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On Some Approach for Finding the Resultant of Two Entire Functions

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One approach for finding the resultant of two entire functions is discussed in the article. It is based on Newton's recurrent formulas.

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Let us consider classic resultant $R(f,g)$ for given polynomials f and g . It can be defined in various ways:

a) using the Sylvester determinant (see, for example, [1–3]);

b) using the formula for the product $R(f,g) = \prod\limits_{\{x: f(x)=0\}} g(x)$ (see, for example, [1–3]);

c) using the Bezout-Cayley method (see, for example, [4]).

See also monograph [5].

In our approach, we take the formula of the product as the main definition.

Let us consider the Sylvester determinant for

$$\begin{cases} f(z) = a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n, \\ g(z) = b_0 + b_1 z + b_2 z^2 + \ldots + b_m z^m. \end{cases} \tag{1}$$

Let us define

$$D_{n,m} = \begin{vmatrix} a_0 & a_1 & a_2 & \ldots & 0 \\ 0 & a_0 & a_1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & a_0 & \ldots & a_n \\ b_0 & b_1 & b_2 & \ldots & 0 \\ 0 & b_0 & b_1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & b_0 & \ldots & b_m \end{vmatrix}. \tag{2}$$

If $a_n \neq 0$ then

$$D_{n,m} = a_n^m \prod_{\{x: f(x)=0\}} g(x),$$

that is, it coincides up to a constant the multiplier with the resultant.

Let us consider the following sums of powers of the values of g at the roots of f

$$S_k = \sum_{\{x: f(x)=0\}} g^k(x), \quad k = 1, 2, \ldots .$$

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It is known that (see [6])

$$R_{f,g} = \frac{1}{n!} \begin{vmatrix} S_1 & 1 & 0 & \dots & 0 \\ S_2 & S_1 & 2 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ S_n & S_{n-1} & S_{n-2} & \dots & S_1 \end{vmatrix}. \qquad (3)$$

A natural generalization of polynomials are entire functions. A number of results are known that extend the definition of resultant in the case of entire functions with a finite number of zeros and with an infinite number of zeros (see [6, 7]). One of the results is the following [6]. Let $g(z)$ be an entire function of the form

$$g(z) = b_0 + b_1 z + b_2 z^2 + \dots + b_m z^m + \dots.$$

then

$$R(f, g) = \lim_{m\to\infty} a_n^{-m} D_{n,m}. \qquad (4)$$

Therefore, to find the resultant $R(f, g)$ it is necessary to calculate determinants of order $m+n$, and then to find their limit. It is certainly quite difficult. Here we propose another approach for finding the resultant.

Let us denote the roots of the polynomial $f(z)$ by $z_1, z_2, \dots z_n$. Multiple roots are taken into account. Then

$$\prod_{i=1}^{n} g(z_i) = g(z_1)\cdot g(z_2)\cdots g(z_n) = \left(\sum_{j=0}^{m} b_j z_1\right)\cdots\left(\sum_{j=0}^{m} b_j z_n\right).$$

This expression is the sum of some symmetric polynomials in variables $z_1, z_2, \dots z_n$. These symmetric polynomials are polynomials of elementary symmetric polynomials of polynomial $f(z)$. Thus, it is possible to find the resultant of polynomials $f(z)$ and $g(z)$ without finding the roots themselves.

Let us consider an example of application of this method to the system

$$\begin{cases} f(z) = a_0 + a_1 z + a_2 z^2, \\ g(z) = b_0 + b_1 z + b_2 z^2 + \dots + b_m z^m. \end{cases} \qquad (5)$$

The roots of polynomial $f(z)$ from (5) are z_1 and z_2. Then

$$\prod_{i=1}^{2} g(z_i) = g(z_1)\cdot g(z_2) = \qquad (6)$$

$$= b_0^2 + b_1^2 z_1 z_2 + b_2^2 z_1^2 z_2^2 + \dots + b_n^2 z_1^n z_2^n +$$
$$+ b_0 b_1 (z_1 + z_2) + b_0 b_2 (z_1^2 + z_2^2) + \dots + b_0 b_n (z_1^n + z_2^n) +$$
$$+ b_1 b_2 z_1 z_2 (z_1 + z_2) + b_1 b_3 z_1 z_2 (z_1^2 + z_2^2) + \dots + b_1 b_n z_1 z_2 (z_1^n + z_2^n) +$$
$$+ b_2 b_3 z_1^2 z_2^2 (z_1 + z_2) + b_2 b_4 z_1^2 z_2^2 (z_1^2 + z_2^2) + \dots + b_2 b_n z_1^2 z_2^2 (z_1^n + z_2^n) +$$
$$\dots\dots\dots$$
$$+ b_{n-1} b_n z_1^{n-1} z_2^{n-1} (z_1^n + z_2^n).$$

Hence we obtain

$$\prod_{i=1}^{2} g(z_i) = g(z_1)\cdot g(z_2) = \sum_{k=0}^{n} b_k^2 z_1^k z_2^k + \sum_{t=0}^{n}\sum_{s=t+1}^{n} b_t b_s (z_1^t z_2^s + z_1^s z_2^t) = \tag{7}$$

$$= \sum_{k=0}^{n} b_k^2 z_1^k z_2^k + \sum_{t=0}^{n}\sum_{s=t+1}^{n} b_t b_s z_1^t z_2^t (z_1^{s-t} + z_2^{s-t}).$$

To simplify the resulting expression we use Vieta's formulas

$$\begin{cases} e_1 = z_1 + z_2 = -a_1, \\ e_2 = z_1 \cdot z_2 = a_0, \end{cases} \tag{8}$$

where e_1, e_2 are elementary symmetric polynomials of polynomial $f(z)$.

