Journal of Siberian Federal University. Mathematics & Physics 2019, 12(4), 475–482

УДК 517.55

The Closure and the Interior of $\mathbb{C}$-convex Sets

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Received 13.01.2019, received in revised form 16.02.2019, accepted 10.04.2019

$\mathbb{C}$-convexity of the closure, interiors and their lineal convexity are considered for $\mathbb{C}$-convex sets under additional conditions of boundedness and nonempty interiors. The following questions on closure and the interior of $\mathbb{C}$-convex sets were tackled
1. The closure of a bounded $\mathbb{C}$-convex domain may not be lineally-convex.
2. The closure of a non-empty interior of a $\mathbb{C}$-convex compact in $\mathbb{C}^n$ may not coincide with the original compact.
3. The interior of the closure of a bounded $\mathbb{C}$-convex domain always coincides with the domain itself.

The questions were formulated by Yu. B. Zelinsky.

Keywords: strong linear convexity, $\mathbb{C}$-convexity, projective convexity, lineal convexity, Fantappie transform, Aizenberg-Martineau duality.
DOI: 10.17516/1997-1397-2019-12-4-475-482.

Introduction

Let $H(M)$ be the space of functions that are holomorphic in an open or a compact subset M of a space $\mathbb{C}^n$ or its projective compactification $\mathbb{CP}^n$. Let us denote the closed subspace $H(M)$ consisting of functions that vanish at points of an infinitely distant hyperplane in CP^n by $H_0(M)$. In the case of $M \in \mathbb{C}^n$ the spaces $H_0(M)$ and $H(M)$ coincide.

A. Martino [1] investigated the *Fantappie correspondence* that associates each analytic functional $\mu \in H'(M)$ with its indicatrix $\hat{\mu}(\zeta) = \mu_z\big(\frac{1}{1+\langle z,\zeta\rangle}\big)$. Here, the index z for the functional μ means that expression in parentheses is a function of z with a fixed parameter ζ. The Fantappie indicatrix belongs to the space $H_0(\widetilde{M})$, where $\widetilde{M} = \mathbb{C}^n \setminus \cup_{z\in D}\{\zeta \in C^n : \langle z,\zeta\rangle \neq -1\}$ is its *dual complement* [2] (also called *le complémentaire projectif* or *conjugate set* [3]). Moreover, $\widetilde{M} \subset \mathbb{C}^n$ if and only if $0 \in M$.

Martino called the open or closed set $M = (\widetilde{\widetilde{M}}) \subset \mathbb{C}^n$ strongly lineally convex (*fortement liné ellement convexe*) if the Fantappie correspondence is an isomorphism of space $H'(M)$ on the function space $H_0(\widetilde{M})$ (which coincides with $H(\widetilde{M})$ in the case of $0 \in M$). With the obvious substitution of $H'(M)$ for $H_0'(M)$ this definition is extended to subsets of the projective space. The natural additional condition is $M = (\widetilde{\widetilde{M}})$. It means that a hyperplane passes through every point of the complement of M that does not enter M. It was termed *liné ellement convexe.*

In much of the subsequent publications this term translates as linear convexity which often causes association with the sum operation. This association is inappropriate because the well-known convexity of the Minkowsky sum of convex sets does not hold for lineal convexity.

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However, a simple equivalent definition of the lineal convexity condition is that the linear functions separate the points of the complement or, in other words, the set coincides with the intersection of the preimages of their linear images.

This interpretation makes the following well-known properties of lineally convex sets obvious:

Any circular (invariant with respect to rotations $z \mapsto e^{i\theta}z$) lineally convex set containing its centre is the intersection of the linear preimages of the circle and therefore is convex.

The maximum circular subset of a given lineally convex set is convex.

There is also no non-convex strongly lineally convex sets [4] among the Cartesian products of complex one-dimensional domains with smooth boundaries. Later an additional restriction on the smoothness of the boundaries was removed [5].

L. A. Aisenberg studied a similar, seemed less restrictive, but as it appears later an equivalent definition of a strongly lineal convexity. He showed that convex domains and compacts are strongly lineally convex [6] and that domains strongly approximated from the inside by lineally convex domains with a smooth boundary are strongly lineally convex [3].

Soon, first examples of limited strongly linearly convex but nonconvex domains with a smooth boundary were presented [7, 8].

Two hypotheses on the geometric meaning of the strong lineal convexity were presented [9].

The first hypothesis consisted in the possibility of approximation by linearly convex domains with a smooth boundary. It is based on the example of a linearly convex region that does not possess such an approximation [10] and it was refuted only almost half a century later by P. Pflug and W. Zwonek [11].

