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Locally Explicit Fundamental Principle for Homogeneous Convolution Equations

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In the present paper a locally explicit version of Ehrenpreis's Fundamental Principle for a system of homogeneous convolution equations $\check{f} * \mu_j = 0$, $j = 1, \dots, m$, $f \in \mathcal{E}(\mathbb{R}^n)$, $\mu_j \in \mathcal{E}'(\mathbb{R}^n)$, is derived, when the Fourier Transforms $\hat{\mu}_j$, $j = 1, \dots, m$ are slowly decreasing entire functions that form a complete intersection in $\mathbb{C}^n$.

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1. Fundamental principle for homogeneous convolution equations

Probably, the monograph [1] was the first in the field that illustrated that both, residues and integral representation formulas in several complex variable, are powerful tools allowing to provide solutions to seemingly untractable otherwise mathematical problems. In the present paper weighted integral representation formulas and different than in [1] realization of residues allow to approach from local point of view the Fundamental principle of convolution equations.

Recall that a pluri-subharmonic function $p(z)$ on $\mathbb{C}^n$ is called a weight function ([3]) if it is satisfying the following conditions: i) $p(z) \geqslant 0$, ii) $\log(1 + ||z||) = O(p(z))$, iii) if $||z - \zeta|| < 1$, then $p(\zeta) \leqslant A_1 p(z) + A_2$ for some constants A_1 and A_2.

Given a weight function $p(z)$ we consider the corresponding subspace of vector space of entire functions $A(\mathbb{C}^n)$:

$$A_p = \left\{ f \in A(\mathbb{C}^n) : \exists A_f, B_f > 0 \ : \ |f(z)| \leqslant A_f e^{B_f p(z)} \right\}.$$

If $(f_1, \dots, f_m)$ are m entire functions on $\mathbb{C}^n$, then $\mathcal{L}$ denotes the family of m-dimensional affine subspaces L of $\mathbb{C}^n$, such that

$$\cup_{L \in \mathcal{L}} \supset \{ z \in \mathbb{C}^n : \ f_i(z) = 0, \ 1 \leqslant i \leqslant n \}.$$

Following [3] we recall the following definition

Definition 1.1. *The family $(f_1, \dots, f_m)$ of m entire functions is slowly decreasing with respect to $\mathcal{L}$ if and only if there exist positive constants ϵ_1, C_1, K_1, K_2 such that*
1) *for each $L \in \mathcal{L}$ the set*

$$\mathcal{O} = \{ z \in L : \ |f_i(z)| < \epsilon_1 \exp(-C_1 p(z)), \ 1 \leqslant i \leqslant m \} \tag{1.1}$$

has relatively compact components,
2) *If $\mathcal{O}$ as in (1.1) and z, ζ belong to the same component of $\mathcal{O}$, then*

$$p(\zeta) \leqslant K_1 p(z) + K_2.$$

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Given a slowly decreasing family of functions $(f_1, f_2, \ldots, f_m)$ with respect to the family $\mathcal{L}$ and the weight function $p(z)$ one defines for a component G of $\mathcal{O}$ the open set

$$\Omega_G = \{z \in \mathbb{C}^n : \text{ there exists } \zeta \in G \text{ so that } |z - \zeta| < \epsilon_2 \exp(-C_2 p(\zeta))\}, \tag{1.2}$$

for some positive constants ϵ_2, C_2. Such an open set is called *good*. Keeping the values of the parameters ϵ_1, C_1 from Definition 1.1 and the values of the parameters ϵ_2, C_2 from (1.2) fixed, one obtains the family of open sets $\mathcal{I} = \{\Omega_G\}_{G \subset \mathcal{O}}$. The family $\mathcal{I}$ is called *good family*. If both parameters ϵ_1 and ϵ_2 decrease, while both parameters C_1 and C_2 increase, then the good family $\mathcal{I}'$ so produced is called a *good refinement* of $\mathcal{I}$. A naturally defined refinement map $\rho : \mathcal{I}' \longrightarrow \mathcal{I}$ corresponds to any open set $\Omega'_{G'} \in \mathcal{I}'$ associated with the component G' of an open subset $\mathcal{O}'$ of a certain line (m-plane) $L \in \mathcal{L}$, the open set Ω associated with the unique component G of the open set $\mathcal{O}$ such that $G' \subset G$. Thus, it is natural to consider the following definition ([3]).

Definition 1.2. *A good family $\mathcal{I}$ is said to be almost parallel if and only if there exists its good refinement $\mathcal{I}'$ such that whenever $\Omega_0, \Omega_1 \in \mathcal{I}'$ and $\Omega_0 \cap \Omega_1 \neq \emptyset$, then $\bar{\Omega}_0 \cup \bar{\Omega}_1 \subset \rho(\Omega_0) \cap \rho(\Omega_1)$, where ρ is the natural refinement map defined above.*

We continue by recalling another necessary definition from ([3]).

