing monomial changes of variables (9) represents more general series, so called Laurent-Puisieux series.

In fact, there is a natural correspondence between the following actions:

- Choosing a $(n-1)$-simplex $\sigma \subset \text{conv}(A)$, i.e. in the Newton polytope of A, which we denote $Q = Q(A)$.
- Choosing a set of n linearly independent column vectors in the matrix A.
- Choosing a set of n rows r_j in the dual matrix R, such that the remaining m rows in R give the unit matrix.

The fact that the chosen position of the unit matrix in R also corresponds to a certain choice of simplex in Q will play an important role when considering the convergence of the above series further into this paper.

3.2. Domains of convergence for the hypergeometric series

We want to find the convergence domain of the series defined in Definition 7. We prove the following result, where $\mathcal{A}_\sigma$ is the amoeba of the reduced principal determinant $E_A(1, a'')$.

Theorem 2. *The domain of convergence D_σ of the series $\phi(1, a'')$ in (8) is a complete Reinhardt domain with the property that the corresponding convex domain $Log(D_\sigma)$ contains all the connected components of the amoeba complement $\mathbb{R}^m \setminus \mathcal{A}_\sigma$, that are associated with the triangulations of $(Q, \mathfrak{A})$ containing the simplex σ (i.e. are associated with a certain vertex in the secondary polytope $\Sigma(A)$), while it is disjoint from all the other components.*

In order to prove this result we want to construct a triangulation of $(Q, \mathfrak{A})$, i.e. a triangulation on Q with the set of vertices on A. We do this in the following way. Take any function $\psi : \mathfrak{A} \to \mathbb{R}$ and consider in the space $\mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R}$ the union of the vertical half-lines

$$\{(\omega, y) \in \mathfrak{A} \times \mathbb{R} : y \leqslant \psi(\omega)\}.$$

Let G_Ψ be the convex hull of all these half-lines. This is an unbounded polyhedron projecting onto Q. The faces of G_ψ which do not contain vertical half-lines (i.e. are bounded) form the bounded part of the boundary of G_ψ, which we call the upper boundary of G_ψ. Clearly the upper boundary projects bijectively onto Q. If the function ψ is chosen to be generic enough, then all the bounded faces of G_ψ are simplices and therefore their projections to Q form a triangulation of $(Q, \mathfrak{A})$.

Let T be an arbitrary triangulation of $(Q, \mathfrak{A})$, and let $\psi : \mathfrak{A} \to \mathbb{R}$ be any function. Then there is a unique T-piecewise-linear function $g_{\psi,T} : Q \to \mathbb{R}$ such that $g_{\psi,T}(\omega) = \psi(\omega)$ when ω is a vertex of the triangulation T. The function $g_{\psi,T}$ is obtained by affine interpolation of ψ inside each simplex. Note that the values of ψ at points that are not vertices of any simplex of T does not affect the function $g_{\psi,T}$.

Definition 8. *Let T be a triangulation of $(Q, \mathfrak{A})$. For each simplex σ of this triangulation we shall denote by $C(\sigma)$ the cone in $\mathbb{R}^{\mathfrak{A}}$ consisting of functions $\psi : \mathfrak{A} \to \mathbb{R}$ with the following properties:*
(a) the function $g_{\psi,T} : Q \to \mathbb{R}$ is concave.
(b) for any $\omega \in \mathfrak{A}$ which is not a vertex of the simplex σ in the triangulation T, we have $g_{\psi,T}(\omega) \geqslant \psi(\omega)$.

Now, let $\mathfrak{A} \subset \mathbb{Z}^{n-1}$ be a finite subset, and Q the convex hull of $\mathfrak{A}$ as before. We assume that $\dim(Q) = n - 1$. Fix a translation invariant volume form Vol on $\mathbb{R}^n$. Let T be a triangulation of $(Q, \mathfrak{A})$. By the characteristic function of T we shall mean the function $\varphi_T(\omega) : \mathfrak{A} \to \mathbb{R}$ defined as follows:

$$\varphi_T(\omega) = \sum_{\delta : \omega \in Vert(\delta)} \mathrm{Vol}(\delta),$$

where the summation is over all (maximal) simplices δ of T for which ω is a vertex. In particular, $\varphi_T(\omega) = 0$ if ω is not a vertex of any simplex of T. Let $\mathbb{R}^{\mathfrak{A}}$ denote the space of all functions $\mathfrak{A} \to \mathbb{R}$.