The second hypothesis is based on results of [12] on connectedness and simply connectedness of sections by complex straight lines of a lineally convex domain with a smooth boundary and on results of [13] on the simply connectedness of the intersection of a strongly lineally convex compact with any complex straight line (in the formulation of L. A. Aizenberg "*For a strong linear convexity of a region (compact), it is necessary and sufficient that the intersection of this region (compact) by any complex straight line is connected and simply connected*" [14].

After the confirmation of the hypothesis [15, 16] this condition was called the $\mathbb{C}$-convexity [2, 17, 18]. It continues to attract attention as a *natural complex analog of convexity*:

1. The geometric definition of $\mathbb{C}$-convexity is a complex topological analogy of the classical definition of convexity from a topological point of view.

2. Equivalent to $\mathbb{C}$-convexity acyclicity of fractional-linear projections of the set [18] is a complex analogy of a topological point of view (the most important property of separability of convex sets).

3. For domains and compacts in CP^n, $\mathbb{C}$-convexity provides an ultimate tool for constructing duality theorems for spaces of holomorphic functions [2, 18] with applications to the theory of convolution equations in the complex region [19] and to the study of supports of analytic functionals [20, 21].

4. Holomorphic antiderivatives in all directions exist in a bounded domain of holomorphy if and only if this region is $\mathbb{C}$-convex [2, 22].

5. A holomorphic function on a compact set is represented by a series of fractional linear functions if and only if the hull of holomorphy of the compact set is $\mathbb{C}$-convex [23].

6. The analytic functional supported on a holomorphically convex compact is uniquely determined by its values of functions of the form

$$f(z) = g(k_1 z_1 + \dots k_n z_n), \qquad k_1, \dots k_n \in \mathbb{C}, \tag{1}$$

where g is an analytic function of one variable, and l is linear, if and only if the compact ℂ-convex [24].

7. Linear combinations of holomorphic functions of the form (1) are dense in $H(K)$ if and only if the compact K is ℂ-convex. (This easily follows from either the previous statement or from sentence 3 in [17])

8. The restriction of ℂ-convexity of a domain or compact in $\mathbb{C}^n$ is natural for Kergin interpolation just as the condition of convexity is natural in the real case [17].

The geometric properties of ℂ-convexity are described in detail in [2, 18, 25] where many unsolved problems are presented. Seven significantly difficult problems on ℂ-convexity were formulated [26]. Problems 2 and 4 of that seven problems were solved [11, 27]. Many other simply and naturally formulated questions concerning geometric properties of ℂ-convexity have not been solved yet.

1. The interior of the closure of a bounded ℂ-convex domain in $\mathbb{C}^n$

Theorem 1. *The interior of the closure of any ℂ-convex bounded domain $D \Subset \mathbb{C}^n$ coincides with the domain itself.*

This theorem answers to the question formulated by Yu. B. Zelinsky [28, 29]. An example of Cartesian product of a complex plane with a cut along a ray and a complex plane shows the significance of the boundness condition.

Proof. Suppose the contrary: there exists a point $w^0 \notin D$ that lies strictly inside the closure $\overline{D}$ of domain D bounded by the radius $R = \sup_{z \in D} |z - 0|$. Since the condition of ℂ-convexity is invariant with respect to automorphisms of the projective space $\mathbb{CP}^n$ and, in particular, shifts, it is sufficient to consider the case $w^0 = 0$. By the condition, there exists a spherical neighbourhood $B_r = \{z : |z| < r\} \Subset \overline{D}\}$ with the centre at this point and $r < R$.

The linear convexity of D in particular means the existence of the hyperplane $h = \{z = (z_1, \dots, z_n) \in \mathbb{C}^n : \sum\limits_{k=1}^{n} a_k z_k = 0\}$ that passes through w^0 outside the domain $0 \in h \subset \mathbb{C}^n \setminus D$. We normalize coefficients by the condition $\sum\limits_{k=1}^{n} |a_k|^2 = 1$ and fix the mapping $f(z) = \sum\limits_{k=1}^{n} a_k z_k$ of the ball B_r to the circle $U_{0,r}$ and domain D to a simply connected domain $f(D) \subset \mathbb{C}$. Let $g(\zeta) = \zeta(\overline{a}_1, \dots, \overline{a}_n)$ be the inverse to f isometry. It immediately follows from the assumption that $f(D)$ is dense in $U_{0,r}$ and $0 \in \partial f(D) \cap U_{0,r}$.