Definition 1.3. *We say that $\mathcal{L}$ is an analytic family of lines (m-planes) if and only if there is good family $\mathcal{I}$ associated to $\mathcal{L}$ with the following property: given $\Omega \in \mathcal{I}$ with the associated line (m-plane) $L \in \mathcal{L}$ there exist local analytic coordinates (s,t) on Ω such that $\Omega \cap \{(s,t) : t = 0\} = \Omega \cap L$ and $\Omega \cap \{(s,t) : t = const\} = \Omega \cap L_t$ for some $L_t \in \mathcal{L}$.*

Furthermore, recall that for $m \leqslant n$, an m-tuple of holomorphic functions $(f_1, \ldots, f_m)$, $f_i : \mathbb{C}^n \longrightarrow \mathbb{C}$, $1 \leqslant i \leqslant m$, defines a complete intersection in $\mathbb{C}^n$ if and only if the complex dimension of the analytic set of common zeroes Z of the functions f_i is equal to $n - m$, that is,

$$\dim_{\mathbb{C}} Z = \dim_{\mathbb{C}}(\cap_{1 \leqslant i \leqslant m} Z_{f_i}) = \dim_{\mathbb{C}} \left(\cap_{1 \leqslant i \leqslant m} \{z \in \mathbb{C}^n : f_i(z) = 0\} \right) = n - m.$$

The following variation of Fundamental Principle for homogeneous system of convolution equation is formulated and proved in [3]

Theorem 1.1. *Assume that $\mu_j \in \mathcal{E}'(\mathbb{R}^n)$, for $j = 1, \ldots, m$, are slowly decreasing and form a complete intersection. That is, for $p(z) = |\Im z| + \log(1 + |z|)$, $z \in \mathbb{C}^n$ there exists an analytic , almost parallel family of lines such that $\hat{\mu}_j$ for $j = 1, \ldots, m$, are slowly decreasing with respect to this family in $A_p(\mathbb{C}^n)$. Then, there exists a locally finite family of closed V_j, $j \in J$ and a partition of the index set J into finite subsets J_k together with partial differential operators ∂_l^z in z with analytic coefficients on the regular points of the set $V = \{z \in \mathbb{C}^n : \hat{\mu}_j(z) = 0, j = 1, \ldots, m\}$ satisfying: 1) $\cup V_k \subset V$, 2) each function $x \longrightarrow \partial_l^z(e^{-ixz})$, with $z \in V$, is a solution to $\check{f} \star \mu_j = 0$, $j = 1, \ldots, m$, where $\check{f}(x) = f(-x)$, $x \in \mathbb{R}^n$, 3) to each solution $f \in \mathcal{E}(\mathbb{R}^n)$ of the system $\check{f} \star \mu_j = 0$, $j = 1, \ldots, m$, there corresponds a family of Borel measures ν_j, whose support is contained in the sets V_j and such that the series*

$$f(x) = \sum_k \left(\sum_{j \in J_k} \int_{V_k} \partial_l(e^{-ixz}) d\nu_j(z) \right) \tag{1.3}$$

is convergent in the space $\mathcal{E}(\mathbb{R}^n)$.

2. Integral representation formula depending on parameter

For $R_0 > 0$, we define the sequence closed balls with doubling radius property

$$K_l = \overline{B(0, 2^l R_0)} = \overline{\{x \in \mathbb{R}^n : \|x\| < 2^l R_0\}},\ l \in \mathbb{N}$$

Then $\{K_l\}_{l\in\mathbb{N}}$ is an increasing sequence of compact convex set s satisfying $K_l \subset intK_{l+1}$ and $\cup K_l = \mathbb{R}^n$. Let also $\{\chi_l\}_{l\in\mathbb{N}}$ be a sequence of elements from $\mathcal{D}(\mathbb{R}^n)$ such that $supp\,\chi_l \subset K_{l+1}$, $\chi_l \equiv 1$ in some neighborhood of K_l, $l \in \mathbb{N}$. The set $\mathcal{U}$ of all continuous functions on $\mathbb{C}^n$ of the form

$$\tau(z) = \sup_{l\in\mathbb{N}} \big(\delta_l \exp(l\ln(2+\|z\|^2) + H_{K_l}(\Im z))\big),$$

where $\{\delta_l\}_{l\in\mathbb{N}}$ is a sequence of positive constants, is a LAU structure for the set $\mathcal{E}'(\mathbb{R}^n)$ ([9,10]).

We now turn to the localization of the solution f to the system $\check{f} * \mu_j = 0$, $j = 1,\ldots,m$. Our purpose is to describe f explicitly in $intK_l$, $l \in \mathbb{N}$. In order to do that we first test f against $u \in \mathcal{D}(\mathbb{C}^n)$, with $supp\,u \subset intK_l$ for some $l \in \mathbb{N}$. Using the definition of the characteristic function χ_l we get from Plancherel theorem that

$$\int_{\mathbb{R}^n} f(t)u(t)dt = \int_{\mathbb{R}^n} (\widehat{f\chi_l})(-\xi)\hat{u}(\xi)d\xi. \tag{2.1}$$

