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Influence of the Interfacial Internal Energy on the Thermocapillary Steady Flow

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Two-dimensional creeping motion of a two immiscible viscous heat-conducting fluids on the interface for which the surface tension depends linearly on the temperature is investigated. On solid walls the temperature has extreme values and this agrees well with the velocity field of the Hiemenz's type. At small Marangoni numbers an exact solution of arising inverse boundary value problem is found. The estimation of degree of influence of the interfacial internal energy on the stationary flow is given.

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1. Statement and transformation of the problem

The system of the two-dimensional stationary motions of viscous heat-conducting fluids in the absence of mass forces has the form

$$u_1 u_{1x} + u_2 u_{1y} + \frac{1}{\rho} p_x = \nu(u_{1xx} + u_{1yy}); \tag{1.1}$$

$$u_1 u_{2x} + u_2 u_{2y} + \frac{1}{\rho} p_y = \nu(u_{2xx} + u_{2yy}); \tag{1.2}$$

$$u_{1x} + u_{2y} = 0; \tag{1.3}$$

$$u_1 \theta_x + u_2 \theta_y = \chi(\theta_{xx} + \theta_{yy}), \tag{1.4}$$

where $u_1(x, y)$, $u_2(x, y)$ are the components of the velocity vector, $p(x, y)$ is the pressure, $\theta(x, y)$ is the temperature, $\rho > 0$, $\nu > 0$, $\chi > 0$ are the density, the kinematic viscosity, the thermal diffusivity, respectively. The values of ρ, ν, χ are represented by constants.

Suppose, that $u_1 = u_1(x, y)$, $u_2 = v(y)$, $p = p(x, y)$, $\theta = \theta(x, y)$ is solution of the system (1.1)–(1.4). Substitution of this solution in equations (1.1)–(1.3) leads to relations

$$\begin{gathered} u_1 = w(y)x + g(y), \quad w + v_y = 0, \\ w_t + v w_y + w^2 = f + \nu w_{yy}, \quad \frac{1}{\rho} p = d(y) - \frac{f x^2}{2}, \\ d_y = \nu v_{yy} - v v_y, \quad v g_y + w g = 0 \end{gathered} \tag{1.5}$$

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with an arbitrary constant f.

Equation (1.4) for the temperature is rewritten as

$$(wx+g)\theta_x + v\theta_y = \chi(\theta_{xx}+\theta_{yy}).$$

Among its solutions there are quadratic one relatively of the variable x

$$\theta = a(y)x^2 + m(y)x + b(y). \tag{1.6}$$

Below, for simplicity, we assume that $g(y) \equiv 0$ and $m(y) \equiv 0$. The latter means that the temperature field has an extremum at the point $x = 0$, more precisely, at $a(y,t) < 0$ it has a maximum and at $a(y,t) > 0$ it has a minimum. Let us apply the solution (1.5), (1.6) to describe the two-layer motion of the viscous heat-conducting fluids in the flat layer with solid walls $y = 0$, $y = h$ and common interface Γ $y = l(x)$, see Fig. 1.

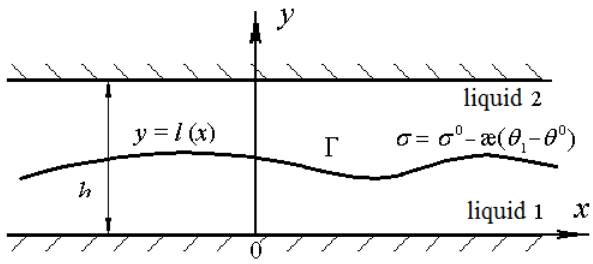

Fig. 1. Schematic diagram of liquid flows in a horizontal layer with interface

Let us introduce index $j = 1, 2$, fixing the fluid. Then in the domain $0 < y < l(x)$ the functions $w_1(y)$, $v_1(y)$ satisfy the equations

$$v_1 w_{1y} + w_1^2 = \nu_1 w_{1yy} + f_1, \quad w_1 + v_{1y} = 0. \tag{1.7}$$

Upon that

$$\frac{1}{\rho_1} p_1 = d_1(y) - \frac{f_1 x^2}{2}, \quad d_{1y} = \nu_1 v_{1yy} - v_1 v_{1y}. \tag{1.8}$$

Similarly, in the domain $l(x) < y < h$ yields

$$v_2 w_{2y} + w_2^2 = \nu_2 w_{2yy} + f_2, \quad w_2 + v_{2y} = 0; \tag{1.9}$$

$$\frac{1}{\rho_2} p_2 = d_2(y) - \frac{f_2 x^2}{2}, \quad d_{2y} = \nu_2 v_{2yy} - v_2 v_{2y}. \tag{1.10}$$

Besides, in the same domains of definition ($j = 1, 2$) the unknowns a_j, b_j satisfy the equations

$$2w_j a_j + v_j a_{jy} = \chi_j a_{jyy}; \tag{1.11}$$

$$v_j b_{jy} = \chi_j b_{jyy} + 2\chi_j a_j. \tag{1.12}$$

On the interface $y = l(x)$ the following conditions are imposed [1]:

$$w_1(l(x)) = w_2(l(x)), \quad v_1(l(x)) = v_2(l(x)); \tag{1.13}$$

$$xw_1(l(x))l_x = v_1(l(x)); \tag{1.14}$$

$$a_1(l(x)) = a_2(l(x)), \quad k_2\frac{\partial a_2}{\partial n} - k_1\frac{\partial a_1}{\partial n} = æa_1w_1; \tag{1.15}$$

$$b_1(l(x)) = b_2(l(x)), \quad k_2\frac{\partial b_2}{\partial n} - k_1\frac{\partial b_1}{\partial n} = æb_1w_1. \tag{1.16}$$

Here k_1, k_2 are the constant coefficients of heat conductivity and normal to curve $y = l(x)$ is the vector $\mathbf{n} = (1 + l_x^2)^{-1/2}(-l_x, 1)$.

