

# THE MOTION OF A BINARY MIXTURE WITH A CYLINDRICAL FREE BOUNDARY AT SMALL MARANGONI NUMBERS

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*We studied the problem of axisymmetric motion of a binary mixture with a cylindrical free boundary at small Marangoni numbers. Using Laplace transformation properties the exact analytical solution is obtained. It is shown that a stationary solution is the limiting one with the growth of time if satisfy certain conditions imposed on the external temperature. Some examples of numerical reconstruction of the velocity, temperature and concentration fields are considered, which correspond well with the theoretical results.*

Key words: binary mixture, free boundary, stationary solution, Laplace transformation, the Marangoni number.

## Introduction

It is well known that at low Reynolds numbers the momentum equation can be simplified by discarding the convective acceleration. Such movement is called *creeping*. The same simplification can be obtained for equations of energy transfer and concentration. One of such problems considered in the work [1, 2], associated with the unsteady motion of a drop (bubble). As the Reynolds number was taken as the number Marangoni, which is small both due to the radius of the droplet, and due to the physical parameters. In another task (the motion of fluid in a cylindrical tube) the small parameter was the product of the number Grashof on the Prandtl number [3].

In the present work we considered a similar problem. It is associated with unsteady motion of a binary mixture with a cylindrical free boundary. Here the analogue of the Reynolds number acts the Marangoni number. This problem is reduced to inverse initial-boundary value problem for parabolic equations. It is solved in quadratures, and it is proved that at certain conditions,

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its solution tends to a stationary regime with a growth time. Numerical treatment of the Laplace transform obtained quantitative results for a model compound, which confirm the theoretical conclusions.

## 1. Statement of the problem

Let us turn to the problem of motion of a binary mixture. Axisymmetric motion of an incompressible viscous heat-conducting mixture is considered. Denote by  $\mathbf{u}(\mathbf{x}, t)$ ,  $p(\mathbf{x}, t)$  — accordingly, the velocity vector and the pressure,  $\theta(\mathbf{x}, t)$  and  $c(\mathbf{x}, t)$  — deviations from the average values of temperature and concentration. Then the system of equations of thermodiffusion motion *in the absence of external forces* ( $\mathbf{g} = 0$ ) has the form:

$$\begin{aligned} \frac{d\mathbf{u}}{dt} + \frac{1}{\rho} \nabla p &= \nu \Delta \mathbf{u}, \quad \text{div } \mathbf{u} = 0, \\ \frac{d\theta}{dt} &= \chi \Delta \theta, \quad \frac{dc}{dt} = d \Delta c + \alpha d \Delta \theta, \end{aligned} \quad (1)$$

where  $\rho$  — the average density,  $\nu$  — the kinematic viscosity,  $\chi$  — the thermal diffusivity,  $d$  — the diffusion coefficient,  $\alpha$  — the thermodiffusion coefficient (the coefficient Soret);  $d/dt = \partial/\partial t + \mathbf{u} \cdot \nabla$ .

After the transition to dimensionless variables (as the scale of length, time, velocity, pressure, temperature and concentration are taken of values  $h_0$ ,  $h_0^2/\nu$ ,  $\alpha_1 A h_0/\mu$ ,  $\alpha_1 A$ ,  $A h_0$ ,  $A h_0 \beta_1/\beta_2$ ) will receive task:

$$\partial \mathbf{u} / \partial t + \text{Ma} \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p = \Delta \mathbf{u}, \quad (2)$$

$$\text{div } \mathbf{u} = 0, \quad (3)$$

$$\text{Pr} (\partial \theta / \partial t + \text{Ma} \mathbf{u} \cdot \nabla \theta) = \Delta \theta, \quad (4)$$

$$\text{Sc} (\partial c / \partial t + \text{Ma} \mathbf{u} \cdot \nabla c) = \Delta c - \varepsilon \Delta c, \quad (5)$$

Here introduced dimensionless parameters:  $\text{Ma} = \alpha_1 A h_0^2 / \mu \nu$  — the thermal Marangoni number,  $\text{Pr} = \nu / \chi$  — Prandtl number,  $\text{Sc} = \nu / d$  — Schmidt number,  $\varepsilon = -\alpha \beta_2 / \beta_1$  — thermodiffusion parameter;  $A$  — temperature gradient,  $\alpha_1$  — some constant. We assume that the Marangoni number is many times less than one  $\text{Ma} \ll 1$  (*the creeping motion*), then in the momentum equations, heat transfer and concentration convective terms can be discarded. With the set physical parameters of a liquid, this condition is carried out if sufficiently small value  $A h_0^2$ . Formally expanding functions  $\mathbf{u}$ ,  $p$ ,  $\theta$ ,  $c$  in series of  $\text{Ma}$ , to get a first approximation task (2)–(5) c  $\text{Ma} = 0$ .

Let  $u(r, z, t)$ ,  $w(r, z, t)$  — projection of velocity vector on the axis of a cylindrical coordinate system  $r$  and  $z$ . Suppose that the free boundary in this system is described by the equation  $r = h(z, t)$ . Then the conditions will take [3]:

$$h_t + w h_z - u = 0; \quad (6)$$

$$(1 - h_z^2)(u_z + w_r) + 2h_z(u_r - w_z) = \frac{L}{\rho \nu} \left[ (h_z \theta_r + \theta_z) \frac{\partial \sigma}{\partial \theta} + (h_z c_r + c_z) \frac{\partial \sigma}{\partial c} \right]; \quad (7)$$

$$p_{gas} - p + 2\rho \nu L^{-2} [u_r - h_z(u_z + w_r) + h_z^2 w_z] = 2\sigma H; \quad (8)$$