The starting point is a weighted Koppelman integral representation formula for the holomorphic function $\hat{u}\theta(\frac{(\cdot)}{R})$, where $\theta \in \mathcal{D}(\mathbb{C}^n)$ so that $\theta \equiv 1$ on an open neighborhood of $\overline{B}(0,1)$ and $supp\,\theta \subset B(0,2)$. This weighted integral representation formula is constructed following the approach developed in [2,7]. Using it, we will produce a division formula involving the functions $\hat{\mu}_j$, $j = 1,2,\ldots,m$, [6]. In $\mathbb{C}^n$ we have the following Heffer functions

$$\begin{aligned} \hat{\mu}_j(z) - \hat{\mu}_j(\zeta) &= \sum_{k=1}^{k=n} g_{j,k}(z,\zeta)(z_k - \zeta_k),\ j = 1,\ldots,n, \\ g_{j,k}(z,\zeta) &= \int_0^1 \frac{\partial\hat{\mu}_j}{\partial\zeta_k}(\zeta + t(z-\zeta))dt \end{aligned}$$

and the corresponding Heffer $(1,0)$-form

$$g_j(z,\zeta) = \sum_{k=1}^{k=n} g_{j,k}(z,\zeta)d\zeta_k,\ j = 1,2,\ldots,m.$$

Furthermore, following [6], within the spirit of constructions in [2,7], we introduce three pairs (Q_1,G_1), (Q_2,G_2), (Q_3,G_3) of auxiliary functions defined as follows

$$\begin{aligned} Q_1(z,\zeta) &= (Q_{11},Q_{12},\ldots,Q_{1n})(z,\zeta) : \overline{D}\times\overline{D} \longrightarrow \mathbb{C}^n, \\ Q_{1i}(z,\zeta) &= \frac{1}{m}\sum_{j=1}^{j=m} |\hat{\mu}_j(\zeta)|^{2\lambda}\frac{g_{j,i}(z,\zeta)}{\hat{\mu}_j(\zeta)},\ i = 1,2,\ldots,n \\ G_1(t) &= \frac{1}{m!}\prod_{j=0}^{j=m-1}(mt - j), \end{aligned}$$

where $D \subset \mathbb{C}^n$ is a bounded domain with $\mathcal{C}^2$ boundary and λ a complex parameter with sufficiently large positive real part, that is, $\Re\lambda >> 0$. Similarly, we have

$$\begin{aligned} Q_2(z,\zeta) &= (Q_{21},Q_{22},\ldots,Q_{2n})(z,\zeta) : \overline{D}\times\overline{D} \longrightarrow \mathbb{C}^n, \\ Q_{2i}(z,\zeta,l) &= Q_{2i}(\zeta,l) = 2\frac{\partial\omega(\zeta)}{\partial\zeta_i},\ i = 1,2,\ldots,n \\ G_2(t) &= t^{\mathcal{N}}, \end{aligned}$$

where $\omega_l(\zeta) = \dfrac{(2+\|\zeta\|^2)^l}{\sigma_l}$, $l \in \mathbb{N}$, $\mathcal{N} = \max(ord(\mathcal{Q}_{J,s})) + n + 1$, $\mathcal{Q}_{J,s}$-being the differential operators describing the action of the residue currents, σ_l are suitably chosen constants. Finally, the third pair of auxiliary functions is defined by

$$\begin{aligned}
Q_3(z,\zeta) &= (Q_{31},Q_{32},\ldots,Q_{3n})(z,\zeta) : \overline{D}\times\overline{D} \longrightarrow \mathbb{C}^n,\\
Q_{3i}(z,\zeta,l) &= Q_{3i}(\zeta,l) = 2\frac{\partial(H_{K_l}(\Im * \rho)(\zeta))}{\partial\zeta_i}, \quad i=1,2,\ldots,n\\
G_3(t) &= \exp(t-1),
\end{aligned}$$

where ρ is a $\mathcal{C}^\infty$ function supported in the unit ball, while having mass equal to 1 there. The function H_{K_l} is the usual support function for compact convex sets K_l introduced in the previous section. These leads us to define the $(1,0)$-forms

$$q_j(z,\zeta) = \sum_{i=1}^{i=n} Q_{ji}(z,\zeta)d\zeta_i, \quad j=1,2,3.$$

In order to derive the Koppelman integral representation formula with a complex parameter λ we have to consider the $\mathcal{C}^1$ map

$$S:\ \overline{D}\times\overline{D} \ \longrightarrow \mathbb{C}^n,$$

satisfying for every compact K contained in D the estimates a) $\|S(z,\zeta)\| \leqslant C_1^K\|z-\zeta\|$, b) $\langle S(z,\zeta), z-\zeta\rangle \geqslant C_2^K\|z-\zeta\|^2$ and the corresponding $(1,0)$-form

$$s(z,\zeta) = \sum_{i=1}^{i=n} S_i(z,\zeta)d\zeta_i.$$

The positive constants C_1^K, C_2^K above depend on K. In order to simplify the notation, we put

$$\begin{aligned}
\Phi_j(z,\zeta) &= \ < Q_j, z-\zeta >, \quad j=1,2,3,\\
G_j^\alpha(t) &= \frac{d^\alpha G_j}{dt^\alpha}|_{t=\Phi_j(z,\zeta)}, \quad j=1,2,3.
\end{aligned}$$

