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Navier-Stokes Equations for Elliptic Complexes

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We continue our study of invariant forms of the classical equations of mathematical physics, such as the Maxwell equations or the Lamé system, on manifold with boundary. To this end we interpret them in terms of the de Rham complex at a certain step. On using the structure of the complex we get an insight to predict a degeneracy deeply encoded in the equations. In the present paper we develop an invariant approach to the classical Navier-Stokes equations.

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Introduction

The problem of describing the dynamics of incompressible viscous fluid is of great importance in applications. In 2006 the Clay Mathematics Institute announced it as the sixth prize millennium problem, see [5]. The dynamics is described by the Navier-Stokes equations and the problem consists in finding a classical solution to the equations. By classical we would mean here a solution of a class which is good motivated by applications and for which a uniqueness theorem is available. Essential contributions are published in the research articles [9, 13, 16, 19, 20, 27] as well as surveys and books [6, 17, 18, 21, 22, 29], etc.

In physics by the Navier-Stokes equations is meant the impulse equation for the flow. In the computational fluid dynamics the impulse equation is enlarged by the continuity and energy equations.

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The impulse equation of dynamics of (compressible) viscous fluid was formulated in differential form independently by Claude Navier (1827) and George Stokes (1845). This is

$$\rho(u'_t + u'_x u) = -\mu\Delta u + (\lambda+\mu)\,\nabla\,\mathrm{div}\,u - \nabla p + f, \qquad (0.1)$$

where $u : \mathcal{X} \times (0,T) \to \mathbb{R}^3$ and $p : \mathcal{X} \times (0,T) \to \mathbb{R}$ are the search-for velocity vector field and pressure of a particle in the flow, respectively, ρ is the mass density, λ and μ are the first Lamé constant and the dynamical viscosity of the fluid under consideration, respectively, by u'_x is meant the Jacobi matrix of u in the spatial variables, $\Delta u = -u''_{x_1x_1} - u''_{x_2x_2} - u''_{x_3x_3}$ is the nonnegative Laplace operator in $\mathbb{R}^3$, and f is the density vector of outer forces, such as gravitation and so on, see [29] and elsewhere.

Here, $\mathcal{X}$ stands for a bounded domain in $\mathbb{R}^3$ whose boundary is assumed to be smooth enough. Hence, to specify a particular solution of (0.1), we consider the first mixed problem in the cylinder $\mathcal{X} \times (0,T)$ by posing the initial conditions on the lower basis of the cylinder and a Dirichlet condition on the lateral surface. To wit,

$$\begin{array}{rcll} u(x,0) & = & u_0(x), & \text{for} \quad x \in \mathcal{X}, \\ u(x,t) & = & u_l(x,t), & \text{for} \quad (x,t) \in \partial\mathcal{X} \times (0,T). \end{array} \qquad (0.2)$$

It is worth pointing out that the pressure p is determined solely from the impulse equation up to an additive constant. To fix this constant it suffices to put a moment condition on p.

If the density ρ does not change along the trajectories of particles, the flow is said to be incompressible. It is the assumption that is most often used in applications. For incompressible fluid the continuity equation takes the especially simple form $\mathrm{div}\,u = 0$ in $\mathcal{X} \times (O,T)$, i.e., the vector field u should be divergence free (solenoidal). In many practical problems the flow is not only incompressible but it has even a constant density. In this case one can divide by ρ in (0.1) which reduces the impulse equation to

$$\begin{array}{rcl} u'_t + u'_x u & = & -\nu\Delta u - \nabla p + f, \\ \mathrm{div}\,u & = & 0 \end{array} \qquad (0.3)$$

in $\mathcal{X} \times (0,T)$, where $\nu = \mu/\rho$ is the so-called kinematic viscosity and we use the same letters p and f to designate p/ρ and f/ρ. In this way we obtain what is referred to as but the Navier-Stokes equations.

Using manipulations of the nonlinear term $u'_x u$ Hopf [9] proved that equations (0.3) under homogeneous data (0.2) have a weak solution satisfying the estimate

$$\|u(\cdot,t)\|^2_{L^2(\mathcal{X},\mathbb{R}^3)} + \int_0^t \|u'(\cdot,t')\|^2_{L^2(\mathcal{X},\mathbb{R}^{3\times 3})} dt' \leqslant \|u_0\|^2_{L^2(\mathcal{X},\mathbb{R}^3)} + \int_0^t \|f(\cdot,t')\|^2_{L^2(\mathcal{X},\mathbb{R}^3)} dt'$$

for all $t < T$.

However, in this full generality no uniqueness theorem for a weak solution has been known. On the other hand, under stronger conditions on the solution, it is unique, cf. [17, 18]. In contrast to [5], we believe that the main problem concerning the Navier-Stokes equations consists in removal of this gap, i.e., in specifying adequate function spaces in which both existence and uniqueness theorems are valid. From the viewpoint of pure mathematics this would initiate new problems similar to that the complex Neumann problem gave rise to the study of subelliptic operators, and even greater ones, let alone phenomena evoked by nonlinear perturbations.

This paper is aimed in elaborating another insight into the Navier-Stokes equations. It consists in specifying this problem within the framework of global analysis of elliptic complexes on manifolds.

In global analysis on manifolds $\mathcal{X}$ one deals with sections of vector bundles rather than with functions on $\mathcal{X}$ with values in $\mathbb{R}^k$ or $\mathbb{C}^k$. A vector bundle F is a topological construction that makes precise the idea of a family of vector spaces parametrised by $\mathcal{X}$. With every point $x \in \mathcal{X}$ we associate a vector space F_x in such a way that these vector spaces fit together to locally admit a basis which depends continuously on x. The simplest example is the case where the family of vector spaces is constant, i.e., F reduces to the Cartesian product $\mathcal{X} \times \mathbb{R}^k$ or $\mathcal{X} \times \mathbb{C}^k$. Such vector bundles are called trivial. Each vector bundle is locally trivial. Sections of a vector bundle F over $\mathcal{X}$ form a vector space and they are locally identified with functions with values in $\mathbb{R}^k$ or $\mathbb{C}^k$, where k stands for the rank of F. We consider only differentiable (of class C^∞) manifolds $\mathcal{X}$ and vector bundles F over $\mathcal{X}$. The tangent bundle $T\mathcal{X}$ of $\mathcal{X}$ and F can be endowed with Riemannian metrics which leads to the space $L^2(\mathcal{X}, F)$ of square integrable sections of F over $\mathcal{X}$. The space $C^\infty(\mathcal{X}, F)$ of smooth sections of F over $\mathcal{X}$ can be specified as a dense subspace of $L^2(\mathcal{X}, F)$. If $\mathcal{X}$ is compact then the distribution sections of F over $\mathcal{X}$ become exhausted by the scale of L^2-based Sobolev spaces $H^s(\mathcal{X}, F)$, where $s \in \mathbb{R}$, the intersection of these spaces just amounts to $C^\infty(\mathcal{X}, F)$. By differential operators A between sections of two vector bundles F and G over $\mathcal{X}$ are meant linear mappings of $C^\infty(\mathcal{X}, F)$ into $C^\infty(\mathcal{X}, G)$ which are local in the sense that $\operatorname{supp} Au \subset \operatorname{supp} u$ for all $u \in C^\infty(\mathcal{X}, F)$. Any differential operator A extends to a continuous mapping of $H^s(\mathcal{X}, F)$ into $H^{s-m}(\mathcal{X}, G)$ for each $s \in \mathbb{R}$, and the order m of A is completely determined by this action.