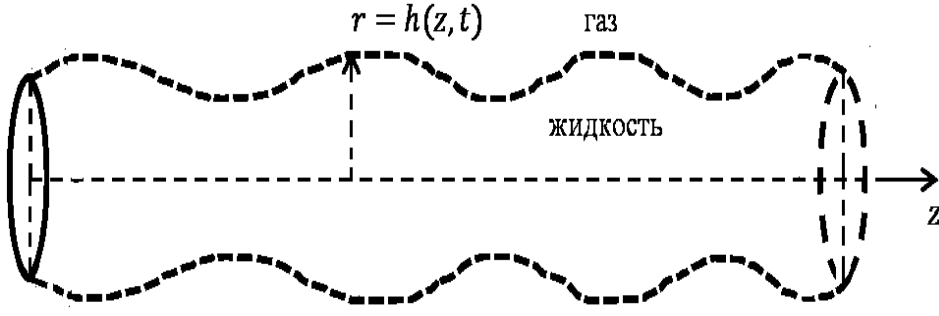


Рис. 1. Схема области течения

$$kL^{-1}(\theta_r - h_z\theta_z) + \gamma(\theta - \theta_{gas}) = Q; \quad (9)$$

$$c_r - h_z c_z + \alpha(\theta_r - h_z\theta_z) = 0, \quad (10)$$

where  $L = (1 + h_z^2)^{1/2}$ ;  $\sigma(\theta, c)$  — the coefficient of surface tension of the mixture and for most real liquids, it is well approximated by a linear dependence

$$\sigma(\theta, c) = \sigma^0 - \alpha_1(\theta - \theta^0) - \alpha_2(c - c^0) \quad (11)$$

where  $\alpha_1 > 0$  — the temperature coefficient,  $\alpha_2$  — the concentration coefficient (typically,  $\alpha_2 < 0$ , since the surface tension increases with increasing concentration),  $\sigma^0, \theta^0, c^0$  — some constant average values;  $p_{gas}$  and  $\theta_{gas}$  — the pressure and the temperature of the ambient gas, which is considered passive;  $\gamma$  — the coefficient of interphase heat transfer,  $k$  — the thermal conductivity coefficient.

In equation (8)  $H$  — the average curvature of the free boundary

$$H = \frac{hh_{zz} - h_z^2 - 1}{2h(1 + h_z^2)^{3/2}}. \quad (12)$$

**Remark 1.** In the recording of the Marangoni number is present the parameter  $\alpha_1$  included in the dependence of surface tension on temperature and concentration (11).

Consider the subgroup generated by four operators  $\partial_z, t\partial_z + \partial_w, \partial_\theta, \partial_c, \partial_p$ . It is easy to check that it is allowed by a system of equations of thermodiffusion. Its invariants  $t, r, u$ , then partially invariant solutions with respect to this subgroup are of the form [5]:

$$u = u(r, t), \quad w = w(r, z, t), \quad p = p(r, z, t), \quad \theta = \theta(r, z, t), \quad c = c(r, z, t). \quad (13)$$

In this case, from the equation of conservation of mass it follows that  $w$  is a linear function of  $z$ . Let

$$w = zv(r, t). \quad (14)$$

General view of the invariant manifolds with respect to the considered subgroup  $\{r, z, t\}$  is  $r = h(t)$  with an arbitrary function  $h(t)$ . Let the dependence  $\sigma(\theta, c)$  has the form (11), then from boundary conditions (7) we get that  $\alpha_1(\theta - \theta^0) - \alpha_2(c - c^0)$  is a quadratic function of  $z$ . It is therefore natural to assume that

$$\theta(r, z, t) = a(r, t)z^2 + b(r, t), \quad c(r, z, t) = l(r, t)z^2 + m(r, t), \quad (15)$$

Interpretation of the solution of (13)–(15) is as follows. Let an axisymmetric heated long enough cylinder of a binary mixture of external temperature on the boundary has a maximum of ( $a < 0$ ) or minimum ( $a > 0$ ) at point  $z = 0$ . Then in a neighbourhood of this point the external temperature can be approximated by a parabolic law.

The substitution of the solution (13)–(15) to the system of equations of thermodiffusion and separation of a variable  $z$  leads to the next task in the region  $t > 0$ ,  $0 < r < h(t)$

$$u_t + p_r = u_{rr} + \frac{1}{r} u_r - \frac{u}{r^2}; \quad (16)$$

$$zv_t + p_z = z \left( v_{rr} + \frac{1}{r} v_r \right); \quad (17)$$

$$u_r + \frac{u}{r} + v = 0; \quad (18)$$

$$a_t = \frac{1}{\text{Pr}} \left( a_{rr} + \frac{1}{r} a_r \right); \quad (19)$$

$$b_t = \frac{1}{\text{Pr}} \left( b_{rr} + \frac{1}{r} b_r + 2a \right); \quad (20)$$

$$l_t = \frac{1}{\text{Sc}} \left( l_{rr} + \frac{1}{r} l_r \right) - \frac{\varepsilon}{\text{Sc}} \left( a_{rr} + \frac{1}{r} a_r \right); \quad (21)$$

$$m_t = \frac{1}{\text{Sc}} \left( m_{rr} + \frac{1}{r} m_r + 2l \right) - \frac{\varepsilon}{\text{Sc}} \left( b_{rr} + \frac{1}{r} b_r + 2a \right). \quad (22)$$

The boundary conditions at  $r = h(t)$  as follows

$$h_t = u; \quad (23)$$

$$v_r = -2a - 2Ml; \quad (24)$$

$$p_{gas} - p + 2v = \text{We} - (a - \theta^0) - M(l - c^0); \quad (25)$$

$$a_r + \text{Bi}(a - a_{gas}) = 0; \quad (26)$$

$$b_r + \text{Bi}(b - b_{gas}) = 0; \quad (27)$$

$$l_r - \varepsilon a_r = 0; \quad (28)$$

$$m_r - \varepsilon b_r = 0. \quad (29)$$

Here introduced dimensionless parameters:  $\text{Bi} = \gamma h_0 / k$  — the number of Bio,  $\text{We} = \sigma_0 / \varkappa_1 A h_0$  — Weber number;  $M = \frac{\text{Mc}}{\text{Ma}} = \varkappa_2 \beta_1 / \varkappa_1 \beta_2$ , where  $\text{Mc} = \varkappa_2 A h_0^2 \beta_1 / \mu \nu \beta_2$  — the concentration Marangoni number;

