УДК 517.9

Fine-analytic Functions in $\mathbb{C}^n$

Azimbai Sadullaev*
Department of Mathematics
National university of Uzbekistan
University, 4, Tashkent, 100174
Uzbekistan

Received 07.09.2018, received in revised form 09.11.2018, accepted 13.04.2019

In this paper we study class of fine-analytic functions in the multidimensional space $\mathbb{C}^n$. The definition of fine-analytic functions in the multidimensional case differs somewhat from the well-known definition of fine-analytic functions on the plane. We give a relationship between classical notion of fine-analyticity and fine-analyticity in $\mathbb{C}^n$.

Keywords: Gonchar class, finite order functions, rational approximation, fine-analytic functions, pluripolar sets.
DOI: 10.17516/1997-1397-2019-12-4-444-448.

1. Introduction and preliminaries

In [17] (see also [16]), we established the following connection between Gonchar class functions with fine-analytic functions in complex space $\mathbb{C}^n$:

Theorem 1.1 (A. Sadullaev, Z. Ibragimov)**.** *Let $K \subset \mathbb{C}^n$ is a nonpluripolar set and $f \in C(K)$ is a continuous funstion on it. If f belong to the Gonchar class R_K of finite order, i.e., there exist a sequence of rational functions*

$$r_m(z) = \frac{p_m(z)}{q_m(z)}, \ \deg r_m \leqslant m, \ m = 1, 2, \ldots,$$

such that

$$\|r_m - f\|_K^{1/m} \leqslant \frac{1}{m^{1/s}}, \ m = 1, 2, \ldots, \ s < \infty,$$

then f fine-analytically continues to the whole space $\mathbb{C}^n$. That is, there is a fine-analytic function $\tilde{f}$ on $\mathbb{C}^n$, such that $\tilde{f}|_K \equiv f$.

The definition of fine-analytic, more specifically $(W2)$fine-analytic functions (see Definition 3) in the multidimensional case differs somewhat from the well-known definition of fine-analytic functions on the plane. In this paper we give comparisons of these definitions in the one-dimensional case, we give examples and indicate the difficulties in determining fine-analytic functions in $\mathbb{C}^n$ by the standard way.

We first recall the definition of fine-analytic functions. They are determined by means of fine(thin) topology. A fine topology in $\mathbb{C}^n$ is the weakest topology in which all plurisubharmonic (psh) functions are continuous. A fine topology is generated by sets of the form $\{u(z) < \alpha\}, \{u(z) > \alpha\}, u \in psh(\mathbb{C}^n)$. Fine neighborhood of a point a is an open set

*sadullaev@mail.ru

$V \subset \mathbb{C}^n$, $a \in V$, in the fine topology for which the complement $W = \mathbb{C}^n \backslash V$ is thin at the point a, i.e. there exists a ball $B = B(a, r)$ and a plurisubharmonic in B function $u \in psh(B) : \overline{\lim\limits_{z \to a, z \in W}} u(z) < u(a)$. A closed fine neighborhood of a point is a compact of the form $\bar{B}(a, r) \backslash G$, where $r > 0$, $G \subset B(a, r)$ is an open, thin set at the point a. In the general case, in a literature, a fine neighborhood is any set $V \subset \mathbb{C}^n$, $a \in V$, for which $\mathbb{C}^n \backslash V$ is thin at the point a (see, for example, [2]).

Definition 1. *A function $f(z)$ is called fine-analytic in a planar domain $D \in \mathbb{C}$ if*

1) *it is defined almost everywhere with respect to the capacity in D i.e., outside of some polar set $E \subset D$ it admits a finite value;*

2) *for each point $a \in D \backslash E$ there is a closed fine neighborhood F of a such that $f|_F \in R(F)$, or, equivalently, the restriction $f|_F$ is uniformly approximated by rational functions on F.*

The notion of fine-analyticity was introduced and used by B. Fuglede [8–10] for a class of functions that have the Mergelyan property in fine neighborhoods of a point. If a function $f(z)$ is analytic in a neighborhood of a point a, then f has the Mergelyan property in arbitrary compact $K \ni a$, i.e f is uniformly approximated on K by rational functions. Fine analyticity of f in the point a means uniformly approximation by rational functions only on some fine neighborhood $K \ni a$. In work of A. Edigarian and J. Wiegerinck [3, 4], A. Edigarian, S. El Marzguioui and J. Wiegerinck [5], S. El Marzguioui and J. Wiegerinck [14], J. Wiegerinck [18], T. Edlund [6], T. Edlund and B. Jöricke [7] fine-analytic functions were studied for their applications in pluripotential theory, more precisely, in the description of pluripolar hulls, in the establishment of pluripolar hulls of graphs $\Gamma = \{w = f(z)\}$. On pluripolar hulls of analytic sets and graphs $\Gamma = \{w = f(z)\}$ see also the papers of N. Levenberg, G. Martin and E. Poletsky [12], N. Levenberg and E. Poletsky [13].