Direct application of the results from [2, 7] leads to the following

Proposition 2.1. *The function $\hat{u}(\xi)\theta(\frac{\xi}{R})$, holomorphic in a neighborhood the closed complex ball $\overline{B}_{\mathbb{C}}(0,R)$, satisfies the following Koppelman integral representation formula for $\xi\in B_{\mathbb{C}}(0,R)$:*

$$\hat{u}(\xi) = \hat{u}(\xi)\theta\Big(\frac{\xi}{R}\Big) = \frac{1}{(2\pi i)^n}\left(\int \hat{u}(\zeta)\theta\Big(\frac{\zeta}{R}\Big)P_\lambda(\xi,\zeta) + \int \hat{u}(\zeta)\bar{\partial}\theta\Big(\frac{\zeta}{R}\Big)K_\lambda(\xi,\zeta)\right), \tag{2.2}$$

whenever the value of the complex parameter having large enough positive part $\Re\lambda >> 0$ is fixed. The kernel $P_\lambda(\xi,\zeta)$ is the (n,n)-form

$$\begin{aligned}
P_\lambda(\xi,\zeta) &= \sum_{k=0}^{k=m} G_1^{(k)}(\Phi_1(\xi,\zeta))B_k(\xi,\zeta)\wedge(\bar{\partial}q_1)^k, \qquad (2.3)\\
B_k(\xi,\zeta) &= \sum_{\alpha_2+\alpha_3=n-k}\frac{G_2^{(\alpha_2)}G_3^{(\alpha_3)}}{\alpha_2!\alpha_3!}(\bar{\partial}q_2)^{\alpha_2}\wedge(\bar{\partial}q_3)^{\alpha_3}, \quad k=0,1,\ldots,n
\end{aligned}$$

and kernel $K_\lambda(\xi,\zeta)$ is the $(n,n-1)$-form defined by

$$\begin{aligned}
K_\lambda(\xi,\zeta) &= \sum_{k=0}^{k=\min\{m,n-1\}} G_1^{(k)}(\Phi_1(\xi,\zeta))A_k(\xi,\zeta)\wedge(\bar{\partial}q_1)^k, \qquad (2.4)\\
A_k(\xi,\zeta) &= \sum_{\alpha_0+\alpha_2+\alpha_3=n-k-1}\frac{G_2^{(\alpha_2)}G_3^{(\alpha_3)}}{\alpha_2!\alpha_3!}\frac{s\wedge(\bar{\partial}s)^{\alpha_0}\wedge(\bar{\partial}q_2)^{\alpha_2}\wedge(\bar{\partial}q_3)^{\alpha_3}}{< S, z-\zeta >^{2(\alpha_0+1)}},
\end{aligned}$$

whenever $k=0,1,\ldots,\min\{m,n-1\}$. Furthermore, the right side of (2.2) has holomorphic extension into the half-plane $\Re\lambda > -\delta$, $\delta > 0$.

Proof. The proof of the first claim is straightforward. First, we apply the method from [2,7] to construct the representation formula (2.2), with summation up to n, where the kernels $P_\lambda(\xi,\zeta)$ and $K_\lambda(\xi,\zeta)$ are defined by the relations (2.3) and (2.4) correspondingly, with only difference that summation is up to n in the first case and up to $n-1$ in the second one. Then the observation that $G_1(t)$ is a polynomial of degree m leads to the desired conclusion.

Furthermore, we observe that

$$\Phi_1(\xi,\zeta) = 1 + < Q_1(\xi,\zeta), \xi-\zeta >= 1 + \frac{1}{m}\sum_{j=1}^{j=m}\sum_{i=1}^{i=n}\frac{|\hat\mu_j(\zeta)|^{2\lambda}}{\hat\mu_j(\zeta)}g_{j,i}(z,\zeta)(\xi_i-\zeta_i) =$$

$$= 1 + \frac{1}{m}\sum_{j=1}^{j=m}\frac{|\hat\mu_j(\zeta)|^{2\lambda}}{\hat\mu_j(\zeta)}\left(\hat\mu_j(\xi)-\hat\mu_j(\zeta)\right) = \frac{1}{m}\sum_{j=1}^{j=m}\frac{|\hat\mu_j(\zeta)|^{2\lambda}}{\hat\mu_j(\zeta)}\hat\mu_j(\xi) - \frac{1}{m}\sum_{j=1}^{j=m}(m-|\hat\mu_j(\zeta)|^{2\lambda}). \quad (2.5)$$

Also, elementary computations imply that

$$\begin{aligned}\bar\partial_\zeta q_1(\xi,\zeta) &= \frac{\lambda}{m}\sum_{j=1}^{j=m}|\hat\mu_j(\zeta)|^{2(\lambda-1)}\bar\partial_\zeta\bar{\hat\mu}_j(\zeta)\sum_{i=1}^{i=n}g_{i,j}(\xi,\zeta)d\zeta_i = \\ &= \frac{\lambda}{m}\sum_{j=1}^{j=m}|\hat\mu_j(\zeta)|^{2(\lambda-1)}\bar\partial_\zeta\bar{\hat\mu}_j(\zeta)g_j(\xi,\zeta).\end{aligned}$$