In addition, it is necessary to require the boundedness of functions on the cylinder axis when  $r = 0$

$$|u| < \infty, \quad |v| < \infty, \quad |p| < \infty, \quad |a| < \infty, \quad |b| < \infty, \quad |l| < \infty, \quad |m| < \infty, \quad (30)$$

and specify initial conditions at  $t = 0$

$$u = u_0(r), \quad v = v_0(r), \quad a = a_0(r), \quad b = b_0(r), \quad l = l_0(r), \quad m = m_0(r), \quad h(0) = h_0 = \text{const} > 0. \quad (31)$$

Note that the functions  $u_0$  and  $v_0$  are related by equation (18),  $v_0, a_0, l_0$  — by condition (25),  $l_0, a_0$  — by condition (29), a  $m_0, b_0$  — by condition (30).

Suppose that at the initial moment of time the free surface satisfies the initial condition:  $r = h(0) = h_0 = \text{const} > 0$ , and decompose the function  $h(t)$  in a series of Marangoni. Then the variable  $r$  will vary in the range  $0 \leq r \leq h_0 + \text{Mah}(t)$ . We introduce dimensionless variables:  $r' = r/h_0$ ,  $z' = z/h_0$ ,  $t' = \nu t/h_0^2$ . The kinematic condition (6) in dimensionless variables has the form (touches for convenience omitted):

$$\text{Mah}_t = \text{Mau}(1 + \text{Mah}(t), t); \quad (32)$$

Assuming that  $\text{Ma} \ll 1$ , from the last equation in  $\text{Ma} \rightarrow 0$  we find that the kinematic condition at  $r = 1$  gets converted to

$$h_t = u(1, t); \quad (33)$$

The function  $u$  is expressed from the continuity equation (18) in the following way

$$u = -\frac{1}{r} \int_0^r r v dr; \quad (34)$$

From equations (16), (17) we express the pressure gradient  $(p_r, p_z)$

$$p_r = u_{rr} + \frac{1}{r} u_r - \frac{u}{r^2} - u_t; \quad (35)$$

$$p_z = z(v_{rr} + \frac{1}{r} v_r - v_t); \quad (36)$$

Differentiating equation (35) for the variable  $z$ , and the equation (36) for the variable  $r$ , and given that a function  $u$  is not depends on  $z$ , we get the equation of compatibility:  $p_{rz} = p_{zr} = 0$ . Whence it follows that the function  $v(r, t)$  is determined from the equation

$$v_t = v_{rr} + \frac{1}{r} v_r + q(t), \quad (37)$$

and the pressure is given by

$$p = -\frac{z^2}{2} q(t) + s(r, t); \quad (38)$$

where  $q(t)$  – arbitrary function, and the derivative in the variable  $r$  of a function  $s(r, t)$  is precisely the right side of equation (35).

Suppose that in the equation (34) when  $r = 1$  running

$$\int_0^1 r v dr = 0. \quad (39)$$

It follows that  $u(1, t) = 0$ . Then, returning to the kinematic condition (33), we have:  $h(t) = 1$ . So a free boundary does not change with time and remains constant value. Converse is also true: if  $h(t) = 1$ , then condition (39).

**Remark 2.** In fact, certain functions  $q(t)$ , i.e. the “gradient pressure” along the axis  $z$ , can be found as follows. Multiply equation (38) on  $r$  and integrating on  $r$  from zero to 1, reducing it thereby to a “loaded” equation

$$\frac{\partial}{\partial t} \int_0^1 r v dr = \int_0^1 (r v_r)_r dr + \frac{1}{2} q(t). \quad (40)$$

Integral, the left-hand side of the equation equal to zero according to the condition (39). It follows that

$$q(t) = -2v_r(1, t). \quad (41)$$

where the right part is set as a boundary condition (24).

Therefore, first solve the equation (19) on the function  $a(r, t)$  with boundary conditions (26), (30) and the initial condition (31); next we define the function  $l(r, t)$  from the equation (21) with conditions (28), (30), (31), while function  $b(r, t)$ ,  $m(r, t)$  are calculated similarly. Thereby recovering the temperature and concentration mixture according to formulas (15). Then solve the problem on determination of the axial components of the velocity vector  $v(r, t)$ , satisfy the equation (38) and the boundary conditions (24), (30), (31). As the result is to find the radial component of  $u(r, t)$  of equation (34) with boundary conditions (30), (31), the function  $q(t)$  - (41) and the pressure  $p$  - from equation (38) and the boundary conditions (8).

## 2. Stationary solution

For this decision all the unknown functions do not depend on time; let them through  $u^0(r)$ ,  $v^0(r)$ ,  $p^0(r)$ ,  $a^0(r)$ ,  $b^0(r)$ ,  $l^0(r)$ ,  $m^0(r)$ . In addition,  $q(t) = q^0 = \text{const}$ . Write out the corresponding boundary value problem for  $0 < r < 1$ :

$$u_{rr}^0 + \frac{1}{r} u_r^0 - \frac{u^0}{r^2} = p_r^0; \quad (42)$$

$$v_{rr}^0 + \frac{1}{r} v_r^0 = \frac{p_z^0}{z} = -q^0; \quad (43)$$

$$u_r^0 + \frac{1}{r} u^0 + v^0 = 0; \quad (44)$$

$$\frac{1}{\text{Pr}} \left( a_{rr}^0 + \frac{1}{r} a_r^0 \right) = 0; \quad (45)$$

$$\frac{1}{\text{Pr}} \left( b_{rr}^0 + \frac{1}{r} b_r^0 + 2a^0 \right) = 0; \quad (46)$$

$$\frac{1}{\text{Sc}} \left( l_{rr}^0 + \frac{1}{r} l_r^0 \right) = 0; \quad (47)$$

$$\frac{1}{\text{Sc}} \left( m_{rr}^0 + \frac{1}{r} m_r^0 + 2l^0 \right) = 0; \quad (48)$$