Now one needs to calculate the sums in brackets in formula (6). To do this we introduce the following notation

$$\widetilde{S}_k = \sum_{i=1}^{2} z_i^k,$$

i.e.,

$$\widetilde{S}_1 = z_1 + z_2,$$
$$\widetilde{S}_2 = z_1^2 + z_2^2, \ \dots$$
$$\widetilde{S}_k = z_1^k + z_2^k.$$

Expressions $\widetilde{S}_k$ are sums of powers of the roots of polynomial $f(z)$.

Let us consider the famous Newton-Girard formula

$$\widetilde{S}_k = \sum_{r_1+2r_2+\ldots+kr_k=k, r_1,r_2,\ldots,r_k\geqslant 0} (-1)^k \frac{k(r_1+\ldots+r_k-1)!}{r_1!\cdot\ldots\cdot r_k!} \prod_{i=1}^{k} (-e_i)^{r_i}.$$

In our case there are only two elementary symmetric polynomials. Therefore

$$\widetilde{S}_{2j} = \sum_{t=0}^{j} (-1)^{j+t} \frac{2j\cdot(j+t-1)!}{(2t)!\cdot(j-t)!} (e_1)^{2t}\cdot(e_2)^{j-t},$$

if j is even, and

$$\widetilde{S}_{2j+1} = \sum_{t=0}^{j} (-1)^{j+t} \frac{(2j+1)\cdot(j+t)!}{(2t+1)!\cdot(j-t)!} (e_1)^{2t+1}\cdot(e_2)^{j-t},$$

if j is odd

For example,

$$\widetilde{S}_2 = e_1^2 - 2e_2 = a_1^2 - 2a_0,$$
$$\widetilde{S}_3 = e_1^3 - 3e_1e_2 = -a_1^3 + 3a_0a_1.$$

As a result, we obtain

$$\prod_{i=1}^{2} g(z_i) = \sum_{k=0}^{n} b_k^2 z_1^k z_2^k + \sum_{t=0}^{n}\sum_{s=t+1}^{n} b_t b_s z_1^t z_2^t (z_1^{s-t} + z_2^{s-t}) = \sum_{k=0}^{n} b_k^2 a_0^k + \sum_{t=0}^{n}\sum_{s=t+1}^{n} b_t b_s a_0^t \widetilde{S}_s. \tag{9}$$

Now, if $g(z)$ is an entire function then formula (9) takes the form

$$\prod_{i=1}^{2} g(z_i) = \sum_{k=0}^{\infty} b_k^2 a_0^k + \sum_{t=0}^{\infty} \sum_{s=t+1}^{\infty} b_t b_s a_0^t \widetilde{S}_s. \tag{10}$$

Example 1. Let us consider the system of equations

$$\begin{cases} f(z) = z^2 - a^2, \\ g(z) = b_0 + b_1 z + b_2 z^2 + \ldots + b_n z^n. \end{cases} \tag{11}$$

The roots of the first equation of the system are $\pm a$. Using formula (7), we have

$$\prod_{i=1}^{2} g(z_i) = \sum_{k=0}^{n} b_k^2 z_1^k z_2^k + \sum_{t=0}^{n} \sum_{s=t+1}^{n} b_t b_s z_1^t z_2^t (z_1^{s-t} + z_2^{s-t}).$$

Then

$$\begin{cases} e_2 = z_1 \cdot z_2 = -a^2, \\ e_1 = z_1 + z_2 = 0, \end{cases} \tag{12}$$

and

$$\widetilde{S}_{s-t} = z_1^{s-t} + z_2^{s-t} = \begin{cases} 2a^{s-t},\ s-t \text{ even}, \\ 0,\ s-t \text{ odd}. \end{cases}$$

Thus

$$\prod_{i=1}^{2} g(z_i) = \sum_{k=0}^{n} (-1)^k b_k^2 a^{2k} + \sum_{t=0}^{n} \sum_{s=t+1}^{n} (-1)^t b_t b_s a^{2t} \widetilde{S}_{s-t} =$$

$$= \sum_{k=0}^{n} (-1)^k b_k^2 a^{2k} + 2 \sum_{t=0}^{n} \sum_{s=t+2}^{n} (-1)^t b_t b_s a^{s-t},$$

provided that $s - t$ is an even number. Let us introduce the following designation $s - t = 2j$. Then

$$\prod_{i=1}^{2} g(z_i) = \sum_{k=0}^{n} (-1)^k b_k^2 a^{2k} + 2 \sum_{t=0}^{n} \sum_{j=1}^{[(n-t)/2]} (-1)^t b_t b_{t+2j} a^{2t+2j}.$$

Example 2. Let us consider the system of equations

$$\begin{cases} f(z) = z^2 - a^2, \\ g(z) = e^{bz} = \displaystyle\sum_{n=1}^{\infty} \frac{(bz)^n}{n!} = 1 + bz + \frac{(bz)^2}{2!} + \ldots + \frac{(bz)^n}{n!} + \ldots. \end{cases} \tag{13}$$

Using formula (4), we obtain

$$\prod_{i=1}^{2} g(z_i) = \sum_{k=0}^{\infty} (-1)^k \frac{(ab)^{2k}}{(k!)^2} + 2 \sum_{t=0}^{\infty} \sum_{j=1}^{\infty} \frac{(-1)^t (ab)^{2t+2j}}{t!(t+2j)!} = 1.$$

The first sum is the Bessel function of the first kind $J_0(2ab)$ (see, for example, [8]).

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О некотором подходе к нахождению результанта двух целых функций

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В статье обсуждается один подход к нахождению результанта двух целых функций, основанный на рекуррентных формулах Ньютона.

Ключевые слова: результирующие, целые функции, формулы Ньютона.