Hence, simplifying the notation, we get the (m,m)-form

$$(\bar\partial q_1)^m = (-1)^{\frac{m(m-1)}{2}}\frac{\lambda^m m!}{m}|\hat\mu|^{2(\underline{\lambda}-\underline{1})}\bar\partial\bar{\hat\mu}\wedge g,$$

where $|\hat\mu| = |\hat\mu_1|\dots|\hat\mu_m|$, $g = g_1\wedge\dots\wedge g_m$, $\underline{\lambda} = \underbrace{(\lambda,\dots,\lambda)}_{m\text{-times}}$, and $\underline{1} = \underbrace{(1,\dots,1)}_{m\text{-times}}$. Now, looking at every term of the kernels P_λ, K_λ described in (2.3), (2.4) one observes that for every k the integral of the corresponding terms in (n,n) or $(n,n-1)$ forms can be continued, as function of λ, holomorphically in a neighborhood of $\lambda=0$. To be more specific, the extensions of distribution valued functions

$$\begin{aligned}\lambda &\longrightarrow G_1^{(k)}A_k\wedge(\bar\partial q_1)^k \\ \lambda &\longrightarrow G_1^{(k)}B_k\wedge(\bar\partial q_1)^k\end{aligned}$$

defining, for every value of λ, terms in the kernels P_λ and K_λ are holomorphic in the neighborhood of $\lambda=0$. This follows from Prop. 3.6 in ([5]) when $\lambda_1=\lambda_2=\dots=\lambda_m$. We claim that the value of holomorphic extension of the above functions at $\lambda=0$ is equal to zero, whenever $k<m-1$. That is, the only terms that have a nonzero contribution in the first integral of (2.2) at $\lambda=0$ are the term of the kernel P_λ that corresponds to $k=m$ and $k=m-1$. Similarly, the only terms that have a nonzero contribution in the second integral of (2.2) at $\lambda=0$ are the terms of the kernel K_λ that corresponds to $k=m$ and $k=m-1$. Actually, the terms in question are (n,n) or $(n,n-1)$ forms, whose coefficients contain factors (or powers of such factors) of the form

$$\lambda^k\prod_{l\in I}\left(\frac{|\hat\mu_{j_l}(\zeta)|^{2\lambda}}{\hat\mu_{j_l}(\zeta)}\right),\quad \prod_{i\in I_1}(1-|\hat\mu_{j_i}(\zeta)|^{2\lambda}),\quad \prod_{d\in J}|\hat\mu_{j_d}(\zeta)|^{2(\lambda-1)}\bigwedge_{d\in J}\overline{\partial\hat\mu}_{j_d},$$

where $|I|+|I_1|=n-k$ or $|I|+|I_1|=n-k-1$, $|J|=k$ and the subsets of indices are mutually disjoint. The vanishing of the corresponding integrals at $\lambda=0$ follows from the application of Prop. 1.5 from ([5]) after the application of Hironaka's de-singularization theorem on $f_1f_2\dots f_m=0$. The only other interesting cases that remain to be seen are those that correspond to the cases when $k=m-1$ or $k=m$. This completes the proof of the proposition. $\square$

3. The division formula

Keeping the notations from previous sections we formulate the following proposition

Proposition 3.1. *Assume that $\hat{\mu}_i \in A_p(\mathbb{C}^n)$, $i = 1, \ldots, m$, $p(z) = \|\Im z\| + \log(1 + \|z\|)$ form a complete intersection and are slowly decreasing with respect to $\mathcal{L}$. Then for $\xi \in \mathbb{C}^n$ the following equality holds*

$$\hat{u}(\xi) = \sum_{j=1}^{j=m} \hat{\mu}_j(\xi) U_j(\xi) + < \bar{\partial}\frac{1}{\hat{\mu}_1}(\cdot) \ldots \bar{\partial}\frac{1}{\hat{\mu}_m}(\cdot), \hat{u}(\cdot) g_1(\xi, \cdot) \wedge \cdots \wedge g_m(\xi, \cdot) \wedge B_l(\xi, \cdot) >, \quad (3.1)$$

where $U_j(\xi)$ is a Fourier transform of distributions with compact supports contained in $K_l = \overline{B(0, 2^l R_0)}$ and $B_l(\xi, \cdot)$ is a $(n-m, n-m)$ differential form given by

$$m! \sum_{\beta_1 + \beta_2 = n-m} \frac{\binom{\mathcal{N}}{\beta_2}}{\beta_1! e} \exp\big(< 2\partial\left(H_{K_l}\Im(\xi) * \rho\right)(\zeta), \xi - \zeta > \big) \big(\varrho(\xi, \zeta)\big)^{\mathcal{N} - \beta_2} \phi_{\beta_1, \beta_2}(\xi, \zeta),$$

where $\varrho(\xi, \zeta) =< 2\dfrac{\partial\omega_l(\zeta)}{\omega_l(\zeta)}, \xi - \zeta > +1$ and

$$\phi_{\beta_1, \beta_2}(\xi, \zeta) = \frac{1}{(2\pi i)^{n-m}} \big(\partial\bar{\partial}(H_{K_l}(\Im\xi) * \rho(\zeta))\big)^{\beta_1} \wedge \big(\partial\bar{\partial}\log(\omega_l(\zeta))\big)^{\beta_2}.$$

