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Upper Half-plane in the Grassmanian $Gr(n; 2n)$

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We investigate the complex geometry of a multidimensional generalization $\mathcal{D}(n)$ of the upper-half-plane, which is homogeneous relative the group $G = SL(2n;\mathbb{R})$. For $n > 1$ it is the pseudo Hermitian symmetric space which is the open orbit of $G = SL(2n;\mathbb{R})$ on the Grassmanian $Gr_{\mathbb{C}}(n; 2n)$ of n-dimensional subspaces of $\mathbb{C}^{2n}$. The basic element of the construction is a canonical covering of $\mathcal{D}(n)$ by maximal Stein submanifolds — horospherical tubes.

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Memory of Lev Aisenberg

I met Leva Aisenberg almost exactly 60 years ago when we both were graduate students. We had a joint interest in integral formulas in many complex variables. Multidimensional complex analysis was the subject of Leva's long mathematical life. I want to especially recognize his results on complex linear convexity. Leva developed this direction in the long period of his work in Krasnoyarsk, Siberia, where he had many remarkable students, before he moved to Israel.

Multidimensional generalizations of the upper half-plane with the transitive actions of the group $SL(2;\mathbb{R})$, starting from the "half-plane" of Siegel, are playing an important role in the multidimensional complex analysis and its applications to the Fourier analysis, Lie groups, its representations and automorphic functions. Following to Bochner, the tube domains have the structure $T = \mathbb{R}^m + iV$ where V is a convex cone without lines. They are biholomorphically equivalent to bounded domains in $\mathbb{C}^m$. If the cone V is homogeneous, then the complex domain T is also homogeneous. Most of Hermitian symmetric spaces (symmetric domains in $\mathbb{C}^m$) have such realizations. However, between these domains there are no domains with the transitive action of the group $SL(m;\mathbb{R})$ for $m > 2$. We will construct in this paper some generalization of the upper half-plane with the transitive action of the group $SL(2n;\mathbb{R})$. For $n > 1$ this domain will be neither Hermitian nor equivalent to a bounded domain.

1. Geometry of the complex Grassmanian $Gr_{\mathbb{C}}(n; 2n)$

We will realize the symmetric space $\mathcal{D}$ as a domain in the complex Grassmanian $Gr = Gr_{\mathbb{C}}(n; 2n)$ of n-dimensional subspaces in $\mathbb{C}^{2n}$. We will use the Stiefel coordinates

$$Z = \begin{pmatrix} z_0 \\ z_1 \end{pmatrix}$$

of complex $(2n \times n)$- matrices of the rank n with the square blocks z_i, $i = 0, 1$, as homogeneous coordinates in the Grassmanian (for $n = 1$ they are homogeneous coordinates on the projective

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line); thus we can interpret Gr as a matrix projective line). Columns of Z is the basis in the corresponding n-subspace. We have the equivalence relation

$$Z \sim Zu, u \in GL(n;\mathbb{C}).$$

So we identify the equivalence classes with the points of Gr. Let us write elements of the group $G_{\mathbb{C}} = GL(2n;\mathbb{C})$ as block matrices

$$g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

with the square matrix block elements of order n. This group acts on Gr (on the equivalence classes!) by the multiplications:

$$Z \mapsto gZ.$$

The dimension of Gr is equal to n^2.

The subgroup $P = P(0) \subset G_{\mathbb{C}}$ of elements g with $C = 0$, is the isotropy subgroup of the point with the (Stiefel) coordinates $Z^0 = \begin{pmatrix} E \\ 0 \end{pmatrix}$ where $E, 0$ are correspondingly the unit and zero matrices. So as a homogeneous space

$$Gr_{\mathbb{C}}(n;2n) = GL(2n;\mathbb{C})/P.$$

The unimodular subgroup $SL(2n;\mathbb{R})$ and the maximal compact subgroup $U(2n)$ (and $SU(2n)$) also act transitively on Gr and induce the structure of the compact Hermitian symmetric space. The Riemannian structure on the Grassmanian is connected with its representation as

$$Gr = SU(2n)/S(U(n) \times U(n)).$$

The points Z with z_0 of the rank n are equivalent to points of the form $\begin{pmatrix} E \\ z \end{pmatrix}$, $z = z_1(z_0^{-1})$. So we have a dense coordinate chart Gr_0 with inhomogeneous coordinates $z \in M(n)$. The subgroup P^+ of elements g with $B = 0$ preserves this chart Gr_0 and its unipotent subgroup N^+ with $A = D = E$ is transitive on it and acts by the addition of matrices ($z \mapsto z + C$).