We distinguish the single-valued branch φ of the function $\arg z$ in $f(D)$. The function φ is defined on a dense subset of the circle $U_{0,r}$ containing the circumference $\partial U_{0,\frac{1}{2}}$. The continuous in $f(D)$ function φ must have some unremovable discontinuity in $\partial U_{0,\frac{1}{2}}$. Let us choose points $\zeta_2, \zeta_3 \in f(D)$ near this discontinuity so that $|\zeta_k - \zeta_1| < \dfrac{r^2}{54R}$, $k = 2.3$ and $\varphi(\zeta_2) - \varphi(\zeta_3) > 6.2 <$

2π. Let $\delta < \dfrac{r^2}{54R}$ be so small that the circular neighbourhoods $U_{\zeta_2,\delta}$ and $U_{\zeta_3,\delta}$ of these points lie in $f(D)$, and values of φ differ by at least 6 in these neighborhoods.

By our assumption, D is dense in the δ-neighbourhood of the point $g(\zeta_2)$ contained in B_r. Therefore, there exists a point $w^2 \in D$ in this neighborhood.

Let us fix a vector of unit length $b \in \mathbb{C}^n$ that is orthogonal to a. D is dense in the δ-neighbourhood of the point $g(\zeta_3) + \dfrac{r}{2}$ which also lies in B_r. Hence, it also has a point $w^3 \in D$.

Due to the $\mathbb{C}$-convexity the section of the domain D by the complex straight line passing through points w^2 and w^3 contains a curve connecting these points. There is a point w^4 on this curve at which the continuous superposition $f \circ \varphi$ takes the intermediate value $\varphi(f(w^4)) = \varphi(w^2) - \pi$. Since arguments of points $f(w^2)$ and $f(w^4)$ differ by π then $|f(w^4) - f(w^2)| > |f(w^2)|$. By the triangle inequality we have

$$\begin{aligned}
|f(w^2)| &\geqslant |\zeta_2| - |\zeta_2 - f(w^2)| \geqslant r/2 - \frac{r^2}{54R} > \frac{r}{3},\\
|f(w^3) - f(w^2)| &\leqslant |f(w^3) - \zeta_3)| + |\zeta_3 - \zeta_2| + |\zeta_2 - f(w^3)| < 3\frac{r^2}{54R} = \frac{r^2}{18R},\\
|w^3 - w^2| &\geqslant |w^3 - g(\zeta_3)| - |g(\zeta_3) - g(\zeta_2)| \geqslant\\
&\geqslant r/2 - \delta - |\zeta_3 - \zeta_1| - |\zeta_2 - \zeta_1| \geqslant \frac{r}{2} - 3\frac{r^2}{72R} > \frac{r}{3}.
\end{aligned}$$

Since points w^k lie on the same complex straight line then

$$|w^4 - w^2| = |w^3 - w^2| \frac{|f(w^4) - f(w^2)|}{|f(w^3) - f(w^2)|} \geqslant \frac{r}{3} \frac{\frac{r}{3}}{\frac{r^2}{18R}} > 2R.$$

This is in contradiction with the definition of R. □

2. Closure of a bounded ℂ-convex domain

For $z = (z_1, \dots, z_n) \in \mathbb{C}^n$, we use the usual notation $z_k = x_k + iy_k$ and $'Z = (z_2, \dots, z_n) \in \mathbb{C}^{n-1}$. Let us introduce the following designations $U_{c,r} = \{z : |z - c| < r\}$ and $U_1 = U_{0,1}$ and $U_2 = U_{1/2,1/2}$.

Lemma 1. *On the circumference ∂U_2 the equality*

$$|x + iy - 1|^2 = 1 - |x + iy|^2 = \frac{y^2}{|x + iy|^2}$$

holds.

Proof. By applying the formula $|x + iy|^2 = x^2 + y^2$ and substituting the equation $y^2 = x - x^2$ of the boundary U_2, each term in the equality can be easily reduced to the form $1 - x$. □

Corollary 1. *Function*

$$\psi(z_1) = \begin{cases} \dfrac{y_1^2}{|z_1|^2}, & z_1 \in U_2,\\ 1 - |z_1|^2, & z_1 \notin U_2 \end{cases}$$

is continuous on $\overline{U_1}$.

Theorem 2.

1. *Domain* $D = \{z \in \mathbb{C}^n : \quad |z_1| < 1, |'z|^2 < \psi(z_1)\}$ *is bounded and* $\mathbb{C}$*-convex.*

2. *Its closure* $\overline{D}$ *is not linearly convex.*

Proof. **1.** Let us first prove the connectedness and the simply connectedness of the sections of the domain by complex lines. Sections with complex lines $z_1 = \text{const}$ are circles. Any other complex line has the form $(z_1 - a_1) \cdot' a$, where parameter z_1 is taken along the coordinate projection on the z_1 axis. We exclude from the further consideration the trivial case $a = 0$.