Part 1. Reformulation within elliptic complexes

1. Generalised Navier-Stokes equations

Let $\mathcal{X}$ be a compact differentiable manifold of dimension n with or without boundary. Consider the de Rham complex

$$0 \to \Omega^0(\mathcal{X}) \stackrel{d}{\to} \Omega^1(\mathcal{X}) \stackrel{d}{\to} \ldots \stackrel{d}{\to} \Omega^n(\mathcal{X}) \to 0$$

on $\mathcal{X}$, where $\Omega^i(\mathcal{X})$ are the spaces of differential forms of degree i with C^∞ coefficients on $\mathcal{X}$. The impulse equation can be immediately rewritten in terms of one-forms as

$$\rho u'_t + \mu\, \Delta u + (\lambda + \mu) dd^* u + dp + (u')^* u = f$$

in $\mathcal{X} \times (0, T)$, where d^* is the formal adjoint of d, $\Delta = d^* d + dd^*$ the Laplace operator of Hodge, and $(u')^*$ the dual of the tangential mapping $u'(x) : T_x\mathcal{X} \to T_x\mathcal{X}$. Every term in the equation makes still sense for differential forms u of arbitrary degree $0 \leqslant i \leqslant n$, except for the nonlinear perturbation $(u')^* u$ which is defined solely for one-forms. On the other hand, the specific form $(u')^* u$ does not survive under simple transforms like a shift $u \mapsto u + v$ which are needed to reduce nonzero initial or boundary data to the zero ones. Hence, to specify the nonlinearity we write it in a more abstract form $N(u)$, where N^i is an unbounded nonlinear operator in the space of differential forms of degree i with square integrable coefficients on $\mathcal{X}$ and, as usual, we set $Nu = N^i u$ for $u \in \Omega^i(\mathcal{X})$. Later on we impose an additional condition on N^i which implies an energy estimate.

Hence, the impulse equation generalises to arbitrary step i of the de Rham complex in the form

$$\rho u'_t + \mu\, \Delta u + (\lambda + \mu) dd^* u + dp + N(u) = f$$

while the continuity equation for incompressible fluid reads $d^* u = 0$.

One studies these evolution equations in the open cylinder $\mathcal{C}_T := \overset{\circ}{\mathcal{X}} \times (0, T)$ whose base is the interior of $\mathcal{X}$. Up to the pressure p the linear part of the impulse equation looks like the generalised Lamé system, cf. [23]. The crucial difference lies in the fact that the impulse equation of hydrodynamics is parabolic while the Lamé system is of hyperbolic type. This is clarified within elasticity theory which proceeds from the assumption that the displacement u befalls along the optical fibres similarly to waves.

We now assume that

$$0 \to C^\infty(\mathcal{X}, F^0) \stackrel{A}{\to} C^\infty(\mathcal{X}, F^1) \stackrel{A}{\to} \dots \stackrel{A}{\to} C^\infty(\mathcal{X}, F^N) \to 0 \tag{1.1}$$

is an arbitrary elliptic complex of first order differential operators between sections of vector bundles F^i over $\mathcal{X}$. The differential A of this complex is given by a sequence $A^i \in \mathrm{Diff}^1(\mathcal{X}; F^i, F^{i+1})$ satisfying $A^{i+1}A^i = 0$, where $A^i \equiv 0$ unless $i = 0, 1, \dots, N-1$. We introduce the generalised Navier-Stokes equations by

$$\begin{array}{rcl} u'_t + \nu\, \Delta u + Ap + N(u) & = & f, \\ A^* u & = & 0 \end{array} \tag{1.2}$$

for unknown sections u and p of the (induced) vector bundles F^i and F^{i-1} over $\mathcal{C}_T$, respectively, where $\Delta = A^*A + AA^*$ is the Laplacian of complex (1.1) and N a graded operator corresponding to a sequence $\{N^i\}$ of unbounded nonlinear operators in the spaces $L^2(\mathcal{X}, F^i)$ of square integrable sections of the vector bundles F^i. By the above, we set $\nu = \nu/\rho$.

Example 1.1. For $i = 0$ equations (1.2) reduce obviously to

$$u'_t + \nu\, \Delta u + N(u) = f$$

in C_T because A acts at step -1 as zero. This equation can be thought of as a far-reaching generalisation of the well-known Burgers equation in one spatial variable, see [2], [8].

When posing initial and boundary conditions for a solution (u, p) of (1.2), we observe that the "pressure" p is no longer determined by u up to a finite-dimensional subspace of $L^2(\mathcal{X}, F^{i-1})$, for the null-space of A^{i-1} need not be of finite dimension. Hence, we have to subject p to certain boundary conditions. Since we are going to project the first equation of (1.2) onto the space of solutions to $A^*u = 0$, we look for a suitable boundary condition within the framework of the Neumann problem after Spencer, see [28]. Given any $v \in L^2(\mathcal{X}, F^i)$, it consists in finding a section $g \in L^2(\mathcal{X}, F^i)$ satisfying $\Delta g = v$ in $\mathcal{X}$ and $n(g) = n(Ag) = 0$ at $\partial\mathcal{X}$ in a weak sense. Here, by $n(g)$ is meant the so-called normal part of g at the boundary which bears the Cauchy data of g with respect to A^*. As already mentioned, the study of this problem stimulated to essential development of analysis and geometry in the 1960s.

Lemma 1.2. *Suppose that the Neumann problem is solvable at step i for the complex (1.1) and H and G are the corresponding harmonic projection and the Green operator. Then the operator $Pg := Hg + A^*A\,Gg$ is an orthogonal projection in $L^2(\mathcal{X}, F^i)$.*

Proof. Under the assumption of the lemma, the space of all $g \in L^2(\mathcal{X}, F^i)$ satisfying $\Delta g = 0$ in $\mathcal{X}$ and $n(g) = n(Ag) = 0$ at $\partial\mathcal{X}$ is finite dimensional. The elements of this space are called harmonic sections and they actually satisfy $Ag = A^*g = 0$ in $\mathcal{X}$. The harmonic sections prove to be C^∞ sections of F^i over $\mathcal{X}$, and so the orthogonal projection H onto the space is a smoothing operator. Moreover, there is a compact selfadjoint operator G in $L^2(\mathcal{X}, F^i)$, such that $n(Gg) = n(AGg) = 0$ at $\partial\mathcal{X}$ for all $g \in L^2(\mathcal{X}, F^i)$ and the identity operator in $L^2(\mathcal{X}, F^i)$ splits into $H + A^*AG + AA^*G$. The Green operator G is of pseudodifferential nature. The decomposition $g = Hg + A^*AGg + AA^*Gg$ valid for all $g \in L^2(\mathcal{X}, F^i)$ is usually referred to as the generalised Hodge decomposition. Since $A^2 = 0$, the summands are pairwise orthogonal, and so both A^*AG and AA^*G are orthogonal projections, too. For a thorough discussion of the Neumann problem we refer the reader to [28, Ch. 4]. □

The projector P is an analogue of the Helmholtz projector onto vector fields which are divergence free. A slightly different approach to this decomposition is presented in [17].

Lemma 1.3. *In order that $Pg = g$ be valid it is necessary and sufficient that $A^*g = 0$ in $\mathcal{X}$ and $n(g) = 0$ at $\partial\mathcal{X}$.*

Proof. Suppose that $Pg = g$. Then $A^*g = A^*(Hg + A^*A\,Gg)$ vanishes in $\mathcal{X}$ and $n(g) = n(Hg + A^*A\,Gg)$ vanishes at $\partial\mathcal{X}$, for $n(AGg) = 0$ implies immediately $n(A^*\,AGg) = 0$, as is easy to check. On the other hand, if $A^*g = 0$ in $\mathcal{X}$ and $n(g) = 0$ at $\partial\mathcal{X}$, then an easy calculation shows that $A^*(Gg) = 0$ whence $Pg = g$, as desired. □

From Lemma 1.3 it follows that P vanishes on sections of the form Ap with $p \in L^2(\mathcal{X}, F^{i-1})$ and $Ap \in L^2(\mathcal{X}, F^i)$. Indeed, to prove this it suffices to show that Ap is orthogonal to all sections g satisfying $A^*g = 0$ in $\mathcal{X}$ and $n(g) = 0$ at the boundary. For such a section g we get

$$(Ap, g)_{L^2(\mathcal{X},F^i)} = (p, A^*g)_{L^2(\mathcal{X},F^{i-1})} = 0,$$

as desired.