Definition 9. *The secondary polytope $\Sigma(\mathfrak{A})$ is the convex hull in the space $\mathbb{R}^{\mathfrak{A}}$ of the vectors φ_T for all the triangulations T of $(Q, \mathfrak{A})$.*

The normal cone to $\Sigma(\mathfrak{A})$ at every φ_T will be called $N_{\varphi_T}\Sigma(\mathfrak{A})$ and consists of all linear forms ψ on $\mathbb{R}^{\mathfrak{A}}$ such that

$$\psi(\varphi_T) = \max_{\varphi \in \Sigma(\mathfrak{A})} \psi(\varphi).$$

The point φ_T is a vertex of $\Sigma(\mathfrak{A})$ if and only if the interior of this cone is non-empty. The union of the normal cones $N_{\varphi_{T_1}}\Sigma(\mathfrak{A}), \ldots, N_{\varphi_{T_k}}\Sigma(\mathfrak{A})$ where $T_1, \ldots, T_k$ are all the triangulations of $(Q, \mathfrak{A})$ that contains the simplex σ will be called the normal cone $N_{\varphi_\sigma}\Sigma(\mathfrak{A})$.

Proof of the theorem 2. We know from the implicit function theorem that D_σ is not empty, and Abel's lemma [17] tells us that whenever a point a belongs to D_σ, then so does the full polydisc centered at a. Therefore D_σ is indeed a complete Reinhardt domain , and the corresponding domain $\mathrm{Log}(D_\sigma)$ will contain the negative orthant $-\mathbb{R}^{n-1}_+$ in its recession cone $C(\sigma)$.

In fact, we will show that $C(\sigma)$ is the negative orthant. This we can see by letting the function $\psi = 0$ on all the points $\alpha^{(j)}$ in the simplex σ. This corresponds to chosing exactly this simplex σ. (We could choose ψ equal to anything at the points in σ.) Now $C(\sigma)$ consists of functions $\psi : \mathfrak{A} \to \mathbb{R}$ such that $g_{\psi,T}$ is concave and $g_{\psi,T}(\omega) \leqslant \psi(\omega)$ for all ω which are not vertices in σ. Hence $\psi(\omega) \leqslant 0$ for all ω and all functons ψ, and thus $C(\sigma)$ is equal to the negative orthant $\mathbb{R}^{n-1}_+$.

Let E be a connected component of $\mathbb{R}^{n-1} \setminus \mathcal{A}_\sigma$ that intersects the domain $\mathrm{Log}(D_\sigma)$. Then we claim that we actually have an inclusion $E \subset \mathrm{Log}(D_\sigma)$.

It follows, from what we have prooved so far, that the domain $\mathrm{Log}(D_\sigma)$ cannot intersect any component of the amoeba complement $\mathbb{R}^{n-1} \setminus \mathcal{A}_\sigma$ whose cone $\mathcal{C}(\sigma)$ is not in the negative orthant. On the other hand every connected component of $\mathbb{R}^{n-1} \setminus \mathcal{A}_\sigma$ with the corresponding cone $C(\sigma)$ contained in $\mathbb{R}^{n-1}_+$ will necessarily intersect, and hence be contained in the domain $\mathrm{Log}(D_\sigma)$. The following proposition therefore suffices to make the proof of Theorem 2 complete. □

Proposition 4. *The normal cone $N_{\varphi_T}\Sigma(\mathfrak{A})$ at a vertex of the secondary polytope $\Sigma(A)$ is contained in the negative orthant $-\mathbb{R}^{n-1}_+$ if and only if the corresponding triangulation of Q contains the simplex σ. In fact the union of such normal cones $N_{\varphi_\sigma}\Sigma(A)$ is equal to $-\mathbb{R}^{n-1}_+$.*

Proof. We will prove this proposition by proving that the normal cone $N_{\varphi_\sigma}\Sigma(A)$ coincides with the cone $C(\sigma)$. See ([11]). We get at once from the definitions of φ_T and $g_{\psi,T}$, and the fact that the integral of an affine-linear function g over a simplex σ is equal to the arithmetic mean of values of g at the vertices of σ times the volume of σ, the following ([11], Ch. 7):

$$(\psi, \varphi_T) = n \int_Q g_{\psi,T}(x)dx. \tag{10}$$

We now fix $\psi \in \mathbb{R}^A$. The upper boundary of G_ψ can be regarded as the graph of a piecewise-linear function $g_\psi : Q \to \mathbb{R}$.

$$g_\psi(x) = \max\{y : (x, y) \in G_\psi\}.$$

We can furthermore state about the function g_ψ the following:
(a) g_ψ is concave.
(b) For any triangulation T of (Q, A) we have $g_\psi(x) \geqslant g_{\psi,T}(x), \forall x \in Q$.
(c) We have

$$\max_{\varphi \in \Sigma(A)} (\psi, \varphi) = n \int_Q g_\psi(x)dx. \tag{11}$$