Proof. Let us begin with the discussion of the terms in the forms P_λ and K_λ of degree $k = m$ and $k = m - 1$. When $k = m$, we have the terms that are residual

$$G_1^{(m)} B_m (\bar{\partial} q_1)^m = const.\lambda^m |\hat{\mu}|^{2(\underline{\lambda} - \underline{1})} \overline{\partial\hat{\mu}} \wedge g, \quad (3.2)$$

$$G_1^{(m)} A_m (\bar{\partial} q_1)^m = const.\lambda^m |\hat{\mu}|^{2(\underline{\lambda} - \underline{1})} \overline{\partial\hat{\mu}} \wedge g\varpi, \quad (3.3)$$

where $\varpi = A_m = \dfrac{1}{< S, z - \zeta >^{2(\alpha_0 + 1)}} = \dfrac{1}{< S, z - \zeta >^2}$. In this case we have forms, whose coefficients, near the set of common zeroes, have growth growth estimates $\left|\dfrac{1}{\hat{\mu}}(z)\overline{\partial\hat{\mu}}\right| \leqslant (1+ +\|z\|)^{-1}\exp(-\|z\|)$, because of the slowly decreasing assumption for the entire functions $\hat{\mu}_j$, $j = 1, \ldots, m$. In the case $k = m - 1$, we have

$$G_1^{(m-1)} B_{m-1} (\bar{\partial} q_1)^{m-1} = (n - m + 1, n - m + 1)\text{-form}, \quad (3.4)$$

$$G_1^{(m-1)} A_{m-1} (\bar{\partial} q_1)^{m-1} = (n - m + 1, n - m + 1)\text{-form}, \quad (3.5)$$

whose support depends on the radius of the ball K_l and whose terms contain, as coefficients, reciprocals of slowly decreasing functions. Thus, since we want to get our division formula to hold over $\mathbb{C}^n$ it is enough to see the convergence of these terms while $R \longrightarrow +\infty$. The forms

$$\frac{G_2^{(\alpha_2)} G_3^{(\alpha_3)}}{\alpha_2! \alpha_3!}, \quad (\bar{\partial} q_2)^{\alpha_2} \wedge (\bar{\partial} q_3)^{\alpha_3}, \quad \frac{s \wedge (\bar{\partial} s)^{\alpha_0}}{< S, z - \zeta >^{2(\alpha_0 + 1)}}$$

are factors in forms $A_m, B_m, A_{m-1}, B_{m-1}$, when $\alpha_2 + \alpha_3 = n - m$ or when $\alpha_0 + \alpha_2 + \alpha_3 = n - m + 1$. Thus, these terms require estimates for the behavior of $G_2^{(\alpha_2)}$, $G_3^{(\alpha_3)}$ and $(\bar{\partial} q_i)^{\alpha_i}$, $i = 1, 2$ and of their products, when $R \longrightarrow +\infty$. Furthermore the same type of estimate is needed for $\bar{\partial}\theta(\frac{\zeta}{R})$. First, we observe that since $u \in \mathcal{D}(\Omega)$, its Fourier transform satisfies for every $k \in \mathbb{N}$ the estimate

$$|\hat{u}(\zeta)| \leqslant C_k \frac{1}{(1 + \|\zeta\|)^k} \exp\big(H_{K_l}(\Im(\zeta))\big), \quad \zeta \in \mathbb{C}^n.$$

On the other hand, the function $\left(H_{K_l}(\Im(\cdot)) * \rho\right)(\zeta)$ is convex, since for a compact convex set K the support function H_K is convex also. Therefore we deduce that

$$\Re < \partial\left(H_{K_l}(\Im(\cdot)) * \rho\right)(\zeta), \xi - \zeta > \leqslant \left(H_K(\Im(\cdot)) * \rho\right)(\xi) - \left(H_K(\Im(\cdot)) * \rho\right)(\zeta),$$

whenever $\xi, \zeta \in \mathbb{C}^n$. Hence

$$|e^{<\partial\left(H_{K_l}(\Im(\cdot))*\rho\right)(\zeta),\xi-\zeta>}| \leqslant e^{\Re<\partial\left(H_{K_l}(\Im(\cdot))*\rho\right)(\zeta),\xi-\zeta>} \leqslant e^{\left(H_{K_l}(\Im(\cdot))*\rho\right)(\xi)-\left(H_{K_l}(\Im(\cdot))*\rho\right)(\zeta)}.$$

Thus

$$|D^{\alpha}_{\zeta,\bar\zeta} e^{<\partial\left(H_{K_l}(\Im(\cdot))*\rho\right)(\zeta),\xi-\zeta>}| \leqslant C_l(z,\alpha)(1+\|\zeta\|)^{|\alpha|} e^{-\left(H_{K_l}(\Im(\cdot))*\rho\right)(\zeta)},$$

where $\alpha \in \mathbb{N}^n$, $\|\zeta - \xi\| \geqslant 1$ and $|\alpha| = \sum \alpha(j)$ = sum of orders of j-directional derivatives. For the positive constant $C_l(\xi,\alpha)$ the following estimate holds

$$|C_l(\xi,\alpha)| \leqslant d(\alpha)\gamma_l^{|\alpha|}\exp(4\gamma_l)(1+\|\xi\|)^{|\alpha|} e^{-\left(H_{K_l}(\Im(\cdot))*\rho\right)(\zeta)},$$