Since equations (1.2) do not contain any derivative of p in t, we formulate an initial condition

$$u(x, 0) = u_0(x), \tag{1.3}$$

for each $x \in \overset{\circ}{\mathcal{X}}$, on the base of the cylinder $\mathcal{C}_T$, and a boundary condition

$$u(x, t) = u_l(x, t) \tag{1.4}$$

for all $(x, t) \in \partial\mathcal{X} \times (0, T)$ on the lateral surface of $\mathcal{C}_T$. If u is affixed to u_0 at the initial moment $t = 0$ strong enough, then u_0 should inherit from u the condition $A^*u_0 = 0$ in some weak sense in $\mathcal{X}$. Moreover, the normal component of the "velocity" u at the lateral surface should vanish, hence $n(u_l) = 0$ at $\partial\mathcal{X} \times (0, T)$. We thus get

$$\begin{array}{rcll} A^*u_0 & = & 0 & \text{in} \quad \mathcal{X}, \\ n(u_l) & = & 0 & \text{at} \quad \partial\mathcal{X} \times (0, T), \end{array} \tag{1.5}$$

a compatibility condition completing the physical interpretation of (1.2) as generalised Navier-Stokes equations.

If the smoothness of u allows one to control the values of u at $\partial\mathcal{X}$ up to $t = 0$, then (1.5) implies, in particular, that u_0 is a solution of the Cauchy problem for the formal adjoint of A^{i-1} with zero data in $\mathcal{X}$. For $i = N$ the differential operator A^{i-1*} is (possibly, overdetermined) elliptic, and so u_0 is specified within a subspace of $C^\infty(\mathcal{X}, F^N)$ of finite dimension. Note that for $i = 0$ both equations of (1.5) are empty.

Neither of equations (1.3) and (1.4) puts any restriction on the "pressure" p, and so p remains still undetermined. If looking for a $p \in L^2(\mathcal{C}_T, F^{i-1})$ within the framework of the Neumann problem, one can determine p uniquely from the condition that p is orthogonal to the space of all solutions $v \in L^2(\mathcal{X}, F^{i-1})$ of the homogeneous equation $Av = 0$ in $\mathcal{X}$. This solution is called canonical (it amounts to $A^*G(Ap)$).

2. Energy estimates

As is known, one of the main relations for incompressible viscous fluid in a bounded domain $\mathcal{X} \subset \mathbb{R}^n$ with smooth boundary is the so-called energy balance relation

$$\frac{1}{2} \int_{\mathcal{X}} |u|^2 dx \,\Big|_{t'}^{t''} + \nu \int_{t'}^{t''}\!\!\int_{\mathcal{X}} |u'_x|^2 dxdt = \int_{t'}^{t''}\!\!\int_{\mathcal{X}} (f, u)\, dxdt$$

for all $t', t'' \in (0, T)$. It is valid for all sufficiently smooth nonstationary vector fields $u(x, t)$ on the cylinder $\mathcal{C}_T$ over $\mathcal{X}$, satisfying (0.3) under the homogeneous boundary condition $u = 0$ at $\partial\mathcal{X}$. The proof is based on a lemma which provides an insight into the nonlinearity.

Lemma 2.1. *For each $u \in C^1(\overline{\mathcal{X}}, \mathbb{R}^n)$ it follows that*

$$\int_{\mathcal{X}} (u'_x u, u)\, dx = - \int_{\mathcal{X}} \frac{1}{2} |u|^2 \operatorname{div} u\, dx + \int_{\partial \mathcal{X}} \frac{1}{2} |u|^2\, (u, \nu)\, ds,$$

where ds is the area form of the hypersurface $\partial \mathcal{X}$ and $\nu(x)$ the outward unit normal vector at a point $x \in \partial \mathcal{X}$.

Proof. Using the Stokes formula, we get

$$\begin{aligned} \int_{\mathcal{X}} (u'_x u, u)\, dx &= \int_{\mathcal{X}} \sum_{j=1}^{n} \sum_{k=1}^{n} (\partial_k u_j)\, u_k u_j dx \\ &= \int_{\mathcal{X}} \sum_{k=1}^{n} \partial_k \Big(\frac{1}{2} \sum_{j=1}^{n} u_j^2 \Big) u_k\, dx \\ &= - \int_{\mathcal{X}} \frac{1}{2} |u|^2 \operatorname{div} u\, dx + \int_{\partial \mathcal{X}} \frac{1}{2} |u|^2\, (u, \nu)\, ds, \end{aligned}$$

as desired. □

To guarantee an energy estimate for generalised Navier-Stokes equations (1.2) we impose a special restriction on the nonlinear term $N(u)$. In the sequel we assume that

$$(N(u), u)_{L^2(\mathcal{X}, F^i)} = 0 \qquad (2.1)$$

for all $u \in L^2(\mathcal{X}, F^i)$ in the domain of N satisfying $A^* u = 0$ in $\mathcal{X}$ and $n(u) = 0$ at $\partial \mathcal{X}$. Equality (2.1) is fulfilled, in particular, if $(N(u), v)_x = (u, AB(v, u))_x$ pointwise for all $u, v \in C^\infty(\mathcal{X}, F^i)$ up to a term vanishing for $u = v$, where B is a smooth sesquilinear form on $F^i \times F^i$ with values in F^{i-1}. In the classical case we have

$$B(u, v) = \frac{1}{2}\, (u, v)_x.$$

Theorem 2.2. *Let u be a bounded section of Slobodetskii space $H^{2,1}(\mathcal{C}_T, F^i)$ satisfying equations (1.2) in $\mathcal{C}_T$ and vanishing at the lateral surface of the cylinder. Then,*

$$\frac{1}{2} \int_{\mathcal{X}} |u|^2 dx \, \Big|_{t'}^{t''} + \nu \int_{t'}^{t''} \!\! \int_{\mathcal{X}} |Au|^2\, dx dt = \Re \int_{t'}^{t''} \!\! \int_{\mathcal{X}} (f, u)\, dx dt \qquad (2.2)$$

for all $t', t'' \in (0, T)$.

A complete theory of spaces $H^{2s,s}(\mathcal{C}_T, F^i)$ were first elaborated in the basic work [26] including both embedding and trace theorems. It allows one to reduce boundary value problems with inhomogeneous boundary conditions to those with homogeneous boundary conditions. Yet another motivation consists in the study of anisotropic elliptic problems, for example, parabolic problems which include the first mixed problem for the heat equation in a cylinder $\mathcal{C}_T$. For a nonnegative integer s, the norm of $H^{2s,s}(\mathcal{C}_T, F^i)$ controls the derivatives $\partial_x^\alpha \partial_t^j u$ with $|\alpha| + 2j \leqslant 2s$ in the L^2-norm on $\mathcal{C}_T$.

Proof. Since u is bounded, we can take the pointwise scalar product of both sides of the impulse equation of (1.2) with u and integrate it over the cylinder $\mathcal{X} \times (t', t'')$. This gives

$$\int_{t'}^{t''} \!\! \int_{\mathcal{X}} (u'_t + \nu \varDelta u + Ap + N(u), u)_x\, dx dt = \int_{t'}^{t''} \!\! \int_{\mathcal{X}} (f, u)_x\, dx dt$$

for all $t', t'' \in (0,T)$.