In [15] the author constructed a function $f \in O(U) \bigcap C^\infty(\bar{U})$, where U is disk, such that the pluripolar hull $\widehat{\Gamma}_f = \Gamma_f$. It is clear that if $f(z)$ holomorphically extended to some point $z^0 \in \partial U$, then the point $(z^0, f(z^0)) \in \widehat{\Gamma}_f$. It is natural to expect the opposite, that if $(z^0, f(z^0)) \in \widehat{\Gamma}_f$, then $f(z)$ will be holomorphic at the point z^0. But this assumption was refuted by A.Edigarian and J. Wiegerinck [4], who constructed a function f, that is holomorphic only inside the unit disk, for which $\widehat{\Gamma}_f \neq \Gamma_f$. J. Siciak, studying this example of Edigarian and Wiegerinck, established that the function f is analytically "pseudo-continued" through boundary points ∂U and the pluripolar hull $\widehat{\Gamma}_f$ always contains a graph of the pseudo-continued function; however, the condition of pseudo-continuity is not necessary for $\widehat{\Gamma}_f \neq \Gamma_f$. In [7] T. Edlund and B. Jöricke showed that a fine-analytic continuation is well suited for describing the pluripolar hull of the graphs.

Unfortunately, the Definition 1 has one drawback, that the elementary functions, such as

$$f(z) = \begin{cases} \exp \dfrac{1}{z}, & \text{for } z \neq 0 \\ 0, & \text{for } z = 0 \end{cases} \tag{1}$$

is not fine-analytical, although outside the point $z = 0$ it is represented as

$$f(z) = 1 + \frac{1}{z} + \frac{1}{2!z} + \ldots .$$

and belong to the Gonchar class R. In connection with this example, we will extend the class of finely analytic functions by slightly weakening the condition.

Definition 2. *A function $f(z)$ is called (W1)fine-analytic in a domain $D \in \mathbb{C}$ if*

1) *it is defined almost everywhere with respect to the capacity in D i.e., outside of some polar set $E \subset D$ it admits a finite value;*
2) *for each point $a \in D \backslash E$ there is a closed fine neighborhood F of a ,such that there exists a sequence of rational functions $r_m(z)$, $m = 1, 2, 3 \ldots$, with poles outside F : $r_m(z^0) \to f(z^0)$ as $m \to \infty$ $\forall z^0 \in F$. Moreover, $r_m(z)$ converges uniformly to $f(z)$ inside of the set $F \backslash \{a\}$, in the sense that for any compact $F' \subset K \backslash \{a\}$, $\|r_m - f\|_{F'} \to 0$ as $m \to \infty$.*

As mentioned in the work of Bedford–Taylor [1], in contrast to the planar case, in the multidimensional space $\mathbb{C}^n$, $n > 1$, the notion of thin set (i.e., a set, that is thin in any of its point) does not coincide with the concept of a pluripolar set. For these and for the complexity of the structures of $R(F)$ in the multidimensional space $\mathbb{C}^n$ we should point out that in $\mathbb{C}^n$, $n > 1$, we could not give definition of fine-analytic functions as in the Definition 1. For example, even the rational function

$$r(z_1, z_2) = \begin{cases} z_1/z_2 & \text{if } (z_1, z_2) \neq (0,0), \\ 0 & \text{if } (z_1, z_2) = (0,0) \end{cases} \tag{2}$$

is not fine-analytic in the sense of Definition 1, since the pluripolar set $\{z_2 = 0\}$ is not thin at the point $(0,0) \in \mathbb{C}^2$. It is convenient for us to define fine-analytic functions in $\mathbb{C}^n$ as follows.