General solution of system (42)–(48) when  $\text{Pr} \neq 0$ ,  $\text{Sc} \neq 0$  are easily found (taken into account the conditions of constraints of (30)):

$$a^0 = a_{gas}^0, \quad b^0 = -\frac{a_{gas}^0 r^2}{2} + b_{gas}^0 + a_{gas}^0 (0.5 + \text{Bi}^{-1}), \quad l^0 = \varepsilon a_{gas}^0, \quad m^0 = -\frac{\varepsilon a_{gas}^0 r^2}{2} + C, \quad (49)$$

where  $a_{gas}^0$ ,  $b_{gas}^0$ ,  $C \equiv \text{const}$ ;

**Remark 3.** The constant  $C$  remains arbitrary, it is not surprising, as the task for the concentration contains only derivatives. On the other hand, this constant can be determine if

ask the average concentration in the cross section of  $z = 0$ , i.e.  $\int_0^1 cr^0(r)dr = 0$ . From this  $C = \varepsilon a_{gas}^0/4$ . Thus, we have the following representation for  $m^0$

$$m^0 = \frac{\varepsilon a_{gas}^0}{2} \left( \frac{1}{2} - r^2 \right). \quad (50)$$

The other functions are as follows

$$u^0 = \frac{q^0}{16} r (r^2 - 1), \quad v^0 = \frac{q^0}{4} \left( \frac{1}{2} - r^2 \right), \quad p^0 = \frac{q^0}{2} \left( z^2 - \frac{r^2}{2} \right), \quad q^0 = 4a_{gas}^0 (1 + M\varepsilon). \quad (51)$$

The obtained solutions satisfy the boundary and initial conditions (23)–(31).

### 3. Determination of the temperature field

For solution of nonstationary problem is used Laplace transform. Believe (assuming the existence of  $\tilde{a}, \tilde{a}_r, \tilde{a}_{rr}, \tilde{a}_{gas}$ )

$$\tilde{a}(r, p) = \int_0^\infty a(r, t) e^{-pt} dt, \quad (52)$$

then the problem for  $a(r, t)$  is reduced to boundary problem for ordinary differential equations

$$\tilde{a}_{rr} + \frac{1}{r} \tilde{a}_r - \text{Pr} p \tilde{a} = -\text{Pr} a_0(r), \quad 0 < r < 1; \quad (53)$$

$$\tilde{a}_r + \text{Bi}(\tilde{a} - \tilde{a}_{gas}) = 0, \quad r = 1; \quad (54)$$

$$|\tilde{a}(0, p)| < \infty, \quad (55)$$

The General solution of the equation (52) it is easy to write (taken into account the conditions of constraints (55))

$$\begin{aligned} \tilde{a} = & C_1 I_0 \left( \sqrt{\text{Pr} p} r \right) + \\ & + \int_0^r \text{Pr} y a_0(y) \left[ I_0 \left( \sqrt{\text{Pr} p} y \right) K_0 \left( \sqrt{\text{Pr} p} r \right) - I_0 \left( \sqrt{\text{Pr} p} r \right) K_0 \left( \sqrt{\text{Pr} p} y \right) \right] dy \quad ; \end{aligned} \quad (56)$$

with a constant  $C_1$ , determined from the boundary conditions:

$$\begin{aligned} C_1 = & \left[ \sqrt{\text{Pr} p} I_1 \left( \sqrt{\text{Pr} p} \right) + \text{Bi} I_0 \left( \sqrt{\text{Pr} p} \right) \right]^{-1} \left\{ \text{Bi} \tilde{a}_{gas} + \right. \\ & + \sqrt{\text{Pr} p} \int_0^1 \text{Pr} y a_0(y) \left[ I_0 \left( \sqrt{\text{Pr} p} y \right) K_1 \left( \sqrt{\text{Pr} p} \right) + I_1 \left( \sqrt{\text{Pr} p} \right) K_0 \left( \sqrt{\text{Pr} p} y \right) \right] dy - \\ & \left. - \text{Bi} \int_0^1 \text{Pr} y a_0(y) \left[ I_0 \left( \sqrt{\text{Pr} p} y \right) K_0 \left( \sqrt{\text{Pr} p} \right) - I_0 \left( \sqrt{\text{Pr} p} \right) K_0 \left( \sqrt{\text{Pr} p} y \right) \right] dy \right\}. \end{aligned} \quad (57)$$

The task is to determine the image of  $\tilde{b}(r, p)$  exactly coincides with the task (53)–(55) with the replacement right parts:  $-\text{Pr } a_0(r)$  for  $-\text{Pr } b_0(r) - 2\tilde{a}$ . Thus, this function is given by

$$\begin{aligned} \tilde{b} = & C_2 I_0 \left( \sqrt{\text{Pr } p r} \right) - \frac{C_1 r I_1 \left( \sqrt{\text{Pr } p r} \right)}{\sqrt{\text{Pr } p}} + \\ & + \int_0^r \text{Pr } y (b_0(y) + 2a_0(y)) \left[ I_0 \left( \sqrt{\text{Pr } p y} \right) K_0 \left( \sqrt{\text{Pr } p r} \right) - I_0 \left( \sqrt{\text{Pr } p r} \right) K_0 \left( \sqrt{\text{Pr } p y} \right) \right] dy \end{aligned} \quad ; \quad (58)$$

with a constant  $C_2$ , determined from the boundary conditions:

$$\begin{aligned} C_2 = & \left[ \sqrt{\text{Pr } p} I_1 \left( \sqrt{\text{Pr } p} \right) + \text{Bi} I_0 \left( \sqrt{\text{Pr } p} \right) \right]^{-1} \left\{ \text{Bi } \tilde{b}_{gas} + C_1 r I_0 \left( \sqrt{\text{Pr } p r} \right) + \right. \\ & + \sqrt{\text{Pr } p} \int_0^1 \text{Pr } y (b_0(y) + 2a_0(y)) \left[ I_0 \left( \sqrt{\text{Pr } p y} \right) K_1 \left( \sqrt{\text{Pr } p} \right) + I_1 \left( \sqrt{\text{Pr } p} \right) K_0 \left( \sqrt{\text{Pr } p y} \right) \right] dy - \\ & \left. - \text{Bi} \int_0^1 \text{Pr } y (b_0(y) + 2a_0(y)) \left[ I_0 \left( \sqrt{\text{Pr } p y} \right) K_0 \left( \sqrt{\text{Pr } p} \right) - I_0 \left( \sqrt{\text{Pr } p} \right) K_0 \left( \sqrt{\text{Pr } p y} \right) \right] dy \right\}, \end{aligned} \quad (59)$$

where  $I_j$ ,  $K_j$  — Bessel functions of the first and the third kind of imaginary argument.

It is possible to show, using the explicit formula (56), (57) that

$$\lim_{t \rightarrow \infty} a(r, t) = \lim_{p \rightarrow 0} p \tilde{a}(r, p) = a_{gas}^0,$$

where  $a_{gas}^0$  — stationary solution for the function  $a(r, t)$ . In the derivation we must assume the existence of the limit  $\lim_{t \rightarrow \infty} a_{gas}(t) = \lim_{p \rightarrow 0} p \tilde{a}_{gas}(p) = a_{gas}^0$ . Note that when  $t \rightarrow 0$  [6]:

$$I_0(t) \sim 1 + \frac{t^2}{4}, \quad K_0(t) \sim -I_0(t) \ln t,$$

$$I_1(t) \sim \frac{t}{2} + \frac{t^3}{16}, \quad K_1(t) \sim \frac{1}{t} + \frac{t}{2} \ln t - \frac{t}{4}.$$

From (56), (57) after long calculations we obtain that  $p \tilde{a}(r, p) \rightarrow a_{gas}^0$  when  $p \rightarrow 0$ .

Similarly, it is shown that

$$\lim_{t \rightarrow \infty} b(r, t) = \lim_{p \rightarrow 0} p \tilde{b}(r, p) = -0.5 a_{gas}^0 r^2 + b_{gas}^0 + a_{gas}^0 (0.5 + \text{Bi}^{-1}),$$

where the right side of the equality is a stationary solution for the function  $b(r, t)$ , assuming the existence of a limit  $\lim_{t \rightarrow \infty} b_{gas}(t) = \lim_{p \rightarrow 0} p \tilde{b}_{gas}(p) = b_{gas}^0$ .

**Lemma 1.** *Problem solving for the functions  $a(r, t)$ ,  $b(r, t)$  are defined by the inverse Laplace transformation according to the formulas (56), (58), and with the growth of time, they reach a stationary regime, if  $a_{gas}(t) \rightarrow a_{gas}^0$ ,  $b_{gas}(t) \rightarrow b_{gas}^0$  when  $t \rightarrow \infty$ .*

## 4. Determination of the concentration of the mixture

Applying to the initial-boundary problem for the concentration the mixture of Laplace transform, obtain for the image  $\tilde{l}(r, p)$  task

$$\tilde{l}_{rr} + \frac{1}{r} \tilde{l}_r - \text{Sc} p \tilde{l} = -\text{Sc} l_0(r) - \varepsilon \text{Pr} a_0(r) + \varepsilon \text{Pr} p \tilde{a}, \quad 0 < r < 1; \quad (60)$$

$$\tilde{l}_r - \varepsilon \tilde{a}_r = 0, \quad r = 1; \quad (61)$$

$$|\tilde{l}(0, p)| < \infty, \quad (62)$$

The General solution of the equation (60) with  $\text{Pr} \neq \text{Sc}$  is represented as follows

$$\begin{aligned} \tilde{l} = & C_3 I_0 \left( \sqrt{\text{Sc} p r} \right) + \\ & + \int_0^r y \left( \text{Sc} l_0(y) + \varepsilon \text{Pr} a_0(y) \right) \left[ I_0 \left( \sqrt{\text{Sc} p y} \right) K_0 \left( \sqrt{\text{Sc} p r} \right) - I_0 \left( \sqrt{\text{Sc} p r} \right) K_0 \left( \sqrt{\text{Sc} p y} \right) \right] dy + \\ & + \frac{\varepsilon \text{Pr}}{\text{Pr} - \text{Sc}} \left[ C_1 I_0 \left( \sqrt{\text{Pr} p r} \right) + \right. \\ & \left. + \int_0^r \text{Pr} y a_0(y) \left[ I_0 \left( \sqrt{\text{Pr} p y} \right) K_0 \left( \sqrt{\text{Pr} p r} \right) - I_0 \left( \sqrt{\text{Pr} p r} \right) K_0 \left( \sqrt{\text{Pr} p y} \right) \right] dy \right] \end{aligned} \quad (63)$$

with a constant  $C_3$ , determined from the boundary conditions (61):

$$\begin{aligned} C_3 = & \left[ \sqrt{\text{Sc} p} I_1 \left( \sqrt{\text{Sc} p} \right) \right]^{-1} \left\{ \frac{\varepsilon \text{Sc}}{\text{Sc} - \text{Pr}} C_1 \sqrt{\text{Pr} p} I_1 \left( \sqrt{\text{Pr} p} \right) + \right. \\ & + \sqrt{\text{Sc} p} \int_0^1 y \left( \text{Sc} l_0(y) + \varepsilon \text{Pr} a_0(y) \right) \left[ I_0 \left( \sqrt{\text{Sc} p y} \right) K_1 \left( \sqrt{\text{Sc} p} \right) + I_1 \left( \sqrt{\text{Sc} p} \right) K_0 \left( \sqrt{\text{Sc} p y} \right) \right] dy - \\ & \left. - \frac{\varepsilon \text{Sc}}{\text{Sc} - \text{Pr}} \sqrt{\text{Pr} p} \int_0^1 \text{Pr} y a_0(y) \left[ I_0 \left( \sqrt{\text{Pr} p y} \right) K_0 \left( \sqrt{\text{Pr} p} \right) - I_0 \left( \sqrt{\text{Pr} p} \right) K_0 \left( \sqrt{\text{Pr} p y} \right) \right] dy \right\}. \end{aligned} \quad (64)$$