It is easily seen that

$$\Re\, (u'_t, u)_x = \frac{1}{2} \frac{\partial}{\partial t} (u,u)_x$$

whence

$$\Re \int_{t'}^{t''} \int_{\mathcal{X}} (u'_t, u)_x dx dt = \frac{1}{2} \int_{\mathcal{X}} |u|^2 dx \,\Big|_{t'}^{t''}$$

by the Newton-Leibniz formula.

Furthermore, using the "continuity equation" $A^* u = 0$ in $\mathcal{X}$ and integration by parts we obtain

$$\int_{\mathcal{X}} (\Delta u, u)_x \, dx \quad = \quad \int_{\mathcal{X}} |Au|_x^2 \, dx + \int_{\partial \mathcal{X}} ((\sigma^i)^* A u, u)_x \, ds,$$

where σ^i is ($\sqrt{-1}$ times) the principal symbol of the differential operator A^i evaluated at the point $(x, \nu) \in T^* \mathcal{X}$. The integral over $\partial \mathcal{X}$ on the right-hand side vanishes, for u has zero Cauchy data with respect to the differential operator A^i at the boundary.

Since $p \in L^2 (\mathcal{X}, F^{i-1})$ and $Ap \in L^2 (\mathcal{X}, F^i)$, it follows from what is said in Section 1. that

$$\int_{\mathcal{X}} (Ap, u)_x \, dx = 0$$

for all $t \in (0,T)$.

Finally, we take into consideration the structure of nonlinearity $N(u)$ described in (2.1) to deduce that

$$\begin{aligned} \int_{\mathcal{X}} (N(u), u)_x \, dx &= \int_{\mathcal{X}} (u, AB(u,u))_x \, dx \\ &= \int_{\mathcal{X}} (A^* u, B(u,u))_x \, dx - \int_{\partial \mathcal{X}} ((\sigma^{i-1})^* u, B(u,u))_x \, ds \\ &= 0 \end{aligned}$$

in much the same way as in the proof of Lemma 2.1. Summarising we arrive at equality (2.2), as desired. □

From Theorem 2.2 it follows readily that for solutions of the generalised Navier-Stokes equations, which vanish at the lateral surface of $\mathcal{C}_T$, one can estimate the energy norm

$$\|u\|_{\mathrm{EN}} := \sup_{0 \leqslant t \leqslant T} \|u\|_{L^2(\mathcal{X}, F^i)} + \|Au\|_{L^2(\mathcal{C}_T, F^{i+1})} \tag{2.3}$$

only through the norms $\|f\|_{L^{2,1}(\mathcal{C}_T, F^i)}$ and $\|u_0\|_{L^2(\mathcal{X}, F^i)}$, where

$$\|f\|_{L^{q,r}(\mathcal{C}_T, F^i)} := \Big(\int_0^T \Big(\int_{\mathcal{X}} |f(x,t)|^q dx \Big)^{r/q} dt \Big)^{1/r},$$

cf. [17].

The set of sections $u(x,t)$ having finite energy norm (2.3) forms a Banach space. Its elements need not be continuous in t in the $L^2(\mathcal{X}, F^i)$-norm. By analogy with other studied problems one might believe that this class is fairly natural for the Navier-Stokes equations. Such a class was first introduced in [9]. However, the class has proved to be too large for the classical Navier-Stokes equations in $\mathbb{R}^3$, for the uniqueness theorem of the first mixed problem is violated in this class. Under finite energy norm, the uniqueness property for the classical Navier-Stokes equations takes place first in $L^{q,r}(\mathcal{C}_T, \mathbb{R}^n)$ with $n/2q + 1/r \leqslant 1/2$, see [17] and [18].

It is worth pointing out that the structure of nonlinearity specified by (2.1) is still too general to introduce weak solutions of class $L^2(\mathcal{C}_T, F^i)$ to the generalised Navier-Stokes equations.

3. First steps towards the solution

On applying the Helmholtz projector to the generalised impulse equation one obtains

$$(Pu)'_t + \nu\, P(\Delta u) + P(Ap) + PN(u) = Pf$$

while the continuity equation means that $Pu = u$ in $\mathcal{C}_T$. Since $P(Ap) = 0$, this allows one to eliminate the "pressure" from the impulse equation, thus obtaining an equivalent form

$$\begin{array}{rcl} (Pu)'_t + \nu\, P(\Delta Pu) + P(Ap) + PN(Pu) & = & Pf, \\ ((I-P)Pu)'_t + \nu\, (I-P)(\Delta Pu) + (I-P)Ap + (I-P)N(Pu) & = & (I-P)f \end{array}$$

of the Navier-Stokes equations, for $(I-P)P = 0$ and

$$\begin{array}{rcl} (I-P)\Delta P & = & (I - H - A^*AG)\, \Delta\, (H + A^*AG) \\ & = & AA^*G\, A^*A\, \Delta G \\ & = & 0, \end{array}$$

the last equality being due to the fact that A^*GA^* vanishes on sections of zero Cauchy data with respect to A^* at $\partial\mathcal{X}$.

In other words, we separate the generalised Navier-Stokes equations into two single problems

$$(Pu)'_t + \nu\, P(\Delta Pu) + P(Ap) + PN(Pu) = Pf$$

in $\mathcal{C}_T$ under the initial and boundary conditions

$$\begin{array}{rcll} Pu\,(x,0) & = & u_0(x), & \text{for} \quad x \in \mathcal{X}, \\ Pu\,(x,t) & = & u_l(x,t), & \text{for} \quad (x,t) \in \partial\mathcal{X} \times (0,T), \end{array}$$

and

$$\begin{array}{rcll} Ap & = & (I-P)\,(f - N(Pu)) & \text{in} \quad \mathcal{C}_T, \\ (p,v)_{L^2(\mathcal{X},F^{i-1})} & = & 0 & \text{for} \quad v \in \ker A. \end{array} \tag{3.1}$$

As already mentioned in Section 1., if the Neumann problem of Spencer is solvable at step i of elliptic complex (1.1), then the only solution of problem (3.1) is given by

$$p = A^*G(I-P)\,(f - N(Pu)).$$

The operator $P\Delta$ is sometimes called the Stokes operator. It is of pseudodifferential nature.

Lemma 3.1. *Suppose that $u \in H^{2,1}(\mathcal{C}_T, F^i)$ is a bounded solution to the first mixed problem*

$$\begin{array}{rcll} u'_t + \nu\, P\Delta u + PN(u) & = & Pf & \text{in} \quad \mathcal{C}_T, \\ u & = & u_0 & \text{at} \quad \mathring{\mathcal{X}} \times \{0\}, \\ u & = & u_l & \text{at} \quad \partial\mathcal{X} \times (0,T) \end{array} \tag{3.2}$$

in the cylinder. Then $A^*u = 0$ in $\mathcal{C}_T$.

Proof. Indeed, from the differential equation of (3.2) it follows that

$$\frac{\partial}{\partial t} A^*u = 0$$

in $\mathcal{C}_T$. Since $A^*u = A^*u_0 = 0$ for $t = 0$, we deduce readily that $A^*u = 0$ for all $t \in (0,T)$, as desired. □

Summarising we choose the following way of solving the generalised Navier-Stokes equations. We first construct a solution u of mixed problem (3.2). According to Lemma 3.1, u satisfies $A^* u = 0$ in $\mathcal{C}_T$, and so $Pu = u$. Substitute this section into equation (3.1) for p. From this equation the "pressure" p is determined uniquely and bears the appropriate regularity of the canonical solution of the Neumann problem for complex (1.1) at step i. Finally, on combining the equations (3.2) and (3.1) we conclude that the pair (u, p) is a solution of (1.2) under conditions (1.3) and (1.4).

In the sequel we focus on the study of operator equation (3.2) by Hilbert space methods.