Definition 3. *A function $f(z)$ is called (W2)fine-analytic in a domain $D \in \mathbb{C}^n$ if there is an increasing sequence of close sets $F_j \subset F_{j+1} \subset D$, $j = 1, 2, \ldots$, such that:*

1) *the condenser capacity $C(B \backslash F_j,\ B) \to 0$ as $j \to \infty$ for each ball $B \subset\subset D$. It follows, that the set $D \backslash \bigcup\limits_j F_j$ is pluripolar, but the convers is not true;*
2) *f admits a finite value everywhere in $\bigcup\limits_j F_j$;*
3) *For each ball $B \subset\subset D$ and for each number j, the restriction $f|_{\bar{B} \bigcap F_j}$ can be uniformly approximated by rational functions on $\bar{B} \bigcap F_j$, i.e. $f|_{\bar{B} \bigcap F_j} \in R(F)$, $j = 1, 2, \ldots$.*

We note that the function

$$f(z) = \begin{cases} \exp \dfrac{1}{z} & \text{for } z \neq 0, \\ 0 & \text{for } z = 0 \end{cases}$$

given in (1), which is not fine-analytic in the sense of Definition 1, is $(W1)$ and $(W2)$ fine-analytic on the plane $\mathbb{C}$: at the point $a = 0$ we can put $F = \{|z| \leqslant 1\}$, and the sequences of rational functions we construct in the following way. We set

$$F_m = \{|z| \leqslant 1\} \setminus \left\{ 0 < |z| < \frac{1}{m},\ -\frac{\pi}{2} - \frac{1}{m} < \arg z < \frac{\pi}{2} + \frac{1}{m} \right\}.$$

Then $f \in R(F_m)$ and there exists a rational function $r_m(z) : \|r_m - f\|_{F_m} < \dfrac{1}{m}$. It is easy to see that the sequence $\{r_m(z),\ m = 1, 2, \ldots\}$ satisfies the conditions of Definitions 2 and 3.

2. Relationship between definitions 1-3 in $\mathbb{C}$

Let a function $f(z)$ is fine-analytic in a domain $D \subset \mathbb{C}$ in the sense of Definition 1, i.e. it admits a finite value outside of some polar set $E \subset D$ and for each point $a \in D \setminus E$ there is a closed thin neighborhood $F \ni a : f|_F \in R(F)$. Then $D \setminus E$ is finely open set, in which the function $f(z)$ is fine-analytic in the sense of B. Fuglede [8] (see, also [5, 10]). It is clear that from Defition 1 follows Definition 2, Def 1 $\Rightarrow$ Def 2. As an example of the function (1) shows, the converse is not true. We prove the following theorem.

Theorem 2.1. *From Defition 3 follows Definition 2, Def 3 ⇒Def 2.*

Proof. Let $f(z)$ be (W2)fine-analytic in D, i.e. there is an increasing sequence of closed sets $F_j \subset F_{j+1} \subset D$, $j = 1, 2, \ldots$, such that, the condenser capacity $C(B\backslash F_j, B) \to 0$ as $j \to \infty$ for each ball $B \subset\subset D$ and the restriction $f|_{\bar{B} \bigcap F_j} \in R(F)$, $j = 1, 2, \ldots$.

We take $a \in \bigcup_j F_j$, assuming without loss of generality, $a \in F_1$. By assumption of Definition 3 there exists a sequence of rational functions $r_j(z) : \|r_j - f\|_{F_j} < \frac{1}{j}$. We can assume that $f(a) = = r_j(a) = 0$. Then for each $j \in \mathbb{N}$ there exists a ball $B_j = B(a, \varepsilon_j) \subset\subset D : \ \|r_j\|_{B_j} \leqslant \frac{1}{j}, \ \varepsilon_j > 0$. We can assume, that $\varepsilon_1 > \varepsilon_2 > \ldots$, and $\varepsilon_j \to 0$ as $j \to \infty$.

Now we use the following Wiener criterion (see [2, 11]): an open set $E \subset \mathbb{C}$ is thin in a point $z^0 \in \partial E$ if and only if

$$\int_0^1 \frac{C(\rho)}{\rho} d\rho < \infty,$$

where $C(\rho)$ is the capasity of $E \cap B(z^0, \rho)$.

Fix a ball $B = B(a, \varepsilon) \subset\subset D$, $\varepsilon > \varepsilon_1$. Since $C(B\backslash F_m) \to 0$ as $m \to \infty$, then there exists an increasing sequence of natural numbers $m_j : \ C\left(B\backslash F_{m_j}\right) < \varepsilon_{j+1}$. We put $K_j = \bar{B}_j \bigcap F_{m_j}$ and $K = \bigcup_{j=1}^{\infty} K_j$. Then K is compact and $a \in K$. For any $\varepsilon_{j+1} \leqslant \rho < \varepsilon_j$ we have $C(\rho) = C\left(B(z^0, \rho)\backslash K\right) \leqslant C\left(B\backslash F_{m_j}\right) < \varepsilon_{j+1} \leqslant \rho$. It follows, that

$$\int_0^1 \frac{C(\rho)}{\rho} d\rho \leqslant 1$$

and by Wiener criterion $B\backslash K$ is thin at the point $a \in K$.