By the action of $G_{\mathbb{C}}$, we can construct other charts. It is enough to consider a finite number of them to cover Gr. If $I = (0 < i_1 < i_2 < \cdots < i_n \leqslant 2n)$ is a sequence of integers, then points Z, for which minors of rows with the numbers from I are not zero determinants, give the coordinate chart Gr_I (it corresponds in $G_{\mathbb{C}}$ to permutations of the rows of Z which makes the rows with the numbers I the first ones.). Gr_0 corresponds to $I = (1, \dots n)$.

2. The symmetric domain $\mathcal{D}$ and its geometry

The domain $\mathcal{D}$, which is the focus of this paper, can be interpreted as an analogue of the "upper half-plane" for the group $SL(2n;\mathbb{R})$. It will be realized as one of two open orbits on the complex Grassmanian $Gr_{\mathbb{C}}(n;2n)$ of the real group $G_{\mathbb{R}} = SL(2n;\mathbb{R}) \subset SL(2n;\mathbb{C})$. For elements of $G_{\mathbb{R}}$ we will use the real $2n$-square matrices with the square blocks of order n, similarly to the complex case and with the same notations. The minimal orbit of the action of the real group $G_{\mathbb{R}}$ on the complex Grassmanian is the real Grassmanian $\Omega = Gr_{\mathbb{R}}(n;2n)$ of n-dimensional subspaces in $\mathbb{R}^{2n}$. It has the real dimension n^2 (equal to the complex dimension of the initial complex Grassmanian).

We will use on the real Grassmanian, the real (homogeneous) Stiefel coordinates $X = \begin{pmatrix} x_0 \\ x_1 \end{pmatrix}$ — the real $(2n \times n)$-matrices of the rank n with the square blocks x_i. Then $\Omega = G_{\mathbb{R}}/P(0)_{\mathbb{R}}$ where the parabolic subgroup $P(0)_{\mathbb{R}}$, the real form of the complex parabolic subgroup $P(0)$ is the isotropy subgroup of $X^0 = \begin{pmatrix} E \\ 0 \end{pmatrix} \in \Omega$. Let us remark that the orthogonal subgroup $SO(2n)$ is transitive on Ω and induces the structure of the (real) compact Riemannian symmetric space $SO(2n)/S(O(n) \times O(n))$.

The intersection with the complex coordinate chart Gr_0 gives the real coordinate chart $\Omega_0 \simeq \mathbb{R}^n$ with the inhomogeneous coordinates $x \in M(n;\mathbb{R}) \sim \begin{pmatrix} E \\ x \end{pmatrix}$. On Ω_0 the group $P(0)_{\mathbb{R}}$ acts transitively; the same is true for its Abelian subgroup $N(0)(A = D = E)$. The point X_0 lies on the boundary of this chart. The action of $G_{\mathbb{R}}$ on Ω_0 gives the other coordinate charts on Ω which are intersections with the complex coordinate charts. We will parameterize them by points of Ω such that $\Omega(\omega)$, $\omega \in \Omega$, is the orbit of the isotropy subgroup $P(\omega)$ of ω.

We will write the complex matrices as $Z = X+iY$, $z = x+iy$. For the complex $(2n\times n)$-matrix Z let us consider the square real matrix

$$Z^{\mathbb{R}} = \begin{pmatrix} X_0 & Y_0 \\ X_1 & Y_1 \end{pmatrix}$$

of order $2n$. The equivalence $Z \sim Zu$, $u = r + is \in Gl(n;\mathbb{C})$ is equivalent to $Z^{\mathbb{R}} \sim Z^{\mathbb{R}}\tilde{u}$ where

$$\tilde{u} = \begin{pmatrix} r & s \\ -s & r \end{pmatrix}.$$

Now we can give the basic construction of the domain $\mathcal{D}$.