(a) follows by construction. To varify (b), it suffices to consider x varying in some fixed simplex σ of T. By definition, $g_{\psi,T}$ is affine-linear over σ and $g_\psi(\omega) \geqslant \psi(\omega) = g_{\psi,T}$ for any vertex $\omega \in \sigma$. Hence the inequality is valid over σ. The maximum in (11) can be taken over the set of the φ_T for all triangulations T of (Q, A), since $\Sigma(A)$ is defined as the convex hull of these φ_T. Hence part (b) together with (10) imply that the left hand side of (11) is greater than or equal to the right hand side. To show the equality, it suffices to exhibit a triangulation T for which $g_\psi = g_{\psi,T}$. To do this, we consider the projections of the bounded faces of the polyhedron G_ψ into Q. These are polytopes with vertices in A. Take a generic ψ' close to ψ. Then the bounded faces of the polyhedron $\mathcal{G}'_\psi$ give a triangulation T of (Q, A) which induces a triangulation of each of the above polytopes. Hence g_ψ is T-piecewise-linear and coincides with $g_{\psi,T}$. This proves (11). Hence the cones coincide. □

Remark 3. Theorem 2 was proven for the special case $n = 2$ in [7].

4. Mellin-Barnes integrals

4.1. General Mellin-Barnes integrals and their domains of convergence

By the multiple Mellin-Barnes integral we mean the integral

$$\Phi(z) = \frac{1}{(2\pi i)^m} \int\limits_{\delta + i\mathbb{R}^m} \frac{\prod\limits_{j=1}^{p} \Gamma(\langle a_j, z\rangle + c_j)}{\prod\limits_{k=1}^{q} \Gamma(\langle b_k, z\rangle + d_k)} z_1^{-s_1} \cdots z_m^{-s_m} ds, \tag{12}$$

where all vector parameters $a_j, b_k \in \mathbb{R}^m$ are real, the scalar parameters $c_j, d_k \in \mathbb{C}$, and $ds = ds_1 \ldots ds_m$. The vector $\delta \in \mathbb{R}^m$ is chosen so that the integration subspace $\delta + i\mathbb{R}^m$ is disjoint from the poles of the gamma-functions in the numerator.

For brevity we rewrite (12) as

$$\Phi(z) = \frac{1}{(2\pi i)^m} \int\limits_{\delta + i\mathbb{R}^m} F(s) z^{-s} ds, \tag{13}$$

denoting by $F(s)$ the ratio of the products of gamma-functions, and z^{-s} denotes the product $z_1^{-s_1} \cdots z_m^{-s_m}$. We suppose that the variable z varies on the Riemann domain over the complex torus $\mathbb{T}^m = (\mathbb{C} \setminus 0)^m$, so we assume that

$$z_j^{-s_j} = e^{-s_j \log z_j}, \quad \arg z_j \in \mathbb{R}.$$

We introduce the following notations:

$$x_j = \mathrm{Re} s_j, \quad y_j = \mathrm{Im} s_j, \quad j = 1, \ldots, m.$$

Let x and y be the vectors in $\mathbb{R}^m$ with coordinates x_j and y_j, correspondingly. Denote by $\theta = \mathrm{Arg} z = (\arg z_1, \ldots, \arg z_m)$, and

$$g(y) = \sum_{j=1}^{p} |\langle a_j, y\rangle| - \sum_{k=1}^{q} |\langle b_k, y\rangle|.$$

.

Theorem 3. *For any $\delta + i\mathbb{R}^m$ outside polar sets, the convergence domain of the Mellin-Barnes integral (12) is equal to* $\mathrm{Arg}^{-1}(U^\circ)$ *where U° is the interior of the set*

$$U = \bigcap_{||y||=1} \{\theta \in \mathbb{R}^m : |\langle y, \theta\rangle| \leqslant \frac{\pi}{2} g(y)\}.$$

In the case when $U^\circ \neq \emptyset$ it coincides with the interior P° of the polytope

$$P = \{\theta \in \mathbb{R}^m : |\langle v_\nu, \theta\rangle| \leqslant \frac{\pi}{2} g(v_\nu), \quad \nu = 1, \dots, d\}$$

where $\pm v_1, \dots, \pm v_d$ is the set of all unit vectors which generate the fan K corresponding to the decomposition of $\mathbb{R}^m$ by hyperplanes

$$\langle a_j, y\rangle = 0, \ \ j = 1, \dots, p \ \ \textit{and} \ \ \langle b_2, y\rangle = 0, \ \ k = 1, \dots, q.$$