The task is to determine the image of  $\tilde{m}(r, p)$  exactly coincides with the task (60)–(62) with the replacement right parts:  $-\text{Sc} l_0(r) - \varepsilon \text{Pr} a_0(r) + \varepsilon \text{Pr} p \tilde{a}$  for  $-\text{Sc} m_0(r) - \varepsilon \text{Pr} b_0(r) + \varepsilon \text{Pr} p \tilde{b}$ .

It is possible to show, using explicit formulas (63), (64), that

$$\lim_{t \rightarrow \infty} l(r, t) = \lim_{p \rightarrow 0} p \tilde{l}(r, p) = \varepsilon a_{gas}^0,$$

where  $\varepsilon a_{gas}^0$  — stationary solution for the function  $l(r, t)$ . If the conclusion is again necessary to assume the existence of the limit  $\lim_{t \rightarrow \infty} a_{gas}(t) = \lim_{p \rightarrow 0} p \tilde{a}_{gas}(p) = a_{gas}^0$ .

Similarly, it is shown that  $\lim_{t \rightarrow \infty} m(r, t) = \lim_{p \rightarrow 0} p \tilde{m}(r, p) = m^0(r)$ , where  $m^0(r)$  — stationary solution for the function  $m(r, t)$ , assuming the existence of a limit  $\lim_{t \rightarrow \infty} b_{gas}(t) = \lim_{p \rightarrow 0} p \tilde{b}_{gas}(p) = b_{gas}^0$ .

Thus, the fair

**Lemma 2.** *Problem solving for the functions  $l(r, t)$ ,  $m(r, t)$  are determined by the inverse Laplace transformation according to the formulas (63), (64), and with the growth of time, they reach a stationary regime, if  $a_{gas}(t) \rightarrow a_{gas}^0$ ,  $b_{gas}(t) \rightarrow b_{gas}^0$  when  $t \rightarrow \infty$ .*

## 5. Determination of velocity field

The application of the Laplace transform to the problem for velocity reduces it to a boundary problem for ordinary differential equations

$$\tilde{v}_{rr} + \frac{1}{r} \tilde{v}_r - \text{Pr } p \tilde{v} = -v_0(r) - \tilde{q}(p), \quad 0 < r < 1; \quad (65)$$

$$\int_0^1 r \tilde{v} dr = 0, \quad r = 1; \quad (66)$$

$$\tilde{v}_r = -2\tilde{a} - 2M\tilde{l}, \quad r = 1; \quad (67)$$

$$|\tilde{v}(0, p)| < \infty, \quad (68)$$

The General solution of the equation (65) is written as follows

$$\begin{aligned} \tilde{v} = & C_5 I_0 \left( \sqrt{\text{Pr } p} r \right) + \frac{\tilde{q}(p)}{p} + \\ & + \int_0^r y v_0(y) \left[ I_0 \left( \sqrt{p} y \right) K_0 \left( \sqrt{p} r \right) - I_0 \left( \sqrt{p} r \right) K_0 \left( \sqrt{p} y \right) \right] dy \end{aligned} \quad ; \quad (69)$$

with a constant  $C_5$ , determined from the boundary conditions (67):

$$\begin{aligned} C_5 = & \left[ \sqrt{p} I_1 \left( \sqrt{p} \right) \right]^{-1} \left\{ -2\tilde{a} - 2M\tilde{l} + \right. \\ & \left. + \sqrt{p} \int_0^1 y v_0(y) \left[ I_0 \left( \sqrt{p} y \right) K_1 \left( \sqrt{p} \right) + I_1 \left( \sqrt{p} \right) K_0 \left( \sqrt{p} y \right) \right] dy \right\}, \end{aligned} \quad (70)$$

where functions  $\tilde{a}(r, p)$ ,  $\tilde{l}(r, p)$  are defined by formula (55), (63) when  $r = 1$ .

In addition, from the boundary conditions (66), (67) defines a function  $\tilde{q}(p)$ , namely

$$\tilde{q}(p) = -2\tilde{v}_r(1, p).$$

Working similarly to the proof of the convergence of the obtained solutions to stationary solutions of temperature and concentration of the expressions (69), (70) we can deduce the equality

$$\lim_{p \rightarrow 0} p \tilde{v}(r, p) = v^0(r), \quad (71)$$

where  $v^0(r)$  – stationary velocity distribution of (51). In the derivation of (71) we must assume the existence of the limit  $\lim_{t \rightarrow \infty} a_{gas}(t) = a_{gas}^0$ .

Thus, the fair

**Lemma 3.** *The solution of the problem for the function  $v(r, t)$  is determined by the inverse Laplace transformation formula (69), (70), and with increasing time it comes on stationary regime, if  $a_{gas}(t) \rightarrow a_{gas}^0$  when  $t \rightarrow \infty$ .*

## 6. Numerical results

The formulas (56), (63), (69) in the image Laplacian were used to numerical finding of the fields of velocity, temperature and concentration mixture under certain conditions imposed on the temperature outside  $a_{gas}(t)$ . This was done by the method of numerical inverse the Laplace transform [7], which was obtained quantitative results for the model system with the following parameter values:  $a_{gas}^0 = 0.2$ ,  $\varepsilon = 0.1$ ,  $M = 1000$ ,  $Bi = 2$ ,  $Pr = 0.2$ ,  $Sc = 0.1$ ,  $\lambda = 10$ ,  $\omega = 10$ .

