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Computation of an Integral of a Rational Function over the Skeleton of Unit Polycylinder in $\mathbb{C}^n$ by Means of the Mellin Transform

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With the help of the Mellin transform we give a simple calculation of an integral of rational functions in several independent parameters aerlier appeared in [2]. The efficiency of this transform is due to the fact that calculation the degree of the polynomial acts as the degree of a monomial. In 2008, G. P. Egorychev and E.V. Zima [5] for the first time successfully used the Mellin transform in the theory of rational summation. The possibility of its application in the analysis and computation of integrals with different types of rational functions is discussed.

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Introduction

In [1] the author obtained an integral representation in bounded n-circular linearly convex domains with piecewise regular boundary. This integral representation is a sum of terms, each of which contains multiple integrals of the following form:

$$J = J_{s,t}(A) := \frac{1}{(2\pi i)^n} \times \int\limits_{|\xi_1|=1} \cdots \int\limits_{|\xi_n|=1} \frac{d\xi_1 \wedge \ldots \wedge d\xi_n}{\xi_1^{s_1+1} \cdot \ldots \cdot \xi_n^{s_n+1} \prod\limits_{j=1}^{m} (a_{j,1} z_1 \xi_1 + \ldots + a_{j,n} z_n \xi_n + c_j)^{t_j}}. \quad (1)$$

Here $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{C}^n$, $s = (s_1, \ldots, s_n) \in \mathbb{N}_0^n$, $A = (a_{jq})$, $a_{jq} \in \mathbb{C}$ is a matrix of dimension $m \times n$, $t = (t_1, \ldots, t_n) \in \mathbb{N}^n$.

We emphasize that for $|\xi_1| = 1, \ldots, |\xi_n| = 1$ the complex parameters $z = (z_1, \ldots, z_n)$ satisfy the conditions

$$\prod_{j=1}^{m} (a_{j,1} z_1 \xi_1 + \ldots + a_{j,n} z_n \xi_n + c_j) \neq 0. \quad (2)$$

Condition (2) implies that for $j = 1, \ldots, m$ the point $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$ **is not in the union** of a family of complex hyperplanes

$$\{V_j\} = \{u = (u_1, \ldots, u_n) \in \mathbb{C}^n : a_{j,1} u_1 \xi_1 + \ldots + a_{j,n} u_n \xi_n + c_j = 0,\ |\xi_1| = 1, \ldots, |\xi_n| = 1\}. \quad (3)$$

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1. Necessary definitions and notation

Because with every point $u = (u_1, \dots, u_n) \in \{V_j\}$ all the points $(u_1 e^{i\psi_1}, \dots, u_n e^{i\psi_n})$, $0 \leqslant \psi_l < 2\pi, l = 1, \dots, n$ also belong to $\{V_j\}$, the family of complex hyperplanes $\{V_j\}$ is a n-**circular set**. If a complex hyperplane $a_1 u_1 + \dots + a_n u_n + c = 0$ passes through the point $u^0 = (u_1^0, \dots, u_n^0)$, then the complex hyperplane $a_1 e^{-i\psi_1} u_1 + \dots + a_n e^{-i\psi_n} u_n + c = 0$ passes through the point $u^0 e^{i\psi} = (u_1^0 e^{i\psi_1}, \dots, u_n^0 e^{i\psi_n})$.

Note that a complex hyperplane $a_1 e^{i\psi_1} u_1 + \dots + a_n e^{i\psi_n} u_n + c = 0$ has real dimension $2n-2$ and is the intersection of two mutually perpendicular real hyperplanes (of dimension $2n-1$). We find that for $0 \leqslant \psi_l < 2\pi, l = 1, \dots, n$ a family of complex hyperplanes $a_1 e^{i\psi_1} u_1 + + \dots + a_n e^{i\psi_n} u_n + c = 0$ is n-circular and is a ruled surface of real dimension $2n-1$ which splits $\mathbb{C}^n = \mathbb{R}^{2n}$ into two disjoint sets.