The order of the relation of the right side of equality (1.15) (or (1.16)) to the first term of its left-hand side is estimated by the parameter $E = æ^2\theta^*/\mu_2k_2$ (μ_1k_1 should be put for the second case), where θ^* is characteristic temperature on the interface. These parameters for ordinary liquids at room temperature are small [2]. So, in the experiments for the air–ethanol system at $\theta^* = 15\,^\circ C$ we have $E \sim 5 \cdot 10^{-4}$. Therefore, the right-hand side in (1.15) and (1.16) is often omitted and it is said about the equalities of the heat flux across the interface. However, for liquids with low viscosity these terms must be taken into account. Calculations [3] carried out for the bubbles motion in various liquids show that values $E = O(1)$ are reached at sufficiently high temperatures. This means that the viscosity rapidly decreases with increasing temperature. Besides, the same fact occurs for some cryogenic liquids, for example, for liquid CO_2. The maximum values of E near the critical points are reached. So for the water $E \sim 0.02$ at $\theta = 303.15\ K$; $E \sim 0.6$ at $\theta = 573.15\ K$; $E \sim 0.7$ at $\theta = 623.15\ K$ (critical point for water $\theta_{\text{кр}} = 647.30\ K$). In the present work the influence of the right-hand side of (1.15) on flow dynamics will be taken into account in the framework of the creeping flow model.

Dynamic condition on the interface has the following form [1]

$$(p_1 - p_2)\mathbf{n} + 2[\mu_2D(\mathbf{u}_2) - \mu_1D(\mathbf{u}_1)]\mathbf{n} = 2\sigma K\mathbf{n} + \nabla_{11}\sigma, \quad \mu_j = \rho_j\nu_j, \tag{1.17}$$

where $\sigma(\theta_1)$ is the surface tension coefficients, K is the average curvature of the interfaces, $\nabla_{11} = \nabla - \mathbf{n}(\mathbf{n}\cdot\nabla)$ is the the surface gradient, $D(\mathbf{u})$ is the velocity–strain tensor. Further, we suppose that (see Fig. 1)

$$\sigma(\theta_1) = \sigma^0 - æ(\theta_1 - \theta^0), \tag{1.18}$$

$\sigma^0 > 0$, $æ > 0$ are the constants, θ^0 is the temperature in the some point of the interface.

Projecting (1.17) on the tangential directions $\boldsymbol{\tau} = (1 + l_x^2)^{-1/2}(1, l_x)$ and using dependence (1.18) we obtain

$$[\mu_2D(\mathbf{u}_2) - \mu_1D(\mathbf{u}_1)]\mathbf{n}\cdot\boldsymbol{\tau} = -æ\nabla_{11}\theta_1\cdot\boldsymbol{\tau} \tag{1.19}$$

at $y = l(x)$, $\mathbf{u}_j = (xw_j(y), v_j(y))$. In our case

$$D(\mathbf{u}_j) = \begin{pmatrix} w_j & \dfrac{x}{2}\,w_{jy} \\ \dfrac{x}{2}\,w_{jy} & v_{jy} \end{pmatrix}. \tag{1.20}$$

Now we rewrite condition (1.19) taking into account representation (1.6) and (1.20) for the temperature at $m = 0$

$$l_x[\mu_2(v_{2y} - w_2) - \mu_1(v_{1y} - w_1)] + \frac{x}{2}\,(1 - l_x^2)(\mu_2w_{2y} - \mu_1w_{1y}) = \\ = -æ(\theta_{1x} + l_x\theta_{1y}) = -æ[2a_1x + l_x(a_{1y}x^2 + b_{1y})]. \tag{1.21}$$

Projection (1.17) on the normal $\mathbf{n}$ with use of formulae for the pressure from (1.5) results in the equality

$$\rho_1 d_1 - \rho_2 d_2 + \frac{[\rho_2 f_2 - \rho_1 f_1]x^2}{2} + 2[\mu_2 D(\mathbf{u}_2) - \mu_1 D(\mathbf{u}_1)]\mathbf{n}\cdot\mathbf{n} = \\ = [\sigma^0 - æ(a_1 x^2 + b_1)]\frac{l_{xx}}{(1+l_x^2)^{3/2}}\,. \quad (1.22)$$

Boundary conditions on the solid walls are the following

$$w_1(0) = 0, \quad v_1(0) = 0, \quad w_2(h) = 0, \quad v_2(h) = 0; \quad (1.23)$$

$$a_1(0) = a_{10}, \quad a_2(h) = a_{20}; \quad (1.24)$$

$$b_1(0) = b_{10}, \quad b_2(h) = b_{20} \quad (1.25)$$

with specified constants a_{j0}, b_{j0}, $j = 1, 2$.