4. A WKB solution

To handle the nonlinear term $N(u)$ in the generalised Navier-Stokes equations it might be useful to gain a small parameter ε multiplying $N(u)$. To this end we restrict our attention to those N which are of the form $N(u) = N(u, u)$, where $N(u, v)$ is a first order bidifferential operator between sections of F^i on $\mathcal{X}$, as in the classical case. Pick an arbitrary $\varepsilon \neq 0$ and change the dependent variable by $u = \varepsilon \tilde{u}$. Substituting u into (3.2) and using the specific form of nonlinearity to divide both sides by ε, we get

$$\begin{array}{rcll} \tilde{u}'_t + \nu\, P\varDelta \tilde{u} + \varepsilon\, PN(\tilde{u}) & = & P\tilde{f} & \text{in} \ \ \mathcal{C}_T, \\ \tilde{u} & = & \tilde{u}_0 & \text{at} \ \ \overset{\circ}{\mathcal{X}} \times \{0\}, \\ \tilde{u} & = & \tilde{u}_l & \text{at} \ \ \partial\mathcal{X} \times (0, T), \end{array} \tag{4.1}$$

where $\tilde{f} = f/\varepsilon$, $\tilde{u}_0 = u_0/\varepsilon$ and $\tilde{u}_l = u_l/\varepsilon$ are as arbitrary as f, u_0 and u_l if the domains for f, u_0 and u_l are invariant under stretching. We have thus arrived at the same mixed problem for the Navier-Stokes equations in the cylinder $\mathcal{C}_T$ but the problem now contains a small parameter ε multiplying the nonlinear term. By abuse of notation we omit the sign "tilde" and write u, f, u_0 and u_l for the new variables.

By experience with other studied mixed problems for parabolic equations, the problem (4.1) for $\varepsilon = 0$ has a unique solution in $H^{2,1}(\mathcal{C}_T, F^i)$ which depends continuously on the data f, u_0 and u_l. Therefore, we may try to exploit a WKB approximation

$$u(x, t) = \sum_{k=1}^{\infty} c_k(x, t)\, \varepsilon^k$$

to construct a solution for nonlinear problem (4.1), the series being asymptotic for $\varepsilon \to 0$. On substituting this expansion into (4.1) and equating the coefficients of the same powers of ε we get a linear mixed problem for determining the initial approximation c_0

$$\begin{array}{rcll} \dfrac{\partial}{\partial t} c_0 + \nu\, P\varDelta c_0 & = & Pf & \text{in} \ \ \mathcal{C}_T, \\ c_0 & = & u_0 & \text{at} \ \ \overset{\circ}{\mathcal{X}} \times \{0\}, \\ c_0 & = & u_l & \text{at} \ \ \partial\mathcal{X} \times (0, T) \end{array} \tag{4.2}$$

and a system of recurrent equations

$$\begin{array}{rcll} \dfrac{\partial}{\partial t} c_k + \nu\, P\varDelta c_k & = & - \displaystyle\sum_{i+j=k-1} PN(c_i, c_j) & \text{in} \ \ \mathcal{C}_T, \\ c_k & = & 0 & \text{at} \ \ \overset{\circ}{\mathcal{X}} \times \{0\}, \\ c_k & = & 0 & \text{at} \ \ \partial\mathcal{X} \times (0, T) \end{array}$$

for $k = 1, 2, \ldots$.

The recurrent equations display once again the main problem in solving the Navier-Stokes equations. We start with data $f \in L^2(\mathcal{C}_T, F^i)$ and u_0, u_l of relevant regularity. The initial approximation c_0 will belong to the Slobodetskii space $H^{2,1}(\mathcal{C}_T, F^i)$. The mixed problem for c_1 has the right-hand side $-N(c_0, c_0)$ and zero initial and boundary data. To evaluate the right-hand side one uses the so-called multiplicative inequalities, see for instance [17]. However, one can see from the very beginning that $N(c_0, c_0)$ fails to belong to $L^2(\mathcal{C}_T, F^i)$ and so no iteration is possible to determine c_1, etc. within the L^2-approach. If c_0 is additionally bounded then $N(c_0, c_0) \in L^2(\mathcal{X}, F^i)$ and one can find c_1 in $H^{2,1}(\mathcal{C}_T, F^i)$, etc. However, $H^{2,1}(\mathcal{C}_T, F^i)$ is embedded into $L^\infty(\mathcal{C}_T, F^i)$ only for $n = 1$ and $n = 2$ while no other criteria for the existence of a bounded solution have been known. Thus, the WKB approximation gives an evidence to the lack of smoothness controlled by L^2-scales.

On the other hand, if we look for a solution $u \in W^{(2,1),q}(\mathcal{X}, F^i)$ of the Navier-Stokes equations with q large enough, so that $W^{(2,1),q}(\mathcal{X}, F^i)$ is embedded continuously into $C(\mathcal{X}, F^i)$, then the construction of a WKB solution goes through. This motivates the study of the linearised Navier-Stokes equations in Banach spaces $W^{(2,1),q}(\mathcal{X}, F^i)$, where q is sufficiently large. (By the Sobolev embedding theorem, $q > n$ is sufficient).

Part 2. Particular cases

5. Analysis in the case of closed manifolds

In this section we consider in detail the case where $\mathcal{X}$ is a smooth compact closed manifold of dimension n. Recall that the classical Hodge theory extends to elliptic complexes on compact closed manifolds without any essential changes, see for instance [28]. As but one byproduct of this theory we mention the fact that the projector P is a classical pseudodifferential operator of order 0 between sections of the vector bundle F^i on $\mathcal{X}$.

Given an arbitrary section $f \in L^2(\mathcal{C}_T, F^i)$, consider the pseudodifferential equation

$$u'_t + \nu\, P\varDelta u + \varepsilon\, PN(u) = Pf \tag{5.1}$$

in $\mathcal{C}_T$ for an unknown section $u \in H^{2,1}(\mathcal{C}_T, F^i)$, where ε is a small parameter. We tacitly assume that the term $N(u)(\cdot, t)$ belongs to $L^2(\mathcal{X}, F^i)$ for almost all $t \in (0, T)$.

To treat (5.1) within the framework of ordinary differential equations with operator-valued coefficients, we should give the operators proper domains. The closure of $\varDelta$ in $L^2(\mathcal{X}, F^i)$ has domain $H^2(\mathcal{X}, F^i)$ and is nonnegative in this domain. The projector P is obviously nonnegative, hence we make $\nu\, P\varDelta$ into a positive operator by adding λI with any $\lambda > 0$. To this end change the dependent variable by $u = e^{\lambda t}\tilde{u}$. Substituting this into (5.1), dividing by $e^{\lambda t}$ and writing $\tilde{u}$ and $\tilde{f} = e^{-\lambda t} f$ simply u and f we get

$$u'_t + Lu + \varepsilon\, e^{\lambda t} PN(u) = Pf$$

where

$$Lu = P(\nu\, \varDelta + \lambda I)u.$$

It is precisely the abstract form in which we study the Navier-Stokes equations in $\mathcal{C}_T$, cf. [14,15].

Remark 5.1. Combining Lemma 1.3 and Lemma 3.1 makes it reasonable to restrict the domains of operators to the subspace of $L^2(\mathcal{X}, F^*)$ consisting of weak solutions to $A^* u = 0$ in $\mathcal{X}$.