If the point $z = (z_1, \dots, z_n) \in \mathbb{C}^n$ do not belong to a family of complex hyperplanes $a_1 e^{i\psi_1} u_1 + + \dots + a_n e^{i\psi_n} u_n + c = 0$, $0 \leqslant \psi_l < 2\pi$, $l = 1, \dots, n$, then all points $z e^{i\varphi} = (z_1 e^{i\varphi_1}, \dots, z_n e^{i\varphi_n}) \in \mathbb{C}^n$, $0 \leqslant \varphi_l < 2\pi$, $l = 1, \dots, n$ do not belong to this family of complex hyperplanes. That is, the set of points $z = (z_1, \dots, z_n) \in \mathbb{C}^n$ that do not belong to the family of complex hyperplanes is also an n-circular set. Now we can investigate the mutual position of two n-circular sets in $\mathbb{C}^n$ using the projection of $\mathbb{C}^n$ in $\mathbb{R}^n_+$ according to

$$\pi : (z_1, \dots, z_n) \to (|z_1|, \dots, |z_n|). \tag{4}$$

Let a complex hyperplane V in $\mathbb{C}^n$ be given by the equation:

$$V = \{u = (u_1, \dots, u_n) \in \mathbb{C}^n : a_1 u_1 + \dots + a_n u_n + c = 0\}. \tag{5}$$

A description of the projection (5) to $\mathbb{R}^n_+$ of a complex hyperplane of $a_1 u_1 + \dots + a_n u_n + c = 0$ in $\mathbb{C}^n$ is given in [2] (Proposition 4.3). Let $|V| = \pi(V)$. It is given by the system of inequalities:

$$|V| = \begin{cases} +|a_1||u_1| - |a_2||u_2| - |a_3||u_3| - \dots - |a_n||u_n| - |c| \leqslant 0, \\ -|a_1||u_1| + |a_2||u_2| - |a_3||u_3| - \dots - |a_n||u_n| - |c| \leqslant 0, \\ \vdots \\ -|a_1||u_1| - |a_2||u_2| - |a_3||u_3| - \dots - |a_n||u_n| + |c| \leqslant 0. \end{cases} \tag{6}$$

If $|\xi_1| = 1, \dots, |\xi_n| = 1$, then the projection π maps each hyperplane of $\{V\} = \{a_1 \xi_1 u_1 + \dots + + a_n \xi_n u_n + c = 0\}$ to the same projection in $\mathbb{R}^n_+$, i.e. $\pi(V) = \pi(\{V\})$. The system of inequalities (6) 'splits' the points of $\mathbb{R}^n_+$ into $n+1$ disjoint parts:

$$\begin{array}{lr} +|a_1||u_1| - |a_2||u_2| - |a_3||u_3| - \dots - |a_n||u_n| - |c| > 0, & (\Pi_1) \\ -|a_1||u_1| + |a_2||u_2| - |a_3||u_3| - \dots - |a_n||u_n| - |c| > 0, & (\Pi_2) \\ \dots\dots\dots \quad \dots\dots\dots \quad \dots\dots & \dots \\ -|a_1||u_1| - |a_2||u_2| - |a_3||u_3| - \dots + |a_n||u_n| - |c| > 0, & (\Pi_n) \\ -|a_1||u_1| - |a_2||u_2| - |a_3||u_3| - \dots - |a_n||u_n| + |c| > 0. & (\Pi_{n+1}) \end{array} \tag{7}$$

If $a_k = 0$ in (5), then $\Pi_k = \varnothing$. If in (5) $c = 0$, then $\Pi_{n+1} = \varnothing$.

If the point $z = (z_1, \dots, z_n) \in \mathbb{C}^n$ does not belong to the family of complex hyperplanes (6), then $\pi(z)$ belongs to one of $\Pi_1, \dots, \Pi_{n+1}$.