Proof. Let $u, v \in \mathbb{R}$. Since the asymptotic equality

$$|v|^{u-1/2} \sim (|v| + 1)^{u-1/2} as |v| \to \infty$$

is valid for every fixed $u \in \mathbb{R}$, Stirling's formula implies that there are constants C_1 and C_2 such that

$$C_1(|v| + 1)^{u-1/2} e^{-\frac{\pi}{2}|v|} \leqslant |\Gamma(u + iv)| \leqslant C_2(|v| + 1)^{u-1/2} e^{-\frac{\pi}{2}|v|}, \tag{14}$$

where $u \in \mathcal{K} \subset \mathbb{R} \setminus \{0, -1, -2, \dots\}$ ($\mathcal{K}$ is a compact set), $v \in \mathbb{R}$, and the constants C_1 and C_2 depend only on the choice of $\mathcal{K}$. Using (14), and our notation $v_j = \mathrm{Im} s_j$, we can make the following estimate for the integrand in (13):

$$|F(s)z^{-s}| \leqslant \mathrm{const} \frac{\prod_{j=1}^p \tau_j}{\prod_{k=1}^q \xi_k} \exp\left\{\langle y, \theta\rangle - \frac{\pi}{2} g(y)\right\} \tag{15}$$

where

$$\tau_j = (|\langle a_j, y\rangle| + 1)^{\langle a_j, x\rangle + c_j - 1/2}, \quad \xi_k = (|\langle b_k, y\rangle| + 1)^{\langle b_k, x\rangle + d_k - 1/2},$$

and $g(y)$ was defined above. Moreover, (15) holds for all $y \in \mathbb{R}^m$ and all x in compact subsets of $\mathbb{R}^m$ disjoint from the polar hyperplanes

$$\{\langle a_j, x\rangle + c_j = -\nu\}, \quad \{\langle b_k, x\rangle + d_k = -\nu\}, \quad \nu = 0, 1, 2, \dots;$$

in particular, (15) is valid for $x = \delta$. It follows from (15) that, for each $\theta = \mathrm{Arg} z$, provided the inequality

$$\langle y, \theta\rangle < \frac{\pi}{2}\left(\sum_{j=1}^m |\langle a_j, y\rangle| - \sum_{k=1}^p |\langle b_k, y\rangle|\right) \text{for any } y \in \mathbb{R}^m \setminus \{0\} \tag{16}$$

the integrand in (13) decreases exponentially as $\|y\| \to \infty$. Therefore for such θ the integral (13) converges absolutely. By homogeneity, (16) is valid for all $y \in \mathbb{R}^m \setminus \{0\}$ whenever it holds for y on the sphere $\{y : \|y\| = 1\}$. It means that this integral converges for all $\theta = \mathrm{Arg} z$ from the intersection of halfspaces:

$$\bigcap_{\|y\|=1} \{\theta : \langle y, \theta\rangle < \frac{\pi}{2} g(y)\}.$$

The unit sphere $||y|| = 1$ is symmetric relative to the origin. Since $g(-y) = g(y)$ this implies that the mentioned intersection of halfspaces coincides with the intersection

$$U := \bigcap_{||g||=1} \{\theta : \, |\langle y, \theta\rangle| < \frac{\pi}{2} g(y)\}$$

of strips.

It is clear that the integral (13) does not converge for $\theta = \arg z$ outside of the closure $\bar{U}$, since the estimate in (14) implies not only (15), but for some other constant the reversed inequality:

$$|F(z)z^{-s}| \geqslant \text{const}\frac{\prod\limits_{j=1}^{p} \tau_j}{\prod\limits_{k=1}^{q} \xi_k} \exp\left\{\langle y, \theta\rangle - \frac{\pi}{2}g(y)\right\}. \qquad (17)$$

Thus if $\theta \in \mathbb{R}^m \setminus \bar{U}$ we have an inequality $\langle \theta, y\rangle > \frac{\pi}{2}g(y)$ for some y on the sphere $\|y\| = 1$. By (17) it means that the integrand in (13) and (12) increases exponentially in some open cone of $\mathbb{R}^m$, and therefore is not integrable. Hence $\mathrm{Arg}^{-1}(U)$ coincides with the doamin of convergence of the integral (12).

Of course, $U \subset P^\circ$. Let us explain why any point $\theta \in P^\circ$ belongs to U. Indeed $g(y)$ is a piece-wise linear function whose graph has corners only over the hyperplanes $\langle a_j, y\rangle = 0$ and $\langle b_k, y\rangle = 0$. Correspondingly, the function $\Psi_\theta(y) := \frac{\langle \theta, y\rangle}{g(y)}$ is a piece-wise fractional linear function with respect to the variable y. Consequently all extremal points of the function $\widetilde{\Psi}_\theta(y) = \Psi_\theta(y)\Big|_{\|y\|=1}$ lie on the vertices set of the spheric polyhedron $K \cap \{\|y\| = 1\}$, i.e. on the set $\{\pm v_1, \ldots, \pm v_d\}$. Therefore using again the property $g(-y) = g(y)$ we get that $\theta \in P^\circ$ implies $\theta \in U$. □

Remark 4. Some partial results on the domains of convergence for integral (12) were obtained in [8, 18, 19], and [4].