If $\varepsilon = 0$, then the unique solution to (5.1) under the initial condition $u(\cdot, 0) = u_0$ is

$$u(\cdot, t) = e^{-tL} u_0 + \int_0^t e^{-(t-t')L}\, Pf(\cdot, t')\, dt'$$

for $t \in (0,T)$, which we denote by $c_0(\cdot,t)$. On using this formula one reduces (5.1) to a nonlinear integral equation of the Fredholm type $u = c_0 + \varepsilon K(u)$ in $(0,T)$, where

$$K(u)(\cdot,t) = -\int_0^t e^{-(t-t')L}\, e^{\lambda t'} PN(u)(\cdot,t')\, dt'.$$

Since

$$K(u)(\cdot,t) - K(v)(\cdot,t) = -\int_0^t e^{-(t-t')L}\, e^{\lambda t'} P(N(u)(\cdot,t') - N(v)(\cdot,t'))\, dt',$$

the small parameter multiplying $K(u)$ may be useful only for Lipschitz nonlinearities $N(u)$, which is not the case for $N(u)$ on the whole space. This gives an evidence to the fact that the contraction mapping principle does not apply on the whole space.

Lemma 5.2. *Suppose that K is a compact operator in a Hilbert space H. If all solutions of the equation $u = c_0 + \varepsilon' K(u)$ with $\varepsilon' \in (0,\varepsilon]$ lie in a ball $B(c_0,R)$ of finite radius, then the equation $u = c_0 + \varepsilon K(u)$ has at least one solution in the closure of the ball.*

Proof. The lemma follows immediately from the mapping degree theory of Leray-Schauder. Indeed, on increasing R, if necessary, one can assume that the mapping family $h_\vartheta(u) = u - u_0 - \vartheta\varepsilon K(u)$ for $\vartheta \in [0,1]$ does not vanish at the boundary of $B(c_0,R)$. Hence, the mapping degree $\deg(h_\vartheta, B(c_0,R))$ is independent of ϑ. For $\vartheta = 0$, the degree just amounts to 1 by the normalisation property. It follows that $\deg(h_1, B(c_0,R)) = 1$, and so $h_1(u) = 0$ has at least one solution in $B(c_0,R)$, as desired. □

In order to prove that all solutions of the equation $u = c_0 + \varepsilon' K(u)$ with $\varepsilon' \in (0,\varepsilon]$ lie in a ball $B(c_0,R)$ with a finite R, one uses the so-called a priori estimates for the solutions.

For a study of the abstract initial value problem $u = c_0 + \varepsilon K(u)$ within the theory of operator semigroups we refer the reader to [10, 12], etc. It exploits fractional powers of the positive selfadjoint operator L in $L^2(\mathcal{X}, F^i)$ and enables one to prove existence and uniqueness theorems for small intervals $(0,T)$ or for small initial data.

6. Potential equations

Assume that the right-hand side $f \in L^2(\mathcal{C}_T, F^i)$ of the generalised impulse equation

$$u'_t + \nu\,\varDelta u + Ap + N(u) = f$$

is potential, i.e., $f = A\varpi$ for some section $\varpi \in L^2(\mathcal{C}_T, F^{i-1})$. Then it is to be expected that the equation possesses a potential solution $u = A\wp$ in the cylinder with $\wp \in L^2(\mathcal{C}_T, F^{i-1})$. On substituting both f and u in the impulse equation we get the equation

$$A\wp'_t + A\,\nu\varDelta\wp + Ap + N(A\wp) = A\varpi$$

for the unknown potential $\wp$. If this equation possesses a solution then $N(A\wp)$ is a potential again.

Hence it follows that the structure condition $NA = AN$ on the nonlinearity is well motivated by applications in natural sciences. On the other hand, this condition is well understood within the framework of homological algebra, for it specifies the so-called cochain mappings (endomorphisms) of complexes. This condition is fulfilled for the classical Navier-Stokes equations if the nonlinear term at step $i = 0$ is defined by

$$N^0(\wp) := \frac{1}{2}\sum_{k=1}^{n} (\partial_k \wp)^2 .$$

Lemma 6.1. *For any vector field u of the form $u = \wp'$ in a domain $\mathcal{X} \subset \mathbb{R}^n$, where $\wp \in C^2(\mathcal{X})$, we have*

$$u'_x u = (N^0(\wp))'_x,$$

i.e., $N^1 d = dN^0$.

Proof. Since

$$u = \begin{pmatrix} \partial_1 \wp \\ \cdots \\ \partial_n \wp \end{pmatrix},$$

it follows that

$$u'_x u = \begin{pmatrix} \displaystyle\sum_{k=1}^{n} (\partial_k \partial_1 \wp)\, \partial_k \wp \\ \cdots \\ \displaystyle\sum_{k=1}^{n} (\partial_k \partial_n \wp)\, \partial_k \wp \end{pmatrix} = \begin{pmatrix} \displaystyle\partial_1 \sum_{k=1}^{n} \frac{(\partial_k \wp)^2}{2} \\ \cdots \\ \displaystyle\partial_n \sum_{k=1}^{n} \frac{(\partial_k \wp)^2}{2} \end{pmatrix} = \big(N^0(\wp)\big)'_x,$$

as desired. □

Our viewpoint sheds some new light on the generalised Navier-Stokes equations (1.2). More precisely, the structure of the classical Navier-Stokes equations actually specifies the nonlinear term $N(u)$ at each step i through the commutative relations $N^i d = dN^{i-1}$. Since the Neumann problem after Spencer is solvable for the de Rham complex at each step i, the space $L^2(\mathcal{X}, \Lambda^i T^* \mathcal{X})$ splits into the range of P and the range of $I - P$. On the range of P the nonlinearity structure is specified by (2.1). And on the range of $I - P$ which coincides with the range of A the nonlinear term $N(u)$ is uniquely determined by the commutative relations $N^i d = dN^{i-1}$ and by the explicit formula for N^1. For arbitrary elliptic complexes (1.1) we may argue in much the same way if the Neumann problem after Spencer is solvable at each step $i > 0$ for (1.1).

To wit, by a cochain mapping of the complex $C^\infty(\mathcal{X}, F^\cdot)$ is meant any sequence of (possibly, nonlinear) self-mappings N^i of $C^\infty(\mathcal{X}, F^i)$ with the property that the diagram

$$\begin{array}{ccccccccccc} 0 & \to & C^\infty(\mathcal{X}, F^0) & \stackrel{A}{\to} & C^\infty(\mathcal{X}, F^1) & \stackrel{A}{\to} & \ldots & \stackrel{A}{\to} & C^\infty(\mathcal{X}, F^N) & \to & 0 \\ & & \downarrow N & & \downarrow N & & & & \downarrow N & & \\ 0 & \to & C^\infty(\mathcal{X}, F^0) & \stackrel{A}{\to} & C^\infty(\mathcal{X}, F^1) & \stackrel{A}{\to} & \ldots & \stackrel{A}{\to} & C^\infty(\mathcal{X}, F^N) & \to & 0 \end{array}$$

commutes. Our standing assumption on the nonlinear terms N^i of the generalised Navier-Stokes equations will be that they constitute a cochain mapping N of complex (1.1), i.e.,

$$N^i A^{i-1} = A^{i-1} N^{i-1} \tag{6.1}$$

for all $i = 1, \ldots, N$.

The interest of the class of Navier-Stokes equations is that it is closed under building potential equations. Namely, for each $i = 1, \ldots, N$, the Navier-Stokes equations at step $i-1$ are potential equations for those at step i, as is easy to check.

Example 6.2. By Lemma 6.1, the Navier-Stokes equations for the de Rham complex at step $i = 0$ read

$$u'_t + \frac{1}{2}\, |u'_x|^2 = \nu \Delta u + f(x,t), \tag{6.2}$$

u being an unknown function in the cylinder $\mathcal{C}_T$.

Equation (6.2) has been frequently studied as a nonlinear model for the motion of an interface under deposition, when the forcing potential f is random, delta-correlated in both space and time, see [11].