Consider a family of complex hyperplanes $V_j = \{z \in \mathbb{C}^n : a_{j,1} u_1 + \dots + a_{j,n} u_n + c_j = 0\}$, $j = 1, \dots, m$. Let $(\Pi_k)_j$ be the set of points of $\mathbb{R}^n_+$ defined by the inequality $|a_{j,1}||u_1| - |a_{j,2}||u_2| - - \dots - |a_{j,(k-1)}||u_{k-1}| + |a_{j,k}||u_k| - |a_{j,(k+1)}||u_{k+1}| - \dots - |a_{j,n}||u_n| - |c| > 0$.

Let for an n-circular set G the set $|G| = \pi(G)$ in $\mathbb{R}^n_+$ belong to $(\Pi_{k_1})_1 \cap \ldots \cap (\Pi_{k_m})_m$. Then the collection of sets $V_1, \ldots, V_m; G$ consider the corresponding set of numbers $(k_1, \ldots, k_m)$, where $1 \leqslant k_j \leqslant n+1$. The result of calculation of the integral (1) depends on this set of numbers.

Note that the number of variants of mutual location of the point $z = (z_1, \ldots, z_n)$ that does not belong to the family of the complex hyperplanes $\{V_1\}, \ldots, \{V_m\}$ equals to $(n+1)^m$.

One approach to evaluation of the integral (1) is the method of binomial expansion of fractions $1/(a_{j,1}z_1\xi_1 + \ldots + a_{j,n}z_n\xi_n + c_j)^{t_j}$ based on one of the inequalities (7).

In [3] the integral (1) is computed in the case when the corresponding set of numbers is $(n+1, \ldots, n+1)$, that is, when

$$-|a_{j,1}||z_1| - \ldots - |a_{j,n}||z_n| + |c| > 0,\ j = 1, \ldots, m. \tag{8}$$

Here in Theorem 1 we compute (1) also for the case $(n+1, \ldots, n+1)$ using the method of coefficients [4], and the Mellin transform for the function under the integral sign.

Note that the calculation results in both cases coincide.

2. Proof of the theorem

Theorem 1. *If the conditions (2) and (8) are satisfied, then the following formula is valid:*

$$J = \frac{(-1)^{(s_1+\ldots+s_n)} \cdot z_1^{s_1} \cdot \ldots \cdot z_n^{s_n}}{(t_1-1)! \cdot \ldots \cdot (t_m-1)! \cdot c_1^{t_1} \cdot \ldots \cdot c_m^{t_m}} \times$$
$$\times \sum_{l_{11},\ldots,l_{m1}\in\mathbb{N}_0}^{l_{11}+\ldots+l_{m1}=s_1} \frac{a_{11}^{l_{11}} \cdot \ldots \cdot a_{m1}^{l_{m1}}}{l_{11}! \cdot \ldots \cdot l_{m1}!} \cdots \sum_{l_{1n},\ldots,l_{mn}\in\mathbb{N}_0}^{l_{1n}+\ldots+l_{mn}=s_n} \frac{a_{1n}^{l_{1n}} \cdot \ldots \cdot a_{mn}^{l_{mn}}}{l_{1n}! \cdot \ldots \cdot l_{mn}!} \times$$
$$\times \frac{(t_1 - 1 + (l_{11} + \ldots + l_{1n}))!}{c_1^{(l_{11}+\ldots+l_{1n})}} \cdot \ldots \cdot \frac{(t_m - 1 + (l_{m1} + \ldots + l_{mn}))!}{c_m^{(l_{m1}+\ldots+l_{mn})}}. \tag{9}$$

If $|\xi_1| = 1, \ldots, |\xi_n| = 1$, then according to (2) and (8) we have

$$\left| \frac{a_{j,1}z_1\xi_1 + \ldots + a_{j,n}z_n\xi_n}{c_j} \right| < 1,\ \ j = 1, \ldots, m. \tag{10}$$

Let us denote

$$\tilde{a}_{j,q} = \frac{a_{j,q}}{c_j},\ j = 1, \ldots, m,\ q = 1, \ldots, n. \tag{11}$$

If $|\xi_1| = 1, \ldots, |\xi_n| = 1$, then according to (10) it follows

$$|\tilde{a}_{j,1}z_1\xi_1 + \ldots + \tilde{a}_{j,n}z_n\xi_n| < 1,\ j = 1, \ldots, m. \tag{12}$$