From $r_j(a) = f(a) = 0$ it follows that $r_j(a) \to f(a)$. In addition, if a compact $F' \subset K\backslash \{a\}$, then $F' \subset F_{j^0}$ for some j^0. Therefore, $\|r_j - f\|_{F'} \to 0$ as $j \to \infty$. Theorem is proved. □

Problem. We do not know if Defition 3 will follows from Defition 1 or 2.

This work was received during my visit to ICTP (International Centre for Theoretical Physics), 17.08-17.09.18. I would like to thank the head of the ICTP for invitation and for creating an excellent condition of stay.

References

[1] E.Bedford, B.A.Taylor, Fine Topology, Šilov Boundary and $(dd^c)^n$, *Journal of Func. Analysis*, **72**(1987), no. 2, 225–251.

[2] M.Brelot, On Topologies and Boundaries in Potential Theory, Lecture Notes in Math., Vol. 175, Springer-Verlage, Berlin, 1971.

[3] A.Edigarian, J.Wiegerinck, The pluripolar hull of the graph of a holomorphic function with polar singularities, *Indiana Univ. Math. J.*, **52**(2003), 1663–1680.

[4] A.Edigarian, J.Wiegerinck, Graphs that are not complete pluripolar, *Proc. Amer. Math. Soc.*, **131**(2003), 2459–2465.

[5] A.Edigarian, S. El Marzguioui, J.Wiegerinck, The image of a finely holomorphic map is pluripolar, *Ann. Polon. Math.*, **97**(2010), 137–149.

[6] T.Edlund, Complete pluripolar curves and graphs, *Ann. Polon. Math.*, **84**(2004), 75–86.

[7] T.Edlund, B. Joricke, The pluripolar hull of a graph and fine analytic continuation, *Ark. Mat.*, **44**(2006), 39–60.

[8] B.Fuglede, Finely harmonic mappings and finely holomorphic functions, *Ann. Acad. Sci. Fennicæ*, **2**(1976), 113–127.

[9] B.Fuglede, Sur les functions finement holomorphes, *Ann. Inst. Fourier (Grenoble)*, **31**(1981), no. 4, 57–88.

[10] B.Fuglede, Finely holomorphic functions. A survey, *Rev. Roumaine Math. Pures Appl.*, **33**(1988), 283–295.

[11] N.S.Landkof, Foundations of Modern Potential theory, Springer–Verlage, 1972.

[12] N.Levenberg, G.Martin, E.A.Poletsky, Analytic disks and pluripolar sets, *Indiana Univ. Math. J.*, **41**(1992), 515–532.

[13] N.Levenberg, E.A.Poletsky, Pluripolar hulls, *Michigan Math. J.*, **46**(1999), 151–162.

[14] S. El Marzguioui, J.Wiegerinck, Continuity properties of finely plurisubharmonic functions, *Indiana Univ Math J.*, **59**(2010), no. 5, 1793–1800.

[15] A.S.Sadullaev, Plurisubharmonic measures and capacities on complex manifolds, *Russian Math. Surveys*, **36**(1981), no. 4, 61–119.

[16] A.S.Sadullaev, Z.S.Ibragimov, The class R and finely analytic functions, *Sbornik: Mathematics*, **209**(2018), no. 8, 1234–1247.

[17] A.S.Sadullaev, Z.S.Ibragimov, Class R of Gonchar in $\mathbb{C}^n$, Springer Proceedings in Mathematics: Statistics, Algebra, Complex Analysis and Pluripotential Theory, 2 USUZCAMP, Urgench, Uzbekistan, August 8–12, 2017, 191–205.

[18] J.Wiegerinck, Pluri fine potential theory, *Ann. Polon. Math.*, **106**(2012), 275–292.

Тонко-аналитические функции в $\mathbb{C}^n$

Азимбай Садуллаев
Department of Mathematics
Национальный университет Узбекистана
Университетская, 4, Ташкент, 100174
Узбекистан

В работе исследуются тонко-аналитические функции в многомерном комплексном пространстве $\mathbb{C}^n$. *Определение тонко-аналитических функций в многомерном случае* $n > 1$ *несколько отличается от плоского случая* $n = 1$. *Мы сравниваем эти определения, в примерах показываем существенные их различия и необходимость использования именно предлагаемого в работе определения при* $n > 1$.

Ключевые слова: класс Гончара, функции конечного порядка, рациональная аппроксимация, тонко-аналитические функции, плюриполярные множества.