7. The homogeneous Burgers equation

The Cole-Hopf transformation was discovered independently by Hopf [8] and Cole [4] around 1950. It changes Burgers' equation $u'_t + uu'_x = u''_{xx}$ into the heat equation $v'_t = v''_{xx}$. To derive the transform, we let $u = \wp'_x$. Then Burgers' equation can be integrated yielding $\wp'_t + (\wp'_x)^2/2 = \wp''_{xx}$ up to a function depending on t only. Let $\wp = -2 \log v$. Thus, $u = -2v'_x/v$. Applying some algebra to this we get $v'_t = v''_{xx}$.

More generally, the n-dimensional impulse equation $u'_t + u'_x u = \nu \Delta u + f'_x(x,t)$ for a vector field $u = \wp'_x$, which describes the dynamics of a stirred, pressure-less and vorticity-free fluid, has found interesting applications in a wide range of non-equilibrium statistical physics problems, see [1]. The associated Hamilton-Jacobi equation, satisfied by the velocity potential $\wp$, just amounts to equation (6.2) of Example 6.2.

Starting with this example, we now consider a quasilinear partial differential equation

$$\wp'_t = \Delta \wp - a(\wp)\, |\wp'_x|^2 \qquad (7.1)$$

in $\mathbb{R}^{n+1}$, where a is a continuous real-valued function on the real axis. Choose a strictly monotone decreasing C^2 function $v = \mathcal{H}(\wp)$ on $\mathbb{R}$, such that

$$-a(\wp) = \frac{\mathcal{H}''(\wp)}{\mathcal{H}'(\wp)}$$

for all $\wp \in \mathbb{R}$. The general solution of this ordinary differential equation satisfying the initial condition $\mathcal{H}'(0) = \mathcal{H}_1 < 0$ is

$$\mathcal{H}'(\wp) = \exp\Big(- \int_0^{\wp} a(\vartheta) d\vartheta \Big) \mathcal{H}_1,$$

which is a smooth function on $\mathbb{R}$ with positive values. The function $v = \mathcal{H}(\wp)$ may be found by integration. In this way we recover what is referred to as the Cole-Hopf transformation.

A simple computation shows that the change of variables $v = \mathcal{H}(\wp)$ reduces (7.1) to the heat equation

$$v'_t = \Delta v \qquad (7.2)$$

for the new unknown function v. Hence, the general solution to (7.1) is $\wp = \mathcal{H}^{-1}(v)$, with v satisfying (7.2).

Example 7.1. Let a be constant. Then

$$\begin{aligned} \mathcal{H}(\wp) &= \mathcal{H}_0 + \mathcal{H}_1 \frac{1 - \exp(-a\wp)}{a}, \\ \mathcal{H}^{-1}(v) &= \frac{-1}{a} \log \Big(1 - a\, \frac{v - \mathcal{H}_0}{\mathcal{H}_1} \Big). \end{aligned}$$

Using the function $\mathcal{H}$ allows one to endow the set of solutions to equation (7.1) with the symmetry $\wp_1 \circ \wp_2 := \mathcal{H}^{-1}(\mathcal{H}(\wp_1) + \mathcal{H}(\wp_2))$.

In [8], the transformation $\mathcal{H}$ is applied to study the Cauchy problem for the homogeneous Burgers equation $u'_t + \nu \Delta u + u'_x u = 0$, cf. Example 1.1. In the last decades, mathematicians become increasingly interested in problems related to the behaviour of solutions to a partial differential equation in which the highest order terms occur linearly with small coefficients. These problems originate from physical applications, mainly from modern fluid dynamics (compressible fluids of small kinematic viscosity $\nu > 0$ and of small heat conductivity λ). Research in these fields has led to some general mathematical observations, such as the following two. The solution of the initial value problem for equations of fluid flow tends for "most" values to a limit function as both ν and λ tend to zero. The limit function is, in general, discontinuous and is pieced together by solutions of the equations in which those highest order coefficients vanish (ideal fluid with contact and shock discontinuities). These observations are perhaps valid in a much wider range of partial differential equations. The second observation is restricted to nonlinear equations, but it seems to point out a typical occurrence in the general case. Exact formulation and rigorous proof of these observations are still tasks for the future. As is noted in [8], a careful study of special problems is still a commendable way towards greater insight into the matter. Among the partial differential equations studied in this direction one meets rarely those in which the totality of solutions is rigorously determined and in which the passage to the limit can thus be studied in detail. On using the Cole-Hopf transformation one obtains a complete solution for the Burgers equation. It was first introduced in [2] as a simple model for the differential equations of fluid flow. Although the Burgers equation is a too simple model to fully illustrate the statistics of free turbulence, a theory of this equation serves as an instructive introduction into some mathematical problems involved. There is a close analogy between the Burgers equation and the Navier-Stokes equations. However, no additional dependent variables such as pressure, density or temperature appear in the Burgers equation. Nevertheless [2] observed certain analogy between some solutions of the Burgers equation and one-dimensional flows of a compressible fluid. According to [8], Burgers had an intuitive picture of the limit case $\nu \to 0$ in the solutions and determined the origin and the law of propagation of discontinuity. Like [2] the paper [8] studies the boundary-free initial value problem, to wit, given u for all x and $t = 0$, one wants u for all x and $t > 0$. The solution is achieved by an exact integration of the Burgers equation. Both problems, the behaviour of the solution as $t \to \infty$ while ν is constant, and its behaviour as $\nu \to 0$ while the initial data are kept fixed, are treated.

8. Linearised Navier-Stokes equations

The study of a nonlinear equation begins with the study of its linearisation. We need only to consider the linearisation of the impulse equation. The nonlinear term is $N(u) := N(u,u)$, where N is a first order bidifferential operator of the type $F^i \times F^i \to F^i$ on $\mathcal{X}$. Hence it follows that the linearisation at a fixed section u_0 of F^i is

$$\begin{aligned} N(u) &= N(u_0,u_0) + N(u_0, u-u_0) + N(u-u_0, u) = \\ &= -N(u_0) + N(u_0,u) + N(u,u_0) + o(\|u-u_0\|) \end{aligned}$$

for u close to u_0. The question still open is of how to choose a domain for the solution.

Note that both partial differential operators $N(u_0,u)$ and $N(u,u_0)$ have discontinuous coefficients unless u_0 bear excess smoothness. Therefore, the study of relevant linearisations of the Navier-Stokes equations requires a fairly delicate analysis. The best general reference here is [17]. Instead of this we consider the linear mixed problem which underlies the construction of a WKB solution to the Navier-Stokes equations, see Section 4. The mixed problem (4.2) is an evolution equation for the Toeplitz operator $P\varDelta$, where P is the Helmholtz projector. To provide an insight into the problem we neglect P and look for a solution of the mixed problem

$$\begin{aligned} u'_t + \nu\,\varDelta u &= f && \text{in } \mathcal{C}_T, \\ u &= u_0 && \text{at } \overset{\circ}{\mathcal{X}} \times \{0\}, \\ u &= u_l && \text{at } \partial\mathcal{X} \times (0,T), \end{aligned} \qquad (8.1)$$

the right-hand side f being a given section of the bundle F^i over the cylinder $\mathcal{C}_T$ and the initial data u_0 and boundary data u_l being prescribed sections of F^i over the lower basis $\mathcal{X} \times \{0\}$ and the lateral boundary $\partial\mathcal{X} \times (0,T)$ of $\mathcal{C}_T$, respectively. As is usual for evolution equations, no condition is posed on the upper basis of the cylinder.

For a recent account of the theory of mixed problems we refer the reader to Chapter 3 of [7]. The energy method for hyperbolic equations takes a considerable part in [7]. This method automatically extends to $2b$-parabolic differential equations with variable coefficients. In this section we specify the general theory for the parabolic equation $u'_t - \nu\, \Delta u = f$ including the Laplacian Δ of elliptic complex (1.1).

By a classical solution of problem (8.1) is meant any section $u \in C^{2,1}_{\mathrm{loc}}(\mathcal{C}_T, F^i)$ which is continuous up to $\mathcal{X} \times \{0\}$ and $\partial\mathcal{X} \times (0,T)$ and satisfies pointwise the equations of (8.1).