Conditions (1.24), (1.25) correspond to the temperature on solid walls is given. Another condition can be specified, for example, the top wall is thermally insulated: $a_{2y}(h) = 0$, $b_{2y}(h) = 0$.

Note the following features of the problem. It is strongly nonlinear and inverse, since constants f_j are unknowns also. It is easy to understand this, if we exclude $v_j(y)$ from the second equations in (1.7), (1.9). Then the problem reduces to the conjugate problem for functions $w_j(y)$, $a_j(y)$ and $l(x)$. The problem for functions $b_j(y,t)$ separates at the known functions $v_j(y,t)$ and $a_j(y,t)$. The functions $d_j(y,t)$ can be restored by quadratures from (1.8), (1.10) up to time functions. The second boundary condition in (1.13) and the last condition in (1.23) are helpful for determining of the constants f_j, $j = 1, 2$.

In real situations, for many liquid media the value σ^0 is very large. Therefore, the relation (1.21) gives $l_{xx} = 0$, i. e. $l(x) = \alpha x + l$ and at $\sigma^0 \to \infty$ the interface can be straight only. Further, we assume that it is parallel to the solid walls $y = 0$, $y = h$, therefore $\alpha = 0$, $l = \text{const}$. The solution of the problem is found in the following form

$$w_j = \varepsilon w_j^{(1)} + \varepsilon^2 w_j^{(2)} + \ldots, \quad v_j = \varepsilon v_j^{(1)} + \varepsilon^2 v_j^{(2)} + \ldots,$$

$$a_j = \varepsilon a_j^{(1)} + \varepsilon^2 a_j^{(2)} + \ldots, \quad b_j = \varepsilon b_j^{(1)} + \varepsilon^2 b_j^{(2)} + \ldots, \quad f_j = \varepsilon f_j^{(1)} + \varepsilon^2 f_j^{(2)} + \ldots,$$

where ε is the formally small parameter. Substituting these expressions into the corresponding equations and boundary conditions and passing to the limit at $\varepsilon \to 0$, we obtain for $w_j^{(1)}$, $v_j^{(1)}$, $a_j^{(1)}$, $b_j^{(1)}$, $f_j^{(1)}$ the linear problem. The boundary conditions (1.15), (1.16) will be homogeneous for the problem, i.e. the effect of interfacial energy on the motion is absent.

In the first approximation the problem has the form

$$w_{jyy}^{(1)} = -\frac{f_j^{(1)}}{\nu_j}, \quad w_j^{(1)} + v_{jy}^{(1)} = 0, \\ a_{jyy}^{(1)} = 0, \quad b_{jyy}^{(1)} = -2a_j^{(1)} \quad (1.26)$$

with boundary conditions

$$w_1^{(1)}(0) = 0, \quad v_1^{(1)}(0) = 0, \quad w_2^{(1)}(h) = 0, \quad v_2^{(1)}(h) = 0,$$

$$a_1^{(1)}(0) = a_{10}, \quad a_2^{(1)}(h) = a_{20}, \quad b_1^{(1)}(0) = b_{10}, \quad b_2^{(1)}(h) = b_{20},$$

$$k_2 a_{2y}^{(1)}(l) = k_1 a_{1y}^{(1)}(l), \quad k_2 b_{2y}^{(1)}(l) = k_1 b_{1y}^{(1)}(l), \quad a_1^{(1)}(l) = a_2^{(1)}(l),$$
$$b_1^{(1)}(l) = b_2^{(1)}(l), \quad v_1^{(1)}(l) = v_2^{(1)}(l) = 0, \quad w_1^{(1)}(l) = w_2^{(1)}(l),$$
$$\mu_2 w_{2y}^{(1)}(l) - \mu_1 w_{1y}^{(1)}(l) = -2æ a_1^{(1)}(l). \tag{1.27}$$

The second approximation leads to inhomogeneous equations within the domains of definition ($0 < y < l$ at $j = 1$ and $l < y < h$ at $j = 2$)

$$w_{jyy}^{(2)} = -\frac{f_j^{(2)}}{\nu_j} + \frac{1}{\nu_j}\left[v_j^{(1)} w_{jy}^{(1)} + \left(w_j^{(1)}\right)^2\right], \quad w_j^{(2)} + v_{jy}^{(2)} = 0,$$
$$a_{jyy}^{(2)} = \frac{1}{\chi_j}\left(2w_j^{(1)} a_j^{(1)} + v_j^{(1)} a_{jy}^{(1)}\right), \quad b_{jyy}^{(2)} = -2a_j^{(2)} + \frac{1}{\chi_j} v_j^{(1)} b_j^{(1)}. \tag{1.28}$$

In boundary conditions (1.27) the following changes takes place (upper index "1" should be changed by index "2")

$$a_1^{(2)}(0) = 0, \quad a_2^{(2)}(h) = 0, \quad b_1^{(2)}(0) = 0, \quad b_2^{(2)}(h) = 0,$$
$$k_2 a_{2y}^{(2)}(l) - k_1 a_{1y}^{(2)}(l) = æ a_1^{(1)}(l) w_1^{(1)}(l), \tag{1.29}$$
$$k_2 b_2^{(2)}(l) - k_1 b_1^{(2)}(l) = æ b_1^{(1)}(l) w_1^{(1)}(l).$$