4.2. Reduction of hypergeometric series to Mellin-Barnes integrals

Given the hypergeometric series (2) with $a^{\gamma+\langle B,k\rangle} = a^\gamma (a^{b^1})^{k_1} \cdots (a^{b^m})^{k_m}$, its formal integral representation is the following Mellin-Barnes integral

$$\hat{\phi}(a) = \frac{a^\gamma}{(2\pi i)^m} \int\limits_{\delta + i\mathbb{R}^m} \frac{\Gamma(s_1)\cdots\Gamma(s_m)}{\prod\limits_{j=1}^{N-m} \Gamma(\gamma_j - \langle b_j, s\rangle + 1)} (-a^{b^1})^{-s_1} \cdots (-a^{b^m})^{-s_m} ds,$$

for some appropriately chosen $\delta \in \mathrm{R}_+^m$. This means that granted that the integral $\hat{\phi}$ converges, it will coincide with ϕ in (8), where the convergence domains of the integral and the series overlap. Let $\mathrm{Re}\gamma_j = c_j$ and choose c_j such that the polyhedron

$$\Pi := \{x_l \geqslant 0, l = 1, \ldots, m; c_j - \langle b_j, x\rangle \leqslant 0, j = 1, \ldots, N - m\}$$

becomes a simplicial m-dimensional polytope. Choose δ in the interior of Π. Using the equation

$$\Gamma(\zeta)\Gamma(1-\zeta) = \frac{2\pi i}{e^{i\pi\zeta} - e^{-i\pi\zeta}}$$

we get

$$\hat{\phi}(a) = \frac{a^\gamma}{(2\pi i)^m} \int_{\delta + i\mathbb{R}^m} \prod_{l=1}^{m} \Gamma(s_l) \prod_{j=1}^{N-m} r_j(s) (-a^{b^1})^{-s_1} \cdots (-a^{b^m})^{-s_m} ds,$$

where

$$r_j(s) = (2\pi i)^{m-N}\Gamma(-\gamma_j + \langle b_j, z\rangle)(e^{i\pi(\langle b_j,s\rangle - \gamma_j)} - e^{-i\pi(\langle b_j,s\rangle-\gamma_j)}).$$

Since $\sum\limits_{j=1}^{N-m} b_j = (-1, \ldots, -1)$ we have that for any division of $\{1, \ldots, N-m\}$ into two groups I and J, we get

$$\sum_{i \in I} b_i - \sum_{j \in J} b_j = (-1, \ldots, -1) - 2\sum_{j \in J} b_j.$$

This implies that

$$\prod_{j=1}^{N-m} (e^{i\pi(\langle b_j, s\rangle - \gamma_j)} - e^{-i\pi(\langle b_j, s\rangle - \gamma_j)})(-a^{b^1})^{-s_1} \cdots (-a^{b^m})^{-s_m} =$$

$$= \prod_{j=1}^{N-m} (e^{i\pi(\langle b_j, s\rangle - \gamma_j)} - e^{-i\pi(\langle b_j, s\rangle - \gamma_j)})e^{-2i\pi(s_1 + \ldots + s_m)}(a^{b^1})^{-s_1} \cdots (a^{b^m})^{-s_m}$$

is a linear combination of terms $(e^{2i\pi\beta_1} a^{b^1})^{-s_1} \ldots (e^{2i\pi\beta_m} a^{b^m})^{-s_m}$, where $\beta = (\beta_1, \ldots, \beta_m) \in \mathbb{Z}^m$; the coefficients of this linear combination depend on β and γ. Hence $\hat{\phi}(a)$ is a finite linear combination of shifted integrals as follows

$$\hat{\phi}(a) = \sum_{\beta} c_\beta(\gamma) \frac{1}{(2\pi i)^m} \int_{\delta + i\mathbb{R}^m} \prod_{j}^{N} \Gamma(-\gamma_j + \langle b_j, s\rangle)(z_1)^{-s_1} \ldots (z_m)^{-s_m} ds,$$

where $z_j = e^{2\pi i\beta_j} a^{b^j}$ and for a uniform notation, we henceforth denote all rows of the matrix $B = (B', E_m)^{tr}$ by b_j.

Therefore the hypergeometric series (8) and (9) can be rewritten as a linear combination of shifted integrals

$$I_\beta(z) = I(e^{2\pi i\beta_1} z_1, \ldots, e^{2\pi i\beta_m} z_m),$$

where

$$I(z) = \frac{1}{(2\pi i)^m} \int_{\delta + i\mathbb{R}^m} \prod_{1}^{N} \Gamma(\langle b_j, s\rangle - \gamma_j) z^{-s} ds. \tag{18}$$

Recall that by the condition on the matrix B we have that $\sum\limits_{1}^{N} b_j = 0$ and that means that (18) is a non-confluent Mellin-Barnes integral [17]. The integral (18) tells us how to define a continuous analogue of the Γ-series.