For the estimate of $\bar\partial\theta\left(\frac{\zeta}{R}\right)$, when $\xi \in \mathbb{C}^n$ is fixed and $\zeta \in \mathbb{C}^n$ is such that $\|\xi - \zeta\| \geqslant 1$, one has that

$$|D^{\alpha}_{\zeta,\bar\zeta}\theta\left(\frac{\zeta}{R}\right)| \leqslant \Gamma_l(\xi,k,\alpha)(1+\|\zeta\|)^{-k} e^{-\left(H_{K_l}(\Im(\cdot))*\rho\right)(\zeta)},$$

where $k > 0$ and $\alpha \in \mathbb{N}^n$ and $\Gamma_l(\xi,k,\alpha)$ is a positive constant. By letting $\bar\partial\dfrac{1}{\hat\mu}(\cdot) = \bar\partial\dfrac{1}{\hat\mu_1}(\cdot) \wedge \dots \cdots \wedge \bar\partial\dfrac{1}{\hat\mu_m}(\cdot)$, the above estimates imply the relation (3.1) when $R \longrightarrow \infty$.

4. Locally explicit version of fundamental principle

Keeping the notation from above, we begin the present section by obtaining localized explicit solutions to a system of homogeneous convolution equations. Namely, we have the following proposition

Proposition 4.1. *Let $\check f * \mu_j = 0$, $j = 1,\dots,m$ be a homogeneous system of convolution equations in $\mathcal{E}(\mathbb{R}^n)$, where $f \in \mathcal{E}(\mathbb{R}^n)$ and $\mu_j \in \mathcal{E}'(\mathbb{R}^n)$, so that the entire functions $\hat\mu_1,\dots,\hat\mu_m$ form a complete intersection in $\mathbb{C}^n$. Let also l be any positive integer so that $supp\mu_j \subset K_l = \{x \in \mathbb{R}^n : \|x\| \leqslant 2^l R_0\}$ for every $j = 1,\dots,m$. Then, for every $t \in intK_l = \{x \in \mathbb{R}^n : \|x\| < 2^l R_0\}$ the solution $f(t)$ is represented by*

$$f(t) = \frac{1}{(2\pi)^n} < \bar\partial\frac{1}{\mu}(\zeta), e^{-i<t,\zeta>}\left(\int\limits_{\xi\in\mathbb{R}^n}\widehat{f\chi_l}(-\xi)g(\xi,\zeta)B_{m,l}(\xi,\zeta)d\lambda(\xi)\right)>, \tag{4.1}$$

where the form $B_{m,l}(\xi,\zeta)$ is the restriction of the form $B_m(\xi,\zeta)$ to K_l.

Proof. Since $\chi_{l+1} \equiv 1$ on K_{l+1} and $supp\,\chi_{l_1} \subset intK_{l+1}$ we have that $(f\check\chi_{l+1}) * \mu_j = \check f * \mu_j = 0$ holds on $K_l = \overline{B(0,2^l R_0)} = \overline{B(0,2^{l+1}R_0 - 2^l R_0)}$ for every $j = 1,\dots,m$. Hence

$$\widehat{(f\check\chi_{l+1}) * \mu_j} = \widehat{(f\check\chi_{l+1})}\cdot\widehat{\mu}_j = 0, \quad j = 1,\dots,m.$$

Therefore, letting $g(\xi,\zeta) = g_1(\xi,\zeta) \wedge \cdots \wedge g_m(\xi,\zeta)$, the Plancherel formula becomes

$$(2\pi)^n \int_{\mathbb{R}^n} f(t)u(t)dt = \int_{\mathbb{R}^n} \widehat{(f\chi_l)}(-\xi)\hat{u}(\xi)d\xi =$$

$$= \int_{\mathbb{R}^n} \widehat{(f\chi_l)}(-\xi) \left(\sum_{j=1}^{j=m} \hat{\mu}_j(\xi)U_j(\xi) \right) d\xi + \int_{\mathbb{R}^n} \widehat{(f\chi_l)}(-\xi) < \bar{\partial}\frac{1}{\mu}(\zeta), \hat{u}(\zeta)g(\xi,\zeta)B_{m,l}(\xi,\zeta)d\xi >=$$

$$=< \bar{\partial}\frac{1}{\mu}(\zeta), \int_{\xi\in\mathbb{R}^n} \widehat{f\chi_l}(-\xi)\hat{u}(\zeta)g(\xi,\zeta)B_{m,l}(\xi,\zeta)d\xi > .$$