2. Solution of boundary value problems of the first and second approximation

Problem (1.26), (1.27) has solution $w_j^{(1)}(y)$, $a_j^{(1)}(y)$, $f_j^{(1)}$:

$$w_1^{(1)}(y) = \frac{æ(1-\gamma)Ah(3y^2/h^2 - 2\gamma y/h)}{2\gamma\mu_2[\gamma + \mu(1-\gamma)]},$$
$$w_2^{(1)}(y) = \frac{æ\gamma Ah(3y^2/h^2 - 2(2+\gamma)y/h + 1 + 2\gamma)}{2(1-\gamma)\mu_2[\gamma + \mu(1-\gamma)]},$$
$$a_1^{(1)}(y) = \frac{(a_{20} - a_{10})}{[\gamma + k(1-\gamma)]}\frac{y}{h} + a_{10}, \tag{2.1}$$
$$a_2^{(1)}(y) = \frac{1}{\gamma + k(1-\gamma)}\left[k(a_{20} - a_{10})\frac{y}{h} + ka_{10} + \gamma(1-k)a_{20}\right],$$
$$f_1^{(1)} = -\frac{3æ\nu(1-\gamma)A}{\gamma h\rho_2[\gamma + \mu(1-\gamma)]}, \quad f_2^{(1)} = -\frac{3æ\gamma A}{(1-\gamma)h\rho_2[\gamma + \mu(1-\gamma)]},$$

where $k = k_1/k_2$, $\nu = \nu_1/\nu_2$, $\gamma = l/h < 1$, $\mu = \mu_1/\mu_2$,

$$A = \frac{(a_{20} - a_{10})\gamma}{\gamma + k(1-\gamma)}. \tag{2.2}$$

Velocities $v_j^{(1)}(y)$ are found by integrating $w_j^{(1)}(y)$:

$$v_1^{(1)}(y) = -\frac{æ(1-\gamma)Ah^2}{2\gamma\mu_2[\gamma + \mu(1-\gamma)]}\left(\frac{y^3}{h^3} - \frac{\gamma y^2}{h^2}\right),$$
$$v_2^{(1)}(y) = -\frac{æ\gamma Ah^2}{2(1-\gamma)\mu_2[\gamma + \mu(1-\gamma)]}\left[\frac{y^3}{h^3} - \gamma^3 - (2+\gamma)\left(\frac{y^2}{h^2} - \gamma^2\right) + (1+2\gamma)\left(\frac{y}{h} - \gamma\right)\right]. \tag{2.3}$$

Functions $b_j^{(1)}(y)$ are obtained also:

$$b_1^{(1)}(y) = -2h^2\left[a_{10}\frac{y^2}{2h^2} + \frac{(a_{20}-a_{10})}{6[\gamma+k(1-\gamma)]}\,\frac{y^3}{h^3}\right] + Cy + b_{10},$$
$$b_2^{(1)}(y) = -\frac{2h^2}{\gamma+k(1-\gamma)}\left\{\frac{k(a_{20}-a_{10})}{6}\,\frac{y^3}{h^3} + [ka_{10}+\gamma(1-k)a_{20}]\frac{y^2}{2h^2}\right\} + D_1y + D_2, \tag{2.4}$$

where $k = k_1/k_2$,

$$C = \frac{b_{20}}{h[\gamma+k(1-\gamma)]} + \frac{h}{3[\gamma+k(1-\gamma)]^2}\times$$
$$\times\left\{2\left[k+\gamma(k-1)\Big(3k(1-\gamma)^2-\gamma^2\Big)\right]a_{10} + \left[k-\gamma(k-1)\Big(4\gamma^2-6\gamma+3\Big)\right]a_{20}\right\},$$
$$D_1 = kC - \frac{2l(k-1)[\gamma a_{20}+k(1-\gamma)a_{10}]}{\gamma+k(1-\gamma)}\,, \tag{2.5}$$
$$D_2 = l(1-k)C + \frac{2l^2(k-1)}{3[\gamma+k(1-\gamma)]}\left\{[\gamma+3k(1-\gamma)]a_{10}+2\gamma a_{20}\right\}.$$

To calculate the second approximation, which is the solution of problem (1.28), (1.29), we introduce the notation

$$F_j(y) = \frac{1}{\nu_j}\left[v_j^{(1)}w_{jy}^{(1)} + \left(w_j^{(1)}\right)^2\right],$$
$$H_j(y) = \frac{1}{\chi_j}\left(2w_j^{(1)}a_j^{(1)} + v_j^{(1)}a_{jy}^{(1)}\right). \tag{2.6}$$

It is clear that $F_j(y)$ are polynomials of the fourth degree by y, a $H_j(y)$ are polynomials of the third degree. Further calculations in comparison with the finding first approximation are rather long and therefore only main stages will be describes below.

Integration of equations for $w_j^{(2)}$, $a_j^{(2)}$ from the system (1.28) leads to representations

$$w_j^{(2)}(y) = m_j^1y + m_j^2 - \frac{f_j^{(2)}y^2}{2\nu_j} + \int_l^y (y-z)F_j(z)\,dz,$$
$$a_j^{(2)}(y) = n_j^1y + n_j^2 + \int_l^y (y-z)H_j(z)\,dz \tag{2.7}$$

with constants m_j^1, m_j^2, n_j^1, n_j^2, $j=1,2$. The constants $f_j^{(2)}$ are also unknown.