The domain $\mathcal{D} = \mathcal{D}_+$ is defined by the condition

$$\det Z^{\mathbb{R}} > 0.$$

Apparently, this definition is compatible with the equivalence relation. Similarly, we define $\mathcal{D}_-$. The closure of the union of the union $\mathcal{D}_+, \mathcal{D}_-$ coincides with the complex Grassmanian Gr. The real Grassmanian Ω is the joint edge of the boundary of these domains.

Let us investigate the domain $\mathcal{D}$ and the action of the group $G_{\mathbb{R}}$ which preserves it. Let $\mathcal{D}_0$ be the intersection with the complex coordinate chart Gr_0.

Then in $\mathcal{D}_0$ we can rewrite the defining condition on $\mathcal{D}$ in inhomogeneous coordinates $z = = x + iy \in M(n;\mathbb{C}) \simeq \mathbb{C}^{n^2}$ in G_0 as

$$\det y > 0.$$

Let us consider in $M(n;\mathbb{R})$ the non convex cone $M_+(n)$ of matrices y with the positive determinant [1, 2].

As a result, we can interpret the domain $\mathcal{D}_0$ as the tube domain

$$T = M(n;\mathbb{R}) + iM_+(n)$$

in the linear space $M(n;\mathbb{C})$ of complex square matrices of order n.

So $\Omega_0 \simeq M(n;\mathbb{R})$ is the edge of this non convex tube domain. The real parabolic group $P^+_{\mathbb{R}}$ (with $B = 0$) of $SL(2n;\mathbb{R})$ acts on Gr_0 by affine transformations as

$$z \mapsto (C + Dz)A^{-1}.$$

Since A, C, D are real and $\det g = 1$, then A, D have non zero determinants of the same sign. As a result the subgroup $P^+_{\mathbb{R}}$ preserves the tube domain $\mathcal{D}_0$ and apparently acts there transitively. All intersections of $\mathcal{D}$ with other coordinate charts $D(\omega)$ are results of actions of elements of $G_{\mathbb{R}}$ to $\mathcal{D}_0$ (where $\mathcal{D}_I$ are the matrices of the permutations).

So $G_{\mathbb{R}} = SL(2n;\mathbb{R})$ is transitive on $\mathcal{D}$ which is covered by the open domains $\mathcal{D}(\omega)$ which are biholomorphically equivalent to the tube $\mathcal{D}_0 = T$ (a finite number of domains $\mathcal{D}(I)$) is enough.

Let us compute the isotropy subgroup H of this action. Let $z = iE \in \mathcal{D}_0 \subset \mathcal{D}$. Then $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G_{\mathbb{R}}$ preserves this point (up to the equivalence) if $C = -B,\ D = A$. So H consists of the elements

$$g = \begin{pmatrix} A & B \\ -B & A \end{pmatrix}$$

and H is isomorphic to $GL(n; \mathbb{C})$.

Finally,

$$\mathcal{D} = SL(2n; \mathbb{R})/GL(n; \mathbb{C}).$$

3. Complex cycles in the domain $\mathcal{D}$

The domain $\mathcal{D}$ relative to the transitive action of the (real) group $G_{\mathbb{R}} = SL(2n, \mathbb{R})$ is the pseudo Hermitian symmetric space. In the analytic picture there are no holomorphic functions on $\mathcal{D}$ which are different from constants.