Then

$$-1 < Re(\tilde{a}_{j,1}z_1\xi_1 + \ldots + \tilde{a}_{j,n}z_n\xi_n) < 1,\ j = 1, \ldots, m. \tag{13}$$

and

$$-1 < 0 < Re(\tilde{a}_{j,1}z_1\xi_1 + \ldots + \tilde{a}_{j,n}z_n\xi_n + 1) < 2,\ j = 1, \ldots, m. \tag{14}$$

Let us denote

$$J = \frac{1}{c_1^{t_1} \cdot \ldots \cdot c_m^{t_m}} res_{\xi_1,\ldots,\xi_n} \left\{ \frac{1}{\xi_1^{s_1+1} \ldots \xi_n^{s_n+1} \prod\limits_{j=1}^{m} (\tilde{a}_{j,1}z_1\xi_1 + \ldots + \tilde{a}_{j,n}z_n\xi_n + 1)^{t_j}} \right\}, \tag{15}$$

where the formal residue $res_{\xi_1,\dots,\xi_n}\{A(\xi_1,\dots,\xi_n)\}$ is the coefficient at $(\xi_1\times\dots\times\xi_n)^{-1}$ of the formal power Laurent series $A(\xi_1,\dots,\xi_n)$ containing only finitely many terms with negative powers (see [4]).

According to the general scheme of the method of coefficients [4] we substitute in (15) each factor under the sign res_ξ by the known Mellin formula

$$\frac{1}{\alpha^j}=\frac{1}{(j-1)!}\int_0^\infty e^{-\alpha z}z^{j-1}dz,\ Re\,\alpha>0, \tag{16}$$

and according to (16) and (12) we have

$$\frac{1}{(\tilde a_{j1}z_1\xi_1+\dots+\tilde a_{jn}z_n\xi_n+1)^{t_j}}=\frac{1}{(t_j-1)!}\int_0^\infty e^{-(\tilde a_{j,1}z_1\xi_1+\dots+\tilde a_{jn}z_n\xi_n+1)w_j}w_j^{t_j-1}dw_j,$$

$j=1,\dots,m,$. Thus

$$J=\frac{1}{c_1^{t_1}\cdot\ldots\cdot c_m^{t_m}\cdot(t_1-1)!\cdot\ldots\cdot(t_m-1)!}\cdot res_{\xi_1,\dots,\xi_n}\left\{\frac{1}{\xi_1^{s_1+1}\dots\xi_n^{s_n+1}}\times\right.$$
$$\left.\times\left\{\prod_{j=1}^m\int_0^\infty e^{-(\tilde a_{j,1}z_1\xi_1+\dots+\tilde a_{j,n}z_n\xi_n+1)w_j}w_j^{t_j-1}dw_j\right\}\right\}=$$
$$=\frac{1}{c_1^{t_1}\cdot\ldots\cdot c_m^{t_m}\cdot(t_1-1)!\cdot\ldots\cdot(t_m-1)!}\cdot res_{\xi_1,\dots,\xi_n}\left\{\frac{1}{\xi_1^{s_1+1}\dots\xi_n^{s_n+1}}\times\right.$$
$$\left.\times\int_0^\infty\dots\int_0^\infty e^{-\sum\limits_{j=1}^m(\tilde a_{j1}z_1\xi_1+\dots+\tilde a_{jn}z_n\xi_n+1)w_j}\cdot w_1^{(t_1-1)}\cdot\ldots\cdot w_m^{(t_m-1)}\cdot dw_1\wedge\dots\wedge dw_m\right\}=$$
$$=\frac{1}{c_1^{t_1}\cdot\ldots\cdot c_m^{t_m}\cdot(t_1-1)!\cdot\ldots\cdot(t_m-1)!}\cdot\int_0^\infty\dots\int_0^\infty e^{-\sum\limits_{p=1}^m w_p}\times$$
$$\times\prod_{q=1}^n res_{\xi_q}\left\{\frac{1}{\xi_q^{s_q+1}}e^{-\xi_q z_q\sum\limits_{j=1}^m w_j\tilde a_{jq}}\right\}\cdot w_1^{(t_1-1)}\cdot\ldots\cdot w_m^{(t_m-1)}\cdot dw_1\wedge\dots\wedge dw_m=$$