Taking into account the sticking to the walls $y=0$ $(j=1)$ и $y=h$ $(j=2)$ the functions $v_j^{(2)}(y)$ are found from the equations of mass conservation in layers:

$$v_1^{(2)}(y) = -\int_0^y w_1^{(2)}(z)\,dz, \quad v_2^{(2)}(y) = -\int_y^h w_2^{(2)}(z)\,dz. \tag{2.8}$$

The following integral equalities are valid

$$\int_0^l w_1^{(2)}(z)\,dz = 0, \quad \int_l^h w_2^{(2)}(z)\,dz = 0, \tag{2.9}$$

since $v_1^{(2)}(l) = 0$, $v_2^{(2)}(l) = 0$ on the interface. Thus, there are ten boundary conditions to determine ten constants m_j^i, n_j^i, $f_j^{(2)}$, $i, j = 1, 2$: (2.9) and

$$\begin{gathered} w_1^{(2)}(0) = 0, \quad w_2^{(2)}(h) = 0, \quad a_1^{(2)}(0) = 0, \quad a_2^{(2)}(h) = 0, \\ w_1^{(2)}(l) = w_2^{(2)}(l), \quad a_1^{(2)}(l) = a_2^{(2)}(l), \\ \mu_2 w_{2y}^{(2)}(l) - \mu_1 w_{1y}^{(2)}(l) = -2æ a_1^{(2)}(l), \quad k_2 a_{2y}^{(2)}(l) - k_1 a_{1y}^{(2)}(l) = æ a_1^{(1)}(l) w_1^{(1)}(l). \end{gathered} \tag{2.10}$$

Substitution of representations (2.7) in conditions (2.9), (2.10) allows one to find the above-mentioned constants uniquely

$$m_1^2 = \int_l^0 zF_1(z)\,dz, \quad n_1^2 = \int_l^0 zH_1(z)\,dz, \quad n_1^1 = \frac{k_2 D_1 - (h-l)D_2}{k_2 h[\gamma + k(1-\gamma)]},$$

$$D_1 = \int_0^l zH_1(z)\,dz - \int_l^h (h-z)H_2(z)\,dz, \quad D_2 = \frac{\gamma(1-\gamma)æ^2 Ah[\gamma a_{20} + k(1-\gamma)a_{10}]}{2\mu_2[\gamma + \mu(1-\gamma)][\gamma + k(1-\gamma)]},$$

$$n_2^1 = \frac{k_1 D_1 + lD_2}{k_2 h[\gamma + k(1-\gamma)]}, \quad n_2^2 = -\int_l^h (h-z)H_2(z)\,dz - \frac{k_1 D_1 + lD_2}{k_2[\gamma + k(1-\gamma)]},$$

$$m_1^1 = \frac{\mu_2 D_3 - (h-l)D_4}{\mu_2 h[\gamma + \mu(1-\gamma)]},$$

$$D_3 = \frac{l^2}{2\nu_1} f_1^{(2)} + \frac{h^2(1-\gamma^2)}{2\nu_2} f_2^{(2)} + \int_0^l zF_1(z)\,dz - \int_l^h (h-z)F_2(z)\,dz,$$

$$D_4 = -2æ(n_1^1 l + n_1^2) + l\rho_2 f_2^{(2)} - l\rho_1 f_1^{(2)},$$

$$m_2^1 = \frac{\mu_1 D_3 + lD_4}{\mu_2 h[\gamma + \mu(1-\gamma)]}, \quad m_2^2 = \frac{h^2}{2\nu_2} f_2^{(2)} - \int_l^h (h-z)F_2(z)\,dz - \frac{\mu_1 D_3 + lD_4}{\mu_2[\gamma + \mu(1-\gamma)]}, \tag{2.11}$$

$$f_1^{(2)} = \frac{1}{\Delta}\left\{(\gamma-1)[4\gamma + \mu(1-\gamma)]K_1 - 3\nu(1-\gamma)^2 K_2\right\}, \quad \nu = \frac{\nu_1}{\nu_2},$$

$$f_2^{(2)} = \frac{1}{\Delta}\left\{3\rho\gamma^2 K_1 + \gamma[\gamma + 4\mu(1-\gamma)]K_2\right\}, \quad \rho = \frac{\rho_1}{\rho_2},$$

$$\Delta = 4\gamma(1-\gamma)[\gamma + \mu(1-\gamma)]^2,$$

$$K_1 = \frac{12\nu_1 æ(\gamma-1)(n_1^1 l + n_1^2)}{\mu_2 h} - \frac{12\nu_1[\gamma + \mu(1-\gamma)]}{hl^2} \int_0^l \int_l^y (y-z)F_1(z)\,dz\,dy + \\ + \frac{6\nu_1[\gamma + 2\mu(1-\gamma)]}{hl} \int_0^l zF_1(z)\,dz + \frac{6\nu_1}{h^2} \int_l^h (h-z)F_2(z)\,dz,$$

$$K_2 = \frac{12æ\gamma(n_1^1 l + n_1^2)}{\rho_2 h} - \frac{6\mu\nu_2}{h^2} \int_0^l zF_1(z)\,dz - \frac{6\nu_2[2\gamma + \mu(1-\gamma)]}{(1-\gamma)h^2} \times \\ \times \int_l^h (h-z)F_2(z)\,dz + \frac{12\nu_2[\gamma + \mu(1-\gamma)]}{(1-\gamma)^2 h^3} \int_l^h \int_l^y (y-z)F_2(z)\,dz\,dy.$$