5. Γ-integrals

5.1. Definition of Γ-integrals and their hypergeometricity

Notice that instead of (1) we could consider a series where all Γ-factors are in the numerator, making use of the fact that the formula $\Gamma(t)\Gamma(1-t) = \pi/\sin \pi t$ enables us to move terms between numerator and denominator. Using the notations

$$\Gamma(k - \gamma) = \prod_{1}^{N} \Gamma(k_j - \gamma_j), \quad a^{\gamma - k} = \prod_{1}^{N} a_j^{\gamma - k_j},$$

we could consider a formal version of the Γ-series on the form

$$\sum_{k\in L}\Gamma(k-\gamma)v^{\gamma-k},$$

which also satisfies the hypergeometric system (5).

Let $L_{\mathbb{C}} = L\otimes\mathbb{C}$ be the subspace Ker$A\subset\mathbb{C}^N$ defined by the linear operator $A:\mathbb{C}^N\to\mathbb{C}^n$. Here A is the integer matrix as before.

Definition 10. *We call the integral*

$$\Phi(a)=\Phi_\gamma(a)=\int_{h\subset L_{\mathbb{C}}}\Gamma(\xi-\gamma)a^{\gamma-\xi}d\mu \tag{19}$$

a Γ*-integral, or an A-hypergeometric integral (of the Mellin-Barnes type). Here h is a cycle of dimension $m=N-n$ with a closed support and $d\mu$ is an m-dimensional measure on $L_{\mathbb{C}}$.*

Usually we take dμ as the differential form

$$\frac{1}{(2\pi i)^m}ds=\frac{1}{(2\pi i)^m}ds_1\wedge\ldots\wedge ds_m,$$

where $s\in\mathbb{C}^m$ is a parameter on $L=\{\xi=Bs\}$ and h as a vertical subspace $\sigma+i\mathbb{R}\subset\mathbb{C}^m$. So we get the integral

$$\Phi(a)=\frac{1}{(2\pi i)^m}\int_{\sigma+i\mathbb{R}^m}\prod_{j=1}^{N}\Gamma(\langle b_j,s\rangle-\gamma_j)a^{\gamma-Bs}ds \tag{20}$$

coinsiding up to a factor a^γ with the integral (18) after substitution $z=a^{Bs}$.

Proposition 5. *Under assumption that the integration set h is homologically equivalent in the holomorphy domain of $\Gamma(\xi-\gamma)|_{L_{\mathbb{C}}}$ to all shifts $h+k$, where k runs over the basis of L, the* Γ*-integral (19) satisfies the hypergeometric system (5) with $\beta=\sum\limits_1^N\gamma_j\alpha^{(j)}$.*

Proof. According to the Remark 1 after Definition 2 we choose any basis vector $k\in L$ and write it as the sum $k=k_+-k_-$ for non-negative vectors k_+ and k_-. Then

$$\frac{\partial^{|k_\pm|}}{\partial a^{k_\pm}}a^{\gamma-\xi}=(-1)^{|k_\pm|}\prod_{j=1}^{N}\prod_{l=1}^{k_{\pm j}}(\xi_j-\gamma_j+l)a^{\gamma-\xi-k_\pm},$$

Using the formula $t\Gamma(t)=\Gamma(t+1)$ this equality for k_- yields the following

$$\frac{\partial^{|k_-|}}{\partial a^{k_-}}\Phi_\gamma=(-1)^{|k_-|}\int_h\Gamma(\xi-\gamma+k_-)a^{\gamma-\xi-k_-}d\mu.$$

The same is true for $k_+=k+k_-$, hence under the assumption $h\sim h+k$, one gets

$$\frac{\partial^{|k_+|}}{\partial a^{k_+}}\Phi_\gamma=(-1)^{|k_+|}\int_h\Gamma(\xi-\gamma+k+k_-)a^{\gamma-\xi-k-k_-}d\mu=$$
$$=(-1)^{|k_+|}\int_h\Gamma(\xi-\gamma+k_-)a^{\gamma-\xi-k_-}d\mu.$$

Now we recall that $|k_+|=|k_-|$ for $k\in L$. Hence $\Box_k\Phi_\gamma=0$ for all basis vectors $k\in L$.

The equations $(\mathcal{E}_i-\beta_i)\Phi=0$ in (5) are satisfied by integral (19), since the action $\xi_i-\beta_i$ on the integrand gives a factor $\sum\limits_{j=1}^N\alpha_i^{(j)}\xi_j$ that vanishes on $h\subset L_{\mathbb{C}}$. □

In the case when the Γ-integral has the form (20), Beukers [20] has given a more effective condition for hypergeometricity.