It is q-pseudo concave, $q = n(n-1)/2$. *This property is connected with the existence inside $\mathcal{D}$ of complex cycles — complex compact submanifolds of dimension q.* To construct such a cycle, let us fix a point $Z \in \mathcal{D}$ and then take the orbit of a maximal compact subgroup $K = SO(2n) \subset G_{\mathbb{R}}$. *It will be the compact complex symmetric space*

$$C(Z) = SO(2n)/U(n),$$

where the unitary group $U(n)$ is realized as the intersection $K \bigcap GL(n; \mathbb{C})$ (realized as the isotropy subgroup of the point Z in the action of $G_{\mathbb{R}} = SL(2n; \mathbb{R})$). Apparently, the whole cycle $C(Z)$ lies inside the domain $\mathcal{D}$. The direct computation shows that this dimension (complex) is equal to $n(n-1)/2$. *The cycle $C(Z)$ will be preserved relative to the induced action of $SO(2n; \mathbb{C})$, the complexification of the compact group $SO(2n)$.*

Acting on the cycle $C(Z)$ by elements $g \in G_{\mathbb{C}}$, we will receive isomorphic cycles in the Grassmanian $Gr_{\mathbb{C}}$. *The parametric space of these cycles is*

$$\mathcal{C}(\mathbb{C}) = SL(2n; \mathbb{C})/SO(2n; \mathbb{C})$$

which is a complex symmetric space. Of course, not all these cycles lie inside $\mathcal{D}$, but cycles from the orbit $\mathcal{C}(\mathbb{R}) \subset \mathcal{C}(\mathbb{C})$ of the real subgroup $G_{\mathbb{R}} = SL(2n; \mathbb{R})$ consists of such cycles; $\mathcal{C}(\mathbb{R})$ *is a Riemannian symmetric space for the group* $SL(2n; \mathbb{R})$.

However, $\mathcal{C}(\mathbb{R})$ does not collect all cycles lying in $\mathcal{D}$: in the construction of $C(Z)$ we can replace $Z \in \mathcal{D}$ by a point $\tilde{Z} \in \mathcal{D}$, but outside $C(Z)$. Parameters of all cycles lying in $\mathcal{D}$ gives a remarkable complex neighborhood of $\mathcal{C}(\mathbb{R})$ which sometime is called *the crown of the symmetric space.* It is Stein manifold invariant (but not homogeneous) relative to $G_{\mathbb{R}}$. It has many remarkable universal properties and can be described explicitly [3, 4] .

From elementary linear algebra, it follows that $\mathcal{C}(\mathbb{C})$ can be realized as the set of complex symmetric matrices of order $2n$, with the unit determinant and correspondingly $\mathcal{C}(\mathbb{R})$ is the subset of real unimodular symmetric matrices. The cycle $C(J)$, corresponded to a symmetric matrix J, in the Stiefel coordinates is represented by the "matrix "-quadric

$$Z'JZ = 0.$$

This matrix equation means $n(2n-1)$ linear equations, since its first part is a symmetric matrix of order $2n$. To the action $Z \mapsto g^{-1}Z$ of $SL(2n; \mathbb{C})$ corresponds the action

$$J \mapsto g'Jg.$$

If we were write J through the square blocks of order n:

$$J = \begin{pmatrix} -L & M \\ M' & N \end{pmatrix},$$

where $L = L'$, $N = N'$, then we have

$$-z_0' L z_0 + z_1' N z_1 + z_0' M' z_1 + z_1' M z_0 = 0.$$

We put the minus for convenience, below.

4. Horospheres in $\mathcal{D}$ and horospherical tubes

Let us connect the cycles with the coordinate charts and inhomogeneous coordinates. Let us consider the initial coordinate chart Gr_0 where $z_0 = E$, $z_1 = z = x + iy$. Its intersection $\mathcal{D}_0$ with the domain $\mathcal{D}$ is defined by the condition $\det y > 0$. Let us consider the cycles $C(J)$ for which in J we have

$$N = 0$$

and M is non degenerate. In inhomogeneous coordinates z we have the matrix form of $n(n+1)/2$ linear equations

$$-L + M'z + z'M = 0.$$

Let us remind that L, M are square matrices and L is the symmetric one. So we have in Gr_0 the planes $F(L, M)$ of dimension $n(n-1)/2$. They are open parts of the corresponding cycles $C(J)$.