In Fig. 2–4 shows the evolution of the dimensionless profiles temperature, concentration and velocity of the mixture. In Fig. 2 a) presented the profile of the dimensionless ‘temperature’  $a(r, t)$ , provided that  $a_{gas}(t) = a_{gas}^0 + \exp(-\lambda t) \sin(\omega t)$ . The function  $a_{gas}(t)$  has a finite limit when  $t \rightarrow \infty$ ,  $a_{gas}^0$ . In this case, there is a convergence to stationary temperature distribution. In Fig. 2 b) presented the dependence  $a(r, t)$ , when  $a_{gas}(t) = \sin(\omega t)$ , i.e.  $a_{gas}(t)$  has no limit when  $t \rightarrow \infty$ . As can be seen from graphics, solution with increase in time does not converge to a stationary regime. The situation is similar with other functions.

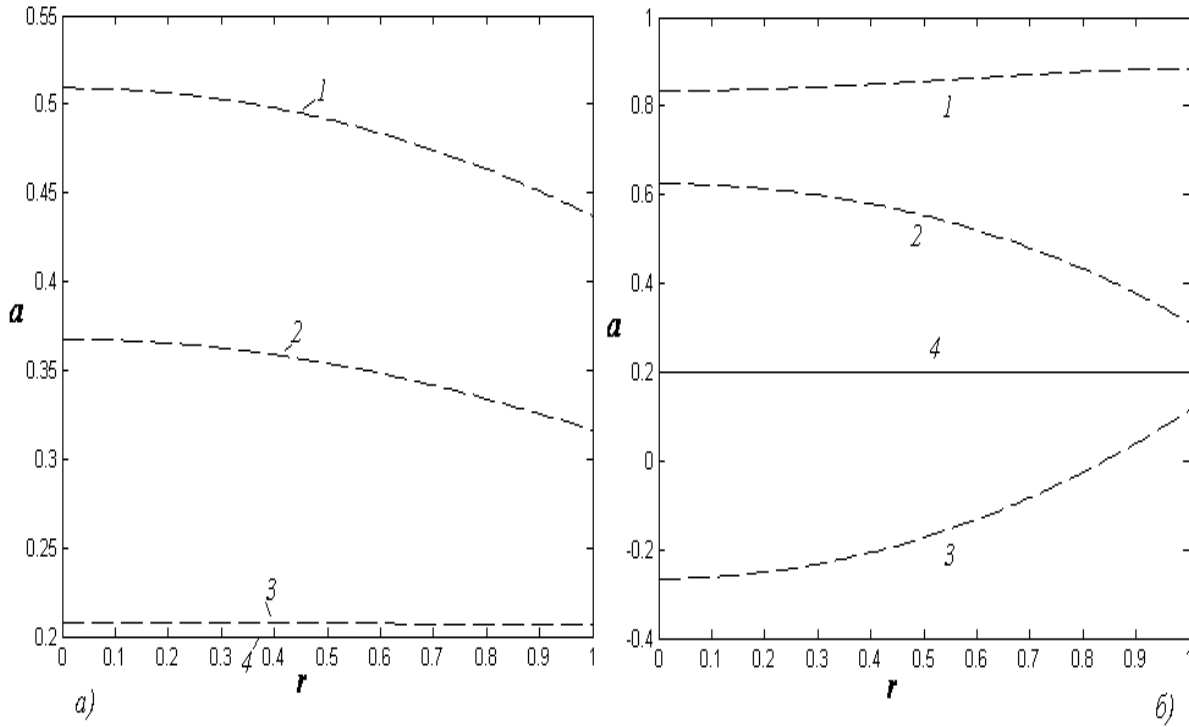


Рис. 2. The temperature profile at different points in time: 1 –  $\tau = 0.21$ , 2 –  $\tau = 0.32$ , 3 –  $\tau = 0.68$ , 4 –  $\tau = \infty$ .

The results of the numerical inverse Laplace transformation confirmed the conclusion that the stationary solution is the limit when large values of time. Thus, by changing the temperature environment, more precisely the function  $a_{gas}(t)$ , can be set various modes of movement of the mixture in the cylinder with a free boundary.