Proposition 6 ([20], Theorem 3.1)**.** *Assume that in integral (3), $\delta = 0$ and $\gamma \in \mathbb{R}^m_-$, i.e. $\gamma_i < 0$ for all coordinates γ_i of γ. Then this integral satisfies the A-hypergeometric system (5) with $\beta = \sum\limits_1^N \gamma_j \alpha^{(j)}$.*

For the proof it is enough to show that in the expressions above for $\dfrac{\partial^{|k_+|}}{\partial a^{k_+}}\Phi_\gamma$ one has

$$\int_{\hat{s}+i\mathbb{R}^m} \Gamma(Bs-\gamma+k_-)a^{\gamma-Bs-k_-}ds = \int_{i\mathbb{R}^m} \Gamma(Bs-\gamma+k_-)a^{\gamma-Bs-k_-}ds,$$

where $\hat{s}$ satisfies the equation $B\hat{s} = k$. It follows from the fact that the family of subspaces $t\hat{s} + i\mathbb{R}^m$, $t \in [0,1]$ gives a homotopy of cycles $\hat{s} + i\mathbb{R}^m$ and $i\mathbb{R}^m$ in the holomorphy domain of the integrand, since we we have the for $\gamma \in \mathbb{R}^m_-$

$$B(t\hat{s}) - \gamma + k_- = tk - \gamma + k_- = tk_+ + (1-t)k_- - \gamma \in \mathbb{R}^m_+.$$

5.2. Domain of convergence for the Γ-integral

Recall also that a Minkowski sum of line segments is called a zonotope (see, for example [21]).

Theorem 4. *The convergence domain of the integral (18) is equal to $\mathrm{Arg}^{-1}(Z_B^\circ)$ with Z_B° being the interior of the zonotope $Z_B = [0,\pi b_1] + \ldots + [0,\pi b_N]$.*

Proof. By Theorem 3 one has to prove that $Z_B = P$ where

$$P = \{\theta \in \mathbb{R}^m : |\langle v, y\rangle| \leqslant \frac{\pi}{2} g(v_\nu),\ \nu = 1,\ldots,d\}.$$

Since $\sum_1^N b_j = 0$ any point $\theta = \pi(\lambda b_1 + \ldots + \lambda_N b_N) \in Z_B$, where $\lambda_j \in [0,1]$, can also be represented as

$$\theta = -\pi((1-\lambda_1)b_1 + \ldots + (1-\lambda_N)b_N).$$

This implies that

$$Z_B = \frac{1}{2}([-\pi b_1, \pi b_1] + \ldots + [-\pi b_N, \pi b_N]).$$

Hence any point $\theta \in Z_B$ can be represented as

$$\theta = \frac{1}{2}\cdot\pi(\lambda_1 b_1 + \ldots + \lambda_N b_N),\ \text{with}\, \lambda_j \in [-1,1]. \tag{21}$$

By the Triangle Inequality we then get for $\theta \in Z_B$

$$|\langle \theta, y\rangle| \leqslant \frac{\pi}{2}\sum |\langle b_j, y\rangle| \ \text{for all}\ y \in \mathbb{R}^m,$$

which means that $Z_B \subset P_B$.

Now we prove that in fact $Z_B = P$. The vector v_ν in the definition of P_B is orthogonal to some subset $B_\nu \subset B$ of $n-1$ linearly independent vectors b_j. Consider the parallelepiped P_ν generated by these vectors,

$$P_\nu = \{\theta = \frac{\pi}{2}\sum_{j:b_j\in B_\nu} \lambda_j b_j : \lambda_j \in [-1,1]\}.$$

The zonotope Z_B has two parallel facets being the two extremal translations of P_ν, namely

$$F_\nu^\pm = P_\nu + \frac{\pi}{2} \sum_{i: b_i \in B \setminus B_\nu} \lambda_i^\pm b_i,$$

where $\lambda_j^\pm = \text{sign}\langle v_\nu, b_j \rangle$. The normal vector to $F_\nu^\pm$ is v_ν and by (21) for each $\theta \in Z_B$

$$\langle v_\nu, \theta \rangle = \frac{\pi}{2} \sum_{i: b_i \in B \setminus B_\nu} \lambda_i^\pm \langle v_\nu, b_i \rangle.$$

We see that

$$\max_{\theta \in Z_B} |\langle v_\nu, \theta \rangle| = \frac{\pi}{2} \sum_{j=1}^{N} |\langle v_{\nu_j}, b_j \rangle|.$$