Let L, M be the real ones. *Then the planes $F(M, N)$ lie in the domain $\mathcal{D}_0$ if the imaginary part of the symmetric matrix L is positive*

$$Im(L) \gg 0.$$

It follows from the generalized Siegel Lemma [1, 2] that real matrices with positive symmetric parts have a positive determinant. Closures of such planes will lie in the closure of $\mathcal{D}_0$ and coincide with the corresponding cycles.

Let us call all these planes $F(L, M)$ *horospheres.* Usually, horospheres are orbits of maximal unipotent subgroups and their conjugate ones. In our case $F(M, L)$ are degenerated horospheres — the orbits of the non maximal subgroups of $GL(2n;\mathbb{C})$ where $A = D = E$, $C = 0$, $B = -B'$ and its conjugate ones.

For each real non degenerate M denote through $T(M)$ the union of corresponding horospheres $L(M, N)$ in $\mathcal{D}_0$ and let us call them horospherical tubes. For example, $T(E)$ is the tube domain of matrices $z = x + iy$, $(y + y') \gg 0$. So it the direct product of the upper half-plane of Siegel (of complex symmetric matrices of order n with positive imaginary parts) and the affine space $\mathbb{C}^m$, $m = n(n-1)/2$, of the skew symmetric matrices. All other $T(M)$ in $\mathcal{D}_0$ are biholomorphically equivalent to this domain. In particular all these tubes are Stein manifolds and we have a remarkable covering of $\mathcal{D}_0$ by Stein manifolds. The horospherical tube $T(E)$ is the orbit of the subgroup G_0 of $SL(2n;\mathbb{R})$ with

$$B = -B', \quad C = 0, \quad D = (A')^{-1}.$$

The conjugate subgroups will be transitive in $T(M)$.

All horospheres $F(g)$ and horospherical tubes $T(g)$ in the domain $\mathcal{D}$, we define as the result of the application of $g \in SL(2n;\mathbb{R})$ to $F(iE, E)$, $T(E)$ correspondingly. The horospheres $F(M, L)$ and tubes $T(L)$ in $\mathcal{D}_0$ above are special cases of this construction. The general horospheres will be planes in the corresponding coordinate charts. The same horospheres or horospherical tubes can correspond to different g. The parametrical space of the horospherical tubes is the homogeneous space G/G_0. If in the construction of G_0 we add the condition of the orthogonality of the matrix A, it will be the parametrization of the horospheres in $\mathcal{D}$. This construction admits the generalization on pseudo Hermitian symmetric spaces connecting with the Jordan algebras [2].

Horospheres and the horospherical tubes are very convenient tools for the development of complex analysis in $\mathcal{D}$. The domain $\mathcal{D}$ is q-pseudo concave domain, $q = n(n-1)$. So the subject of the complex analysis in $\mathcal{D}$ are $\bar{\partial}$-cohomology. Our language of the smoothly parameterized Čech cohomology [5–7], using the covering of $\mathcal{D}$ by the horospherical tubes, gives a canonical explicit realization of this cohomology. In turn, this gives a cohomological realization of the Speh realization of the group $SL(2n;\mathbb{R})$, but that will be the subject of another paper.

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Верхняя полуплоскость в грассманиане $Gr(n;2n)$

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США

Исследуется комплексная геометрия многомерного обобщения $\mathcal{D}(n)$ верхней полуплоскости, которая однородна относительно группы $G = SL(2n;\mathbb{R})$. При $n > 1$ это псевдоэрмитово симметрическое пространство является открытой орбитой $G = SL(2n;\mathbb{R})$ на грассманиане $Gr_{\mathbb{C}}(n;2n)$ n-мерных подпространств в $\mathbb{C}^{2n}$. Основным элементом конструкции является каноническое покрытие $\mathcal{D}(n)$ максимальными подмногообразиями Штейна — орисферическими трубками.

Ключевые слова: грассманиан, псевдоэрмитово симметрическое пространство, цикл, орисфера, орисферическая труба.