So, the values n_j^i $(i, j = 1, 2)$, m_1^2 are calculated from the first approximation and the input data. Then $f_j^{(1)}$, $j = 1, 2$ are found; with their help we determine the constants D_3, D_4, and hence, m_1^1, m_2^1, m_2^2. Functions $w_j^{(2)}(y)$, $a_j^{(2)}(y)$ are given by formula (2.7), and $v_j^{(2)}$ are given by equalities (2.8). It should be noted that the functions $w_j^{(1)}$, $v_j^{(1)}$ are proportional to æ, and $w_j^{(2)}$, $v_j^{(2)}$ to æ^2.

3. Analysis of the influence of internal energy on the example of liquids layers of equal thickness

Suppose for simplicity that $\gamma = l/h = 1/2$ and upper wall $y = h$ is under the influence of a constant temperature, i. e. $a_{20} = 0$. We introduce the notation of the Marangoni number in a convenient form for computation ($a_{10} \neq 0$)

$$\mathrm{M} = \frac{æ a_{10} h^3}{\chi_1 \mu_2 (k+1)(\mu+1)}\,, \quad k = \frac{k_1}{k_2}\,, \quad \mu = \frac{\mu_1}{\mu_2}\,. \tag{3.1}$$

In this case, the characteristic temperature is $\theta^* = a_{10}h^2$. Then, in the dimensionless form, the formulas of the first approximation (2.1), (2.3) (recall that $b_j^{(1)}$ from (2.4) does not affect convection) are

$$\begin{gathered} W_1^{(1)}(\xi) = \frac{w_1^{(1)}(y)h^2}{\chi_1} = -\mathrm{M}(3\xi^2 - \xi), \\ A_1^{(1)}(\xi) = \frac{a_1^{(1)}(y)}{a_{10}} = 1 - \frac{2\xi}{k+1}\,, \\ V_1^{(1)}(\xi) = \frac{v_1^{(1)}(y)h}{\chi_1} = -\mathrm{M}\Big(\xi^3 - \frac{1}{2}\,\xi^2\Big), \\ F_1^{(1)} = \frac{f_1^{(1)}h^4}{\chi_1^2} = 6P_1\mathrm{M} \end{gathered} \tag{3.2}$$

at $0 \leqslant \xi = y/h \leqslant 1/2$, $P_1 = \nu_1/\chi_1$ is Prandtl number of the first liquid;

$$\begin{gathered} W_2^{(1)}(\xi) = \frac{w_2^{(1)}(y)h^2}{\chi_1} = -\mathrm{M}(3\xi^2 - 5\xi + 2), \\ A_2^{(1)}(\xi) = \frac{a_2^{(1)}(y)}{a_{10}} = \frac{2k(1-\xi)}{k+1}\,, \\ V_2^{(1)}(\xi) = \frac{v_2^{(1)}(y)h}{\chi_1} = \mathrm{M}\Big(\xi^3 - \frac{5}{2}\,\xi^2 + 2\xi - \frac{1}{2}\Big), \\ F_2^{(1)} = \frac{f_2^{(1)}h^4}{\chi_1^2} = \frac{6P_2\mathrm{M}}{\chi} \end{gathered} \tag{3.3}$$

at $1/2 \leqslant \xi = y/h \leqslant 1$, $P_2 = \nu_2/\chi_2$ is the Prandtl number of the second liquid, and $\chi = \chi_1/\chi_2$ is the ratio of thermal diffusivity coefficients.

To calculate the second approximation, we give the form of the functions F_j, H_j. The integrals of these functions are included in the representations for $w_j^{(2)}$, $v_j^{(2)}$ and all the constants. Denoting for brevity

$$B = \frac{æ a_{10} h}{\mu_2 (k+1)(\mu+1)}\,, \tag{3.4}$$

we get

$$F_1(\xi) = \frac{B^2}{\nu_1}\Big(3\xi^4 - 2\xi^3 + \frac{\xi^2}{2}\Big),$$
$$H_1(\xi) = -\frac{2a_{10}B}{\chi_1(k+1)}\Big[-5\xi^3 + \Big(3k + \frac{9}{2}\Big)\xi^2 - (k+1)\xi\Big] \tag{3.5}$$

at $0 \leqslant \xi \leqslant 1/2$;

$$F_2(\xi) = \frac{B^2}{\nu_2}\Big(3\xi^4 - 10\xi^3 + \frac{25}{2}\xi^2 - 7\xi + \frac{3}{2}\Big),$$
$$H_2(\xi) = -\frac{2ka_{10}B}{\chi_2(k+1)}\Big(-5\xi^3 + \frac{27}{2}\xi^2 - 12\xi + \frac{7}{2}\Big) \tag{3.6}$$

at $1/2 \leqslant \xi \leqslant 1$. Taking into account the first part of the formulas (2.11) and functions $H_1(\xi)$, $H_2(\xi)$ from (3.5), (3.6) we find