Since $Z_B = \cap_{\nu=1}^d S_\nu$, where S_ν is the strip

$$S_\nu = \{\theta; |\langle v_\nu, \theta \rangle| \leqslant \frac{\pi}{2} \sum_{j=1}^{N} \langle v_\nu, b_j \rangle = \frac{\pi}{2} g(v_\nu)\}$$

we get our statement $Z_B = P_B$. □

5.3. Independence of Γ-integrals

Now, if we choose coordinates $s = (s_1, \dots, s_m) \in L_{\mathbb{C}}$ and represent each $\xi \in L_{\mathbb{C}}$ as a linear combination of column vectors b^j of the matrix B,

$$\xi = s_1 b^1 + \ldots + s_m b^m,$$

then we reduce the Γ-integral (19) into an integral of type (18) with $z = a^{b^j}$. Hence the integral (18) satisfies the hypergeometric system. Since the domain of convergence of the shifted integrals $I_\beta(z)$ are shifted zonotopes $Z^\beta = -\beta + Z_B$, any collection of shifted integrals with non-empty common domain of convergence is linearly independent. More precisely we have the following lemma:

Lemma 1. *Let $\{Z^\beta\}_{\beta \in J}$ be a family of pairwise different shifts of the zonotope Z_B. If their intersection*

$$Z^J := \bigcap_{\beta \in J} Z^\beta$$

has a non-empty interior, then the corresponding shifts $I_\beta(z)$ of the integral (3) is linearly independent.

Proof. Let Z^{β_0}, $\beta_0 \in J$, be an arbitrary zonotope from the chosen family. Suppose that it contributes to the boundary of the intersection Z^J, that is, there exists a point $\theta_0 \in \partial Z^{\beta_0}$, which is an interior point of the intersection $\bigcap_{\beta \in J \setminus \beta_0} Z^\beta$. Then we claim that in any linear relation between the integrals $\{I_\beta\}_{\beta \in J}$ the coefficient of I_{β_0} must be zero. Indeed, otherwise from the representation

$$I_{\beta_0}(z) = \sum_{\beta \in J \setminus \beta_0} c_\beta I_\beta(z)$$

we would get from Theorem 4 that I_{β_0} is holomorphic in the sectorial domain over

$$\Big(\bigcap_{\beta \in J \setminus \beta_0} Z^\beta \Big)^\circ,$$

which contains points from $\mathbb{R}^n \setminus Z^{\beta_0}$. Then it would follow from Theorem 4 and Bochner's theorem that I_{β_0} would be holomorphic in a sectorial domain over a bigger convex set than Z_{β_0}. But this would contradict Theorem 4. From this argument we conclude that, if there exists a linear relation between the I_β, not involving I_{β_0}, then Z^{β_0} contains the intersection of all the other Z^β, $\beta \in J \setminus \beta_0$. Applying the same reasoning to this latter family, we arrive at a situation where all the zonotopes in the family contribute to the boundary of the intersection, and for which the corresponding family of integrals is linearly independent. □

Finally we will need two results from [22] that we for completeness list below. The following theorem reveals a close connection between the discriminant coamoeba and the zonotope.

Theorem 5. *The summed chain $\mathcal{A}'_B + Z_B$ is a cycle, and hence equal to $m_B\mathbf{T}^2$, with some integer multiplicity m_B. In fact, provided that the vectors b_k are ordered clockwise projectively, this multiplicity is given by the formula*

$$m_B = \frac{1}{2}\sum_{j<k} \det{}^+(b_j, b_k)\,.$$

Theorem 6. *The multiplicity m_B from Theorem 5 coincides with d_B, where d_B is the normalized volume of the convex hull of the point configuration $A \subset \mathbf{Z}^2$.*

Using these results of [22], we arrive at the following result in two dimensions. We use the notation m for the normalized area of $\operatorname{conv}(A)$ which is equal to the maximal number of linearly independent solutions to the hypergeometric system of differential equations (5).

Theorem 7. *Assume $n = 2$. Let F_i be a component of the torus $(\mathbb{R}/2\pi\mathbb{Z})^2$ that is covered by the coamoeba $\mathcal{A}'_{E_A}$ exactly i times. Then there will be exactly $m - i$ integrals of the type (18) converging in the sectorial domain over F_i. In particular for the complement of the coamoeba, that is for $i = 0$, these integrals provide a basis for the whole solution space to (5).*

Proof. Follows from Lemma 1, Theorem 5 and Theorem 6. □

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Области сходимости A-гипергеометрических рядов и интегралов

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Доказываются две теоремы об областях сходимости для A-гипергеометрических рядов и ассоциированных с ними интегралов типа Меллина–Барнса. Точные области сходимости описаны в терминах амеб и коамеб соответствующих главных A-детерминантов.

Ключевые слова: A-гипергеометрический ряд, интеграл Меллина-Барнса, Γ*-интеграл, главный A-дискриминант.*