Since the case of inhomogeneous boundary conditions reduces in a familiar manner to the case of homogeneous boundary conditions, we will assume in the sequel that $u_l = 0$.

Lemma 8.1. *Suppose $f \in L^2(\mathcal{C}_T, F^i)$. If $u \in C^{1,0}(\mathcal{X} \times (0,T), F^i)$ is a classical solution of problem (8.1), then $u \in H^{1,0}(\mathcal{C}_T, F^i)$.*

Proof. We pick arbitrary $\varepsilon, t' \in (0,T)$ satisfying $\varepsilon < t'$, multiply the differential equation in (8.1) by u^* and integrate the equality over the cylinder $\mathcal{X} \times (\varepsilon, t')$. Since

$$\Re\, (u'_t, u)_x = \frac{1}{2} \frac{\partial}{\partial t} (u,u)_x$$

for all $t \in (0,T)$, the Stokes formula implies

$$\frac{1}{2} \int\limits_{\mathcal{X}} \left(|u(\cdot,t')|^2 - |u(\cdot,\varepsilon)|^2 \right) dx + \nu \int\limits_{\mathcal{X}} \int\limits_{\varepsilon}^{t'} \left(|Au|^2 + |A^*u|^2 \right) dxdt = \Re \int\limits_{\mathcal{X}} \int\limits_{\varepsilon}^{t'} (f,u)_x \, dxdt.$$

Hence it follows that

$$\begin{aligned} \frac{1}{2} \int_{\mathcal{X}} |u(\cdot,t')|^2\, dx + \nu \int_{\mathcal{X}} \int_{\varepsilon}^{t'} \left(|Au|^2 + |A^*u|^2 \right) dxdt \ &\leqslant \\ \leqslant\ \frac{1}{2} \int_{\mathcal{X}} |u(\cdot,\varepsilon)|^2\, dx + \int_{\mathcal{X}} \int_{\varepsilon}^{t'} |f|\, |u|\, dxdt \ &\leqslant \\ \leqslant\ \frac{1}{2} \int_{\mathcal{X}} |u(\cdot,\varepsilon)|^2\, dx + \|f\|_{L^2(\mathcal{C}_T,F^i)}\, \|u\|_{L^2(\mathcal{X}\times(\varepsilon,t'),F^i)}, & \end{aligned}$$

and so on passing in this inequality to the limit as $\varepsilon \to 0$ we get

$$\begin{aligned} \frac{1}{2} \int_{\mathcal{X}} |u(\cdot,t')|^2\, dx \ &\leqslant\ \frac{1}{2} \|u_0\|^2_{L^2(\mathcal{X},F^i)} + \|f\|_{L^2(\mathcal{C}_T,F^i)}\, \|u\|_{L^2(\mathcal{C}_{t'},F^i)}, \\ \nu \int_{\mathcal{C}_{t'}} \left(|Au|^2 + |A^*u|^2 \right) dxdt \ &\leqslant\ \frac{1}{2} \|u_0\|^2_{L^2(\mathcal{X},F^i)} + \|f\|_{L^2(\mathcal{C}_T,F^i)}\, \|u\|_{L^2(\mathcal{C}_{t'},F^i)}. \end{aligned} \tag{8.2}$$

Choose an arbitrary $t \in (0,T)$ and integrate the first inequality of (8.2) in $t' \in (0,t)$. This yields

$$\begin{aligned} \int_0^t \Big(\int_{\mathcal{X}} |u(\cdot,t')|^2\, dx \Big) dt' \ &\leqslant\ T\, \|u_0\|^2_{L^2(\mathcal{X},F^i)} + 2T\, \|f\|_{L^2(\mathcal{C}_T,F^i)}\, \|u\|_{L^2(\mathcal{C}_t,F^i)} \ \leqslant \\ &\leqslant\ T\, \|u_0\|^2_{L^2(\mathcal{X},F^i)} + 2T^2\, \|f\|^2_{L^2(\mathcal{C}_T,F^i)} + \frac{1}{2} \|u\|^2_{L^2(\mathcal{C}_t,F^i)} \end{aligned}$$

whence

$$\|u\|^2_{L^2(\mathcal{C}_t,F^i)} \ \leqslant\ 2T\, \|u_0\|^2_{L^2(\mathcal{X},F^i)} + 4T^2\, \|f\|^2_{L^2(\mathcal{C}_T,F^i)} =: C$$

for all $t \in (0,T)$, with C a nonnegative constant independent of t. We have thus proved that $\|u\|_{L^2(\mathcal{C}_T,F^i)} \leqslant \sqrt{C}$, and so the second inequality of (8.2) shows readily that

$$\|Au\|^2_{L^2(\mathcal{C}_{t'},F^{i+1})} + \|A^*u\|^2_{L^2(\mathcal{C}_{t'},F^{i-1})} \leqslant \frac{1}{2\nu}\,\|u_0\|^2_{L^2(\mathcal{X},F^i)} + \frac{\sqrt{C}}{\nu}\,\|f\|_{L^2(\mathcal{C}_T,F^i)}$$

for all $t' \in (0,T)$. On using a familiar argument with the Dirichlet scalar product on $\overset{\circ}{H}{}^1(\mathcal{X},F^i)$ we now conclude that $|u'_x|$ is square integrable on $\mathcal{C}_T$, which establishes the lemma. □

Remark 8.2. Combining the first inequality of (8.2) and $\|u\|_{L^2(\mathcal{C}_T,F^i)} \leqslant \sqrt{C}$ one sees that the classical solution $u \in C^{1,0}(\mathcal{X}\times(0,T),F^i)$ of problem (8.1) satisfies $\|u(\cdot,t')\|_{L^2(\mathcal{X},F^i)} \leqslant C'$ for all $t' \in (0,T)$, where the constant C' depends only on T and $\|u_0\|_{L^2(\mathcal{X},F^i)}$, $\|f\|_{L^2(\mathcal{C}_T,F^i)}$.

Let $f \in L^2(\mathcal{C}_T,F^i)$ and let $u \in C^{1,0}(\mathcal{X}\times(0,T),F^i)$ be a classical solution of mixed problem (8.1). We multiply the differential equation of (8.1) by v^*, where v is a C^1 section of F^i over the closure of $\mathcal{C}_T$ satisfying $v(x,T)=0$ for all $x\in\mathcal{X}$ and $v=0$ at $\partial\mathcal{X}\times(0,T)$, and integrate the resulting equality over the cylinder $\mathcal{X}\times(\varepsilon,t')$, where $0<\varepsilon<t'<T$. On applying the Stokes formula we arrive at the equality

$$\int_{\mathcal{X}} \big(u(\cdot,t'),v(\cdot,t')\big)_x dx + \int_{\mathcal{X}}\int_{\varepsilon}^{t'} (-(u,v'_t)_x + \nu((Au,Av)_x + (A^*u,A^*v)_x))\,dxdt =$$
$$= \int_{\mathcal{X}} (u(\cdot,\varepsilon),v(\cdot,\varepsilon))_x dx + \int_{\mathcal{X}}\int_{\varepsilon}^{t'} (f,v)_x\,dxdt.$$

By Lemma 8.1, $u \in H^{1,0}(\mathcal{C}_T,F^i)$, and so the restriction of u to the lateral surface belongs to $L^2(\partial\mathcal{X}\times(0,T),F^i)$. Using Remark 8.2, we pass in the last equality to the limit as $\varepsilon\to 0$ and $t'\to T$, thus obtaining

$$\int\limits_{\mathcal{C}_T}(-(u,v'_t)_x + \nu((Au,Av)_x + (A^*u,A^*v)_x))dxdt = \int\limits_{\mathcal{X}}(u_0