$$n_1^1 = \frac{a_{10}\mathrm{M}}{4(k+1)^2h}\Big(\frac{3k\chi - 2k + 1}{24} + kM_0\Big), \quad n_1^2 = \frac{a_{10}\mathrm{M}(2k-1)}{192(k+1)},$$
$$n_2^1 = \frac{ka_{10}\mathrm{M}}{4(k+1)^2h}\Big(\frac{3k\chi - 2k + 1}{24} - M_0\Big), \tag{3.7}$$
$$n_2^2 = \frac{ka_{10}\mathrm{M}}{64(k+1)}\Big[\chi - \frac{16}{k+1}\Big(\frac{3k\chi - 2k + 1}{24} - M_0\Big)\Big],$$

with dimensionless constant

$$M_0 = \frac{æ\chi_1}{k_2h}. \tag{3.8}$$

Therefore

$$A_1^{(2)}(\xi) = \frac{a_1^{(2)}(y)}{a_{10}} = \frac{\mathrm{M}}{k+1}\Big\{\frac{1}{4}\Big[\frac{1}{k+1}\Big(\frac{3k\chi - 2k + 1}{24} + kM_0\Big) - \frac{1}{8}\Big]\xi + $$
$$+ \frac{(k+1)}{3}\xi^3 - \frac{(2k+3)}{4}\xi^4 + \frac{1}{2}\xi^5\Big\}, \quad 0 \leqslant \xi \leqslant \frac{1}{2},$$
$$A_2^{(2)}(\xi) = \frac{a_2^{(2)}(y)}{a_{10}} = \frac{k\mathrm{M}}{k+1}\Big[\chi\Big(-\frac{7}{32} + \frac{47}{32}\xi - \frac{7}{2}\xi^2 + 4\xi^3 - \frac{9}{4}\xi^4 + \frac{1}{2}\xi^5\Big) + \tag{3.9}$$
$$+ \frac{1}{4(k+1)}\Big(\frac{3k\chi - 2k + 1}{24} + kM_0\Big)(\xi - 1)\Big], \quad \frac{1}{2} \leqslant \xi \leqslant 1.$$

Similarly, is calculated $W_j^{(2)}(\xi)$. The final form of the functions $A_j(\xi) = A_j^{(1)}(\xi) + A_j^{(2)}(\xi)$ and $W_j(\xi) = W_j^{(1)}(\xi) + W_j^{(2)}(\xi)$ are

$$A_1(\xi) = 1 - \frac{2\xi}{k+1} + \frac{\mathrm{M}}{k+1}\Big\{\frac{1}{32(k+1)}\Big[\frac{1}{3}(3k\chi - 2k + 1) - 1\Big]\xi + $$
$$+ \frac{(k+1)}{3}\xi^3 - \frac{(2k+3)}{4}\xi^4 + \frac{1}{2}\xi^5\Big\} + \frac{k}{4(k+1)^2}\mathrm{M}_1\xi, \quad 0 \leqslant \xi \leqslant \frac{1}{2}, \tag{3.10}$$

$$A_2(\xi) = \frac{2k}{k+1}(1 - \xi) + \frac{k\mathrm{M}}{k+1}\Big\{\chi\Big(-\frac{7}{32} + \frac{47}{32}\xi - \frac{7}{2}\xi^2 + 4\xi^3 - \frac{9}{4}\xi^4 + \frac{1}{2}\xi^5\Big) + $$
$$+ \frac{3k\chi - 2k + 1}{96(k+1)}(\xi - 1)\Big\} - \frac{k}{4(k+1)^2}\mathrm{M}_1(\xi - 1), \quad \frac{1}{2} \leqslant \xi \leqslant 1,$$

$$
\begin{gathered}
\mathrm{M}_1=\frac{æ^2 a_{10}h^2}{\mu_2 k_2(\mu+1)(k+1)}=\frac{\mathrm{E}}{(\mu+1)(k+1)}\,,\\
W_1(\xi)=\frac{w_1(y)h^2}{\chi_1\mathrm{M}}=-3\xi^2+\xi-\frac{1}{2}F_1\xi^2+m_1\xi+\frac{\mathrm{M}}{\mathrm{P}_1}\Big(\frac{1}{10}\xi^6-\frac{1}{10}\xi^5+\frac{1}{24}\xi^4\Big),\quad 0\leqslant\xi\leqslant\frac{1}{2}\,,\\
W_2(\xi)=\frac{w_2(y)h^2}{\chi_1\mathrm{M}}=-3\xi^2+5\xi-2+\frac{1}{2}F_2(1-\xi^2)+m_2(\xi-1)+\\
+\frac{\nu\mathrm{M}}{\mathrm{P}_1}\Big(\frac{1}{10}\xi^6-\frac{1}{2}\xi^5+\frac{25}{24}\xi^4-\frac{7}{6}\xi^3+\frac{3}{4}\xi^2-\frac{9}{40}\Big),\quad \frac{1}{2}\leqslant\xi\leqslant 1,\\
F_1=\frac{\mathrm{M}}{(\mu+1)\mathrm{P}_1}\Big(\frac{1}{70}+\frac{5\mu}{224}+\frac{9\nu}{1120}-\frac{3k(\mu+1)\mathrm{P}_1}{4(k+1)}(3\chi+2k-1)\Big)-\frac{3k\mathrm{M}_1}{4(k+1)}\,,\\
F_2=\frac{\mathrm{M}}{70\mathrm{P}_1}(\nu-1)+F_1,\ m_1=\frac{1}{6}F_1-\frac{\mathrm{M}}{1120\mathrm{P}_1}\,,\ m_2=\frac{5}{6}F_2-\frac{893}{3360}\frac{\nu\mathrm{M}}{\mathrm{M}_1}\,,\\
V_1(\xi)=\frac{v_1(y)h}{\chi_1\mathrm{M}}=-\int_0^{\xi}W_1(\xi)\,d\xi,\quad V_2(\xi)=\frac{v_2(y)h}{\chi_1\mathrm{M}}=-\int_{1/2}^{\xi}W_2(\xi)\,d\xi.
\end{gathered}
\tag{3.11}
$$