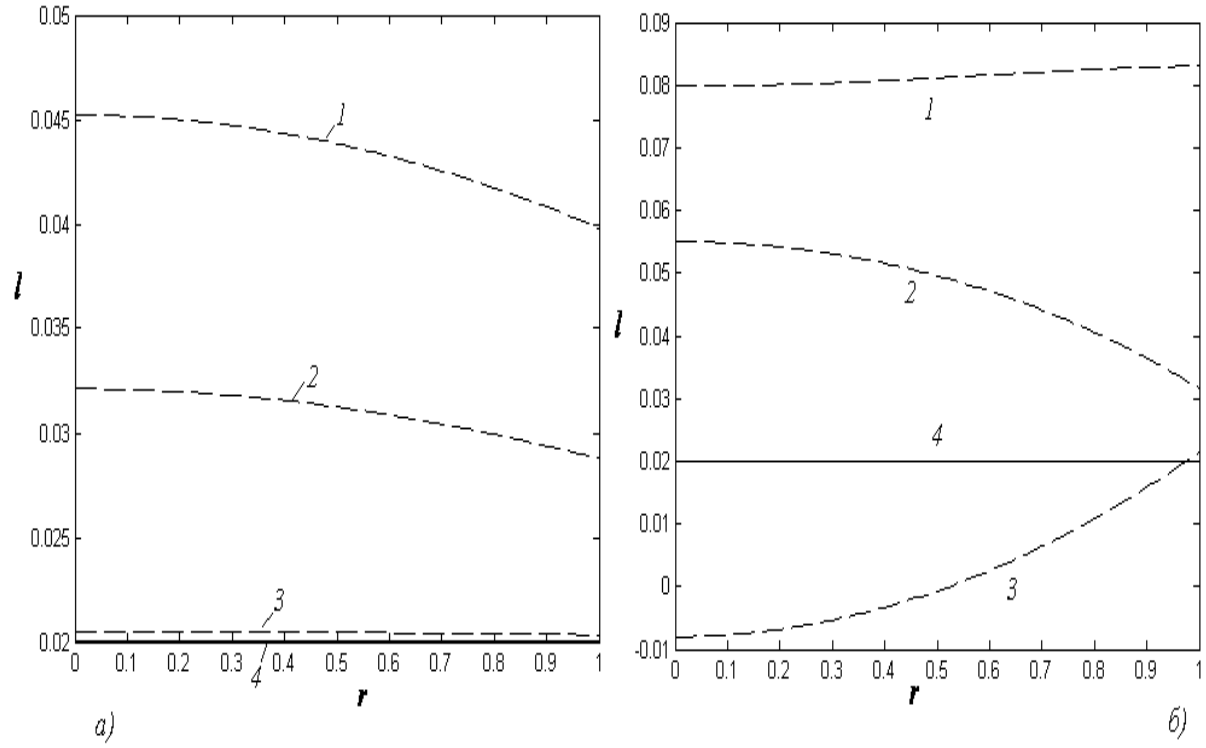


Рис. 3. Concentration profile at different points in time: 1 –  $\tau = 0.21$ , 2 –  $\tau = 0.32$ , 3 –  $\tau = 0.68$ , 4 –  $\tau = \infty$ .

## Conclusion

Studied partially-invariant solution of the axisymmetric motion of a binary mixture with a cylindrical free the boundary at small Marangoni numbers. The problem is reduced to initial-boundary value problem for parabolic equations. The images by Laplace obtained the exact analytical solution. Found the stationary solution of the problem and proved that it is the limit when  $t \rightarrow \infty$ , if you satisfy certain conditions imposed by on the temperature outside. The examples of numerical reconstruction of fields the velocity, temperature and concentration, confirming the solution in stationary regime.

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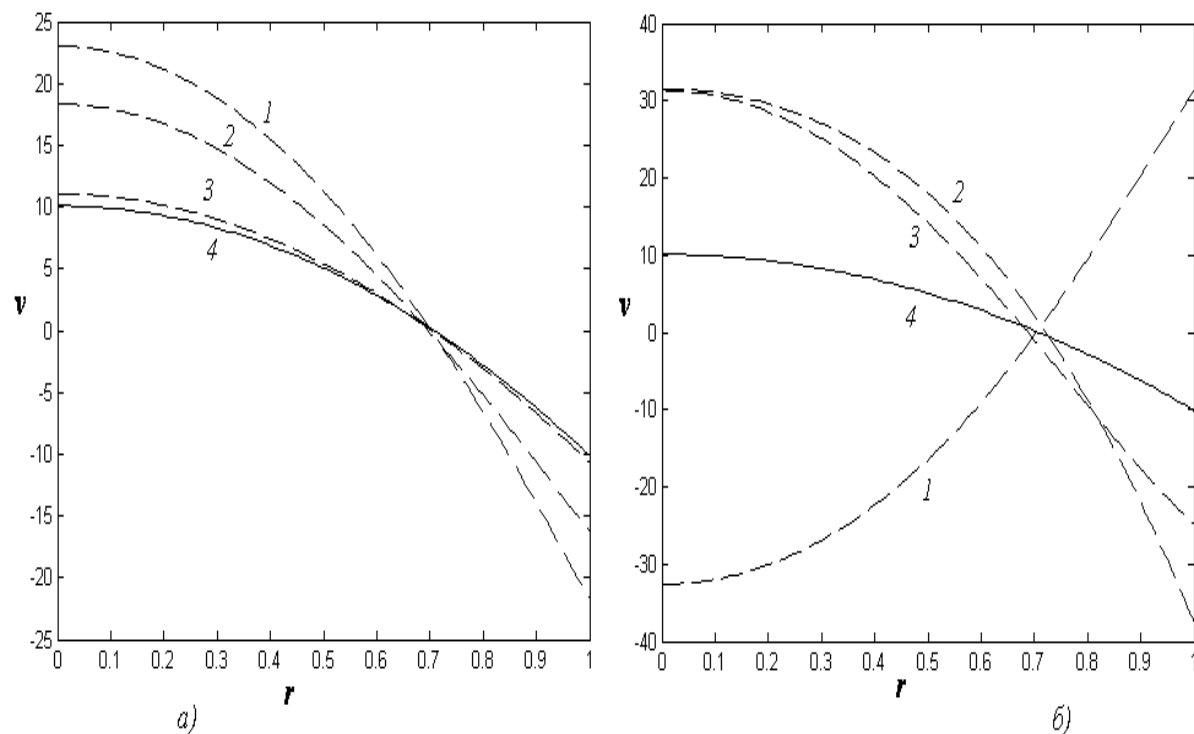


Рис. 4. The velocity profile at different points in time: 1 –  $\tau = 0.21$ , 2 –  $\tau = 0.32$ , 3 –  $\tau = 0.68$ , 4 –  $\tau = \infty$ .

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**THE MOTION OF A BINARY MIXTURE  
WITH A CYLINDRICAL FREE BOUNDARY  
AT SMALL MARANGONI NUMBERS**

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*We studied the problem of axisymmetric motion of a binary mixture with a cylindrical free boundary at small Marangoni numbers. Using Laplace transformation properties the exact analytical solution is obtained. It is shown that a stationary solution is the limiting one with the growth of time if satisfy certain conditions imposed on the external temperature. Some examples of numerical reconstruction of the velocity, temperature and concentration fields are considered, which correspond well with the theoretical results.*

Key words: binary mixture, free boundary, stationary solution, Laplace transformation, the Marangoni number.