$(res_x\{e^{\alpha x}/x^{k+1}\}=\alpha^k/k!)$

$$=\frac{1}{c_1^{t_1}\cdot\ldots\cdot c_m^{t_m}\cdot(t_1-1)!\cdot\ldots\cdot(t_m-1)!}\cdot\int_0^\infty\dots\int_0^\infty e^{-\sum\limits_{p=1}^m w_p}\cdot\frac{(-z_1)^{s_1}\cdot(\sum_{j=1}^m w_j\tilde a_{j,1})^{s_1}}{s_1!}\cdot\ldots\times$$
$$\times\frac{(-z_n)^{s_n}\cdot(\sum_{j=1}^m w_j\tilde a_{j,n})^{s_n}}{s_n!}\cdot w_1^{(t_1-1)}\cdot\ldots\cdot w_m^{(t_m-1)}\cdot dw_1\wedge\dots\wedge dw_m=$$

$\Big($Since $(w_1\tilde a_{1q}+\dots+w_m\tilde a_{mq})^{s_q}=\sum\limits_{l_{1q},\dots,l_{mq}\in\mathbb N_0}^{l_{1q}+\dots+l_{mq}=s_q}\frac{s_q!\cdot(w_1\tilde a_{1q})^{l_{1q}}\cdot\ldots\cdot(w_m\tilde a_{mi})^{l_{mq}}}{l_{1q}!\cdot\ldots\cdot l_{mq}!}$, we have$\Big)$

$$=\frac{(-1)^{(s_1+\dots+s_n)}\cdot z_1^{s_1}\cdot\ldots\cdot z_n^{s_n}}{c_1^{t_1}\cdot\ldots\cdot c_m^{t_m}\cdot(t_1-1)!\cdot\ldots\cdot(t_m-1)!}\cdot\int_0^\infty\dots\int_0^\infty e^{-\sum_{p=1}^m w_p}\cdot w_1^{(t_1-1)}\cdot\ldots\cdot w_m^{(t_m-1)}\times$$
$$\times\sum_{l_{11},\dots,l_{m1}\in\mathbb N_0}^{l_{11}+\dots+l_{m1}=s_1}\frac{(w_1\tilde a_{11})^{l_{11}}\cdot\ldots\cdot(w_m\tilde a_{m1})^{l_{m1}}}{l_{11}!\cdot\ldots\cdot l_{m1}!}\cdot\ldots\times$$
$$\times\sum_{l_{1n},\dots,l_{mn}\in\mathbb N_0}^{l_{1n}+\dots+l_{mn}=s_n}\frac{(w_1\tilde a_{1n})^{l_{1n}}\cdot\ldots\cdot(w_m\tilde a_{mn})^{l_{mn}}}{l_{1n}!\cdot\ldots\cdot l_{mn}!}\cdot dw_1\wedge\dots\wedge dw_m=$$
$$=\frac{(-1)^{(s_1+\dots+s_n)}\cdot z_1^{s_1}\cdot\ldots\cdot z_n^{s_n}}{c_1^{t_1}\cdot\ldots\cdot c_m^{t_m}\cdot(t_1-1)!\cdot\ldots\cdot(t_m-1)!}\times$$

$$\times \sum_{l_{11},\dots,l_{m1}\in\mathbb{N}_0}^{l_{11}+\dots+l_{m1}=s_1} \cdots \sum_{l_{1n},\dots,l_{mn}\in\mathbb{N}_0}^{l_{1n}+\dots+l_{mn}=s_n} \frac{\tilde{a}_{11}^{l_{11}}\cdot\ldots\cdot\tilde{a}_{m1}^{l_{m1}}}{l_{11}!\cdot\ldots\cdot l_{m1}!}\cdot\ldots\cdot\frac{\tilde{a}_{1n}^{l_{1n}}\cdot\ldots\cdot\tilde{a}_{mn}^{l_{mn}}}{l_{1n}!\cdot\ldots\cdot l_{mn}!}\times$$
$$\times \int_0^\infty e^{-w_1} w_1^{(t_1-1+l_{11}+\dots+l_{1n})}dw_1\cdot\ldots\cdot\int_0^\infty e^{-w_m} w_m^{(t_m-1+l_{m1}+\dots+l_{mn})}dw_m =$$