In the formulas the terms including M_1 show the contribution of influence of interfacial internal energy on the functions $A_j(\xi)$ and profiles of the longitudinal velocities. We emphasize important feature of the formation of Marangoni finite stresses through increments of the interfacial internal energy [1]. It does not require the inflow of energy into the system from outside in a thermal or chemical form. Such stresses can also be formed in the isothermal state of the interface.

The profiles of the functions $W_j(\xi)$ and the vertical velocities $V_j(\xi)$ are shown in Fig. 2, 3. Under the influence of the parameter, the velocity profiles deform, but this deformation is not significant: $\max\limits_{\xi}|V_j(\xi,\mathrm{M},\mathrm{M}_1)-V_j(\xi,\mathrm{M},0)|\sim 0{,}1$. Of course, this smallness is due to the smallness of the Marangoni number. It is of interest to study the general nonlinear problem even for the isothermal case of the interphase boundary.

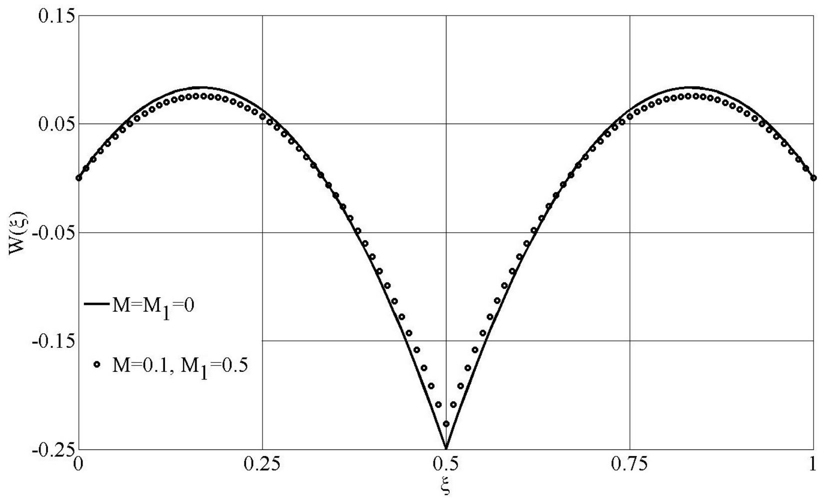

Fig. 2. Profiles of the functions $W_j(\xi)$ at $a_{10}>0$ for model liquids

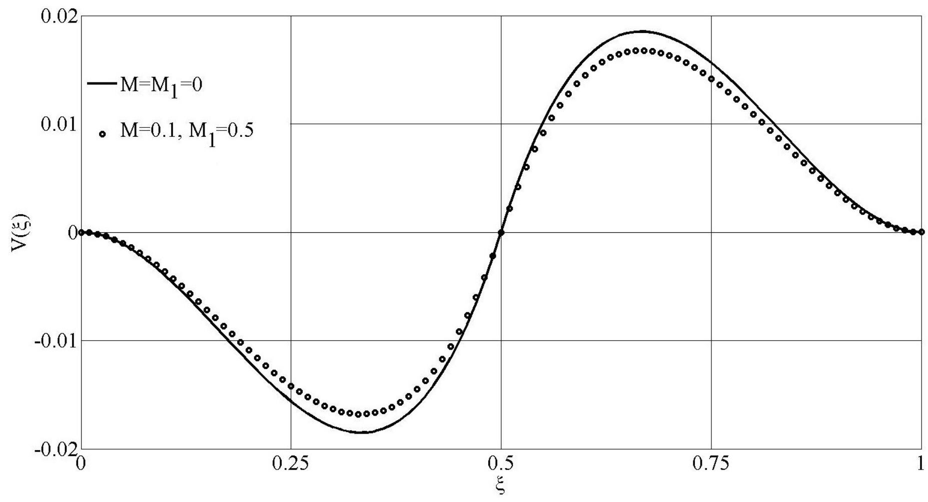

Fig. 3. Profiles of the vertical velocities $V_j(\xi)$ at $a_{10} > 0$

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О влиянии внутренней энергии границы раздела на термокапиллярное течение

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Изучается двумерное ползущее движение двух несмешивающихся вязких теплопроводных жидкостей, на границе раздела которых поверхностное натяжение линейно зависит от температуры. На твердых стенках температура имеет экстремальные значения, что хорошо согласуется с полем скоростей типа Хименца. При малых числах Марангони найдено точное решение возникающей обратной краевой задачи и дана оценка степени влияния внутренней энергии границы раздела на стационарное течение.

Ключевые слова: граница раздела, термокапиллярность, внутренняя энергия границы раздела, обратная задача.