The boundary of the domain D is contained in the union of the boundary of the unit ball B and the boundary of the domain $Q = \left\{z \in \mathbb{C}^n : \ |z_1| < 1, |'z| < \frac{|y_1|}{|z_1|}\right\}$. To understand the geometry of sections by complex lines of the domain Q we apply to it a linear fractional transformation of the form $(w,'w) = f(z,'z) = \dfrac{1}{z_1}(1,'z)$ which is an involution so that $(z,'z) = = f(w)$. This transformation translates complex lines into complex lines. We obtain $f(D) = = \{(w_1,'w) \in \mathbb{C} \times \mathbb{C}^{n-1} : \ |w_1| > 1, \ |'w| < |\operatorname{Im} w_1|\}$.

Now in each quadrant $\operatorname{Re} w_1 > 1, \operatorname{Im} w_1 \neq 0$, representing the semicircles $U_\pm = U_2 \cap \{\operatorname{Im} z_1 \gtrless 0\}$, the coordinate projection of the complex section boundary the straight line appear either as a branch of a hyperbola, a branch of a parabola or an ellipse that does not contain real points, or as a pair of lines with a real intersection point. In any case, it has an axis of symmetry which is parallel to the imaginary axis. Therefore, non-empty part of the boundary of the original projection in any half of U_2 always match one of the following cases:

- an inversion of either the ellipse or its part with the ends on their open semicircumference,
- an inversion of the part of the parabola or hyperbola branch with one end at zero and the other on the open semicircle,
- a pair of circular arcs with a common end on the real axis and equal angles between the tangents at the common end and the axis and the other ends are zero and a non-real point of ∂U_2.

Moreover, the presence of a closed boundary curve in one of the sets U_+, U_- or $U_1 \setminus U_2$ is possible only if the point b of the intersection of the section line with the z_1 axis lies in the same sets.

If a part of the boundary of a section on the unit sphere is an incomplete arc of a circle then it must be closed by one or two curves described above in $U_\pm$ and self-intersections are impossible. If the arc is the complete circumference then it must be complemented by circular arcs with common ends 0 and 1 in both $U_\pm$.

If the sign $\operatorname{Im} z_1$ is the same in the part of the boundary of the section on the unit sphere then the centre of the circle has the same sign as well as $\operatorname{Im} b$.

Taking into account the described features, an easy complete enumeration of possible variants of sections shows the connectedness and simple connectedness of the section.

2. Since the domain $\overline{D}$ contains single disks on the coordinate axes its lineally convex hull $\widetilde{\widetilde{D}}$ contains the unit hyper-cone $\left\{z \in \mathbb{C}^n : \sum\limits_{k=1}^{n} |z_k| < 1\right\}$. The point $\left(\frac{1}{2}, \frac{1}{4}, 0, \ldots, 0\right)$ lies strictly inside this hyper-cone but outside $\overline{D}$. □

Let us denote the shift of the region containing the origin by $D_2 = \{(z_1 + \frac{1}{3}, z_2) : z \in D\}$.

Corollary 2. *The compact* $K = \widetilde{D}_2$ *gives an example of a* $\mathbb{C}$*-convex compact in* $\mathbb{C}^n$ *with non-empty interior that is not dense in* K.

Conclusion

Theorems 1,2 and Corollary 2 answer to questions formulated by Yu. B. Zelinsky [28, 29]. They once more confirmed the mysteriousness of the geometry of ℂ-convex domains and compact sets.

It is well known [30] that the sections of bounded ℂ-convex sets cannot be arbitrary acyclic domains. They always satisfy the additional condition of spiral connection.

Problem 1. *Describe simply connected flat domains that are straight linear sections of bounded ℂ-convex domains.*

Until now, not a single sufficient condition (except for convexity and particular examples) on a bounded simply connected domain guarantees the existence of a bounded ℂ-convex domain with a given section by the coordinate line. Moreover, there is no idea in what terms to formulate similar sufficient and necessary conditions.

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Замыкание и внутренность ℂ-выпуклых множеств

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Для ℂ-выпуклых множеств также и при дополнительных условиях ограниченности и непустоты внутренности исследованы ℂ-выпуклость замыкания и внутренности и их линейчатая выпуклость. Получены следующие ответы на цикл вопросов Ю. Б. Зелинского о замыкании и внутренности ℂ-выпуклых множеств:

1. Замыкание ограниченной ℂ-выпуклой области может не быть линейчато выпуклым.

2. Замыкание непустой внутренности ℂ-выпуклого компакта в $\mathbb{C}^n$ может не совпасть с исходным компактом.

3. Внутренность замыкания ограниченной ℂ-выпуклой области всегда совпадает с самой областью.

Ключевые слова: сильная линейная выпуклость, ℂ-выпуклость, проективная выпуклость, линейчатая выпуклость, двойственность Айзенберга-Мартино.