$\Big($the computation of each factor in the last expression by the formula: $\int\limits_0^\infty e^{-\alpha z} z^{j-1}dz = \dfrac{(j-1)!}{\alpha^j}\Big)$

$$= \frac{(-1)^{(s_1+\dots+s_n)}\cdot z_1^{s_1}\cdot\ldots\cdot z_n^{s_n}}{c_1^{t_1}\cdot\ldots\cdot c_m^{t_m}\cdot(t_1-1)!\cdot\ldots\cdot(t_m-1)!}\times$$
$$\times \sum_{l_{11},\dots,l_{m1}\in\mathbb{N}_0}^{l_{11}+\dots+l_{m1}=s_1} \cdots \sum_{l_{1n},\dots,l_{mn}\in\mathbb{N}_0}^{l_{1n}+\dots+l_{mn}=s_n} \frac{\tilde{a}_{11}^{l_{11}}\cdot\ldots\cdot\tilde{a}_{m1}^{l_{m1}}}{l_{11}!\cdot\ldots\cdot l_{m1}!}\cdot\ldots\cdot\frac{\tilde{a}_{1n}^{l_{1n}}\cdot\ldots\cdot\tilde{a}_{mn}^{l_{mn}}}{l_{1n}!\cdot\ldots\cdot l_{mn}!}\times$$
$$\times(t_1-1+l_{11}+\dots+l_{1n})!\cdot\ldots\cdot(t_m-1+l_{m1}+\dots+l_{mn})! =$$

(the change of the variables (11))

$$= \frac{(-1)^{(s_1+\dots+s_n)}\cdot z_1^{s_1}\cdot\ldots\cdot z_n^{s_n}}{c_1^{t_1}\cdot\ldots\cdot c_m^{t_m}\cdot(t_1-1)!\cdot\ldots\cdot(t_m-1)!}\times$$
$$\times \sum_{l_{11},\dots,l_{m1}\in\mathbb{N}_0}^{l_{11}+\dots+l_{m1}=s_1} \cdots \sum_{l_{1n},\dots,l_{mn}\in\mathbb{N}_0}^{l_{1n}+\dots+l_{mn}=s_n} \frac{\left(\frac{a_{11}}{c_1}\right)^{l_{11}}\cdot\ldots\cdot\left(\frac{a_{m1}}{c_m}\right)^{l_{m1}}}{l_{11}!\cdot\ldots\cdot l_{m1}!}\cdot\ldots\cdot\frac{\left(\frac{a_{1n}}{c_1}\right)^{l_{1n}}\cdot\ldots\cdot\left(\frac{a_{mn}}{c_m}\right)^{l_{mn}}}{l_{1n}!\cdot\ldots\cdot l_{mn}!}\times$$
$$\times(t_1-1+l_{11}+\dots+l_{1n})!\cdot\ldots\cdot(t_m-1+l_{m1}+\dots+l_{mn})! =$$
$$= \frac{(-1)^{(s_1+\dots+s_n)}\cdot z_1^{s_1}\cdot\ldots\cdot z_n^{s_n}}{c_1^{t_1}\cdot\ldots\cdot c_m^{t_m}\cdot(t_1-1)!\cdot\ldots\cdot(t_m-1)!}\times$$
$$\times \sum_{l_{11},\dots,l_{m1}\in\mathbb{N}_0}^{l_{11}+\dots+l_{m1}=s_1} \cdots \sum_{l_{1n},\dots,l_{mn}\in\mathbb{N}_0}^{l_{1n}+\dots+l_{mn}=s_n} \frac{a_{11}^{l_{11}}\cdot\ldots\cdot a_{m1}^{l_{m1}}}{l_{11}!\cdot\ldots\cdot l_{m1}!}\cdot\ldots\cdot\frac{a_{1n}^{l_{1n}}\cdot\ldots\cdot a_{mn}^{l_{mn}}}{l_{1n}!\cdot\ldots\cdot l_{mn}!}\times$$
$$\times\frac{(t_1-1+l_{11}+\dots+l_{1n})!}{c_1^{(l_{11}+\dots+l_{1n})}}\cdot\ldots\cdot\frac{(t_m-1+l_{m1}+\dots+l_{mn})!}{c_m^{(l_{m1}+\dots+l_{mn})}}.$$

Remark 1. *If the conditions $a_1z_1\xi_1+\dots+a_nz_n\xi_n+c\neq 0$ are fulfilled for $|\xi_1|=1,\dots,|\xi_n|=1$ and $-|a_1||z_1|-\dots-|a_n||z_n|+|c|>0$, then from (9) we get*

$$\frac{1}{(2\pi i)^n}\int_{|\xi_1|=1}\cdots\int_{|\xi_n|=1}\frac{d\xi_1\wedge\ldots\wedge d\xi_n}{\xi_1^{s_1+1}\cdot\ldots\cdot\xi_n^{s_n+1}(a_1z_1\xi_1+\dots+a_nz_n\xi_n+c)^t} =$$
$$= \frac{(-1)^{(s_1+\dots+s_n)}\cdot z_1^{s_1}\cdot\ldots\cdot z_n^{s_n}}{(t-1)!\cdot c^t}\cdot\frac{a_1^{s_1}\cdot\ldots\cdot a_n^{s_n}}{s_1!\cdot\ldots\cdot s_n!}\cdot\frac{(t-1+(s_1+\dots+s_n))!}{c^{(s_1+\dots+s_n)}}. \quad (17)$$

Corollary. The following combinatorial identity is valid:

$$\frac{1}{(t_1-1)!\cdot\ldots\cdot(t_m-1)!}\cdot\sum_{l_{11},\dots,l_{m1}\in\mathbb{N}_0}^{l_{11}+\dots+l_{m1}=s_1}\frac{(t_1-1+(l_{11}+\dots+l_{1n}))!}{l_{11}!\cdot\ldots\cdot l_{m1}!}\cdot\ldots\times$$
$$\times\sum_{l_{1n},\dots,l_{mn}\in\mathbb{N}_0}^{l_{1n}+\dots+l_{mn}=s_n}\frac{(t_m-1+(l_{m1}+\dots+l_{mn}))!}{l_{1n}!\cdot\ldots\cdot l_{mn}!} = \frac{(t_1+\dots+t_m-1+(s_1+\dots+s_n))!}{(t_1+\dots+t_m-1)!\cdot s_1!\cdot\ldots\cdot s_n!}. \quad (18)$$

Conclusion

Earlier in [5] the method of coefficients and the Mellin transform were first effectively used in the theory of rational summation. Here we give an original example of application of the same apparatus for computing multiple integrals of a certain type depending on parameters and obtain new non-trivial combinatorial identities.

Authors consider a possibility of application of the Mellin transform in analysis and calculation of integrals of different type of rational functions arising in various areas of mathematics.

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Вычисление интеграла от рациональной функции по остову единичного полицилиндра в $\mathbb{C}^n$ с помощью преобразования Меллина

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С помощью преобразования Меллина приведено простое вычисление одного кратного интеграла от рациональной функции от нескольких независимых параметров, возникшего в работе [2]. Эффективность этого преобразования обусловлена тем, что в вычислениях степень полинома выступает как степень монома. Ранее в [5] преобразование Меллина впервые было успешно использовано в теории рационального суммирования. Рассматривается возможность применения преобразования Меллина при анализе и вычислении интегралов различного типа от рациональной функции.

Ключевые слова: интегральные представления, преобразования Меллина, комбинаторные тождества.