Now, recall that $\hat{u}(\zeta) = \int\limits_{K_l} u(t)e^{-i<\zeta,t>}dt$ for a test function $u \in \mathcal{D}(intK_l)$. If we assume also that the interior of the set $supp\, u \subset intK_l$ is not empty and that $u \equiv 1$, then Fubini's Theorem implies that

$$(2\pi)^n \int\limits_{K_l} f(t)u(t)dt =< \bar{\partial}\frac{1}{\mu}(\zeta), \hat{u}(\zeta) \int\limits_{\xi\in\mathbb{R}^n} \widehat{f\chi_l}(-\xi)g(\xi,\zeta)B_{m,l}(\xi,\zeta)d\xi >=$$

$$= \int\limits_{K_l} u(t)e^{-i<\zeta,t>} \Big(< \bar{\partial}\frac{1}{\mu}(\zeta), \int\limits_{\xi\in\mathbb{R}^n} \widehat{f\chi_l}(-\xi)\widehat{f\chi_l}(-\xi)g(\xi,\zeta)B_{m,l}(\xi,\zeta)d\xi > \Big)dt.$$

Note that one can use Fubini's Theorem because the action of the residue current $\bar{\partial}\frac{1}{\mu} = \bar{\partial}\frac{1}{\mu_1} \wedge \bar{\partial}\frac{1}{\mu_2} \wedge \cdots \wedge \bar{\partial}\frac{1}{\mu_m}$ is independent of $e^{-i<\zeta,t>}$, because this function does not contribute to the set of common zeroes of the functions $\hat{\mu}_j$, $j = 1, \ldots, m$. We now apply Lebesgue' Differentiation Theorem to deduce that for almost all $t \in int(supp\, u)$ one has that (4.1) holds. This concludes the proof of the proposition. □

We now formulate the main result of the paper.

Theorem 4.1. *Let $\check{f} * \mu_j = 0$, $j = 1, \ldots, m$ be a homogeneous system of convolution equations in $\mathcal{E}(\mathbb{R}^n)$, where $f \in \mathcal{E}(\mathbb{R}^n)$ and $\mu_j \in \mathcal{E}'(\mathbb{R}^n)$, so that the entire functions $\hat{\mu}_1, \ldots, \hat{\mu}_m$ are slowly decreasing with respect to $\mathcal{L}$ and form a complete intersection in $\mathbb{C}^n$. Then, there exists an $(n, n-m)$ differential form*

$$\Psi(\zeta) = \int_{\xi\in\mathbb{R}^n} \widehat{f\chi_l}(-\xi)\hat{g}(\xi,\zeta)B_l(\xi,\zeta)d\xi,$$

whose coefficients are in the space $L^1(\mathbb{C}^n, \tau)$, where τ is some element in LAU structure, so that

$$f(t) = \frac{1}{(2\pi)^n} < \bar{\partial}\frac{1}{\mu}(\zeta), e^{-i<t,\zeta>}\Psi(\zeta) >, \quad t \in \mathbb{R}^n. \tag{4.2}$$

Proof. For the sequence of compact closed balls

$$K_l = \overline{\{x \in \mathbb{R}^n : \|x\| < 2^l R_0\}},$$

where $R_0 > 0$ and $l \in \mathbb{N}$ exhausting $\mathbb{R}^n$, we describe the LAU structure $\mathcal{U}$. Let $\rho_l = e^4\gamma_l(l\gamma_l)^{\mathcal{N}}\sigma_l^2$ be a sequence of positive constants, where the constant γ_l depends on K_l and the constant σ_l depends on the solution f that we want to express explicitly. For example $\gamma_l = \max\{\|t\|,\ t \in K_l\} = 2^l R_0$ and $\mathcal{N}$ is the order of differential operator involved in computation residual term $\bar{\partial}\frac{1}{\mu_1} \wedge \bar{\partial}\frac{1}{\mu_2} \wedge \cdots \wedge \bar{\partial}\frac{1}{\mu_m}$. Let τ be some function in LAU structure that dominates all the functions

$$\sup_{l\in\mathbb{N}} \left(e^{-4\gamma_l}(l\gamma_l)^{-\mathcal{N}}\sigma_l^{-2}(\omega_l)^p e^{H_{K_l}(\Im(\cdot))} \right), \quad p = 1, \ldots, \mathcal{N}+1.$$

From the preceding proposition we know that for every $l \in \mathbb{N}$ there exists a differential $(n, n-m)$ form $\Psi_l(\zeta)$ such that

$$f(t) = \frac{1}{(2\pi)^n} < \bar{\partial}\frac{1}{\mu}(\zeta), e^{-i<t,\zeta>}\Psi_l(\zeta) > .$$

Applying Ascoli's Theorem, we extract a subsequence of $(n, n-m)$ forms from the sequence on $\{\Psi_l\}$ of $(n, n-m)$ forms that converge in the space of differential forms with continuous coefficients in the space $L^1(\mathbb{C}^n, \tau)$. □

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Локально явный фундаментальный принцип для однородных уравнений в свертках

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В настоящей статье локально явная версия основополагающего принципа Эренпрейса для системы однородных уравнений в свертках $\check{f} * \mu_j = 0$, $j = 1, \ldots, m$, $f \in \mathcal{E}(\mathbb{R}^n)$, $\mu_j \in \mathcal{E}'(\mathbb{R}^n)$, получается, когда преобразования Фурье $\hat{\mu}_j$, $j = 1, \ldots, m$ — медленно убывающие входные функции, которые образуют полное пересечение в $\mathbb{C}^n$.

Ключевые слова: фундаментальный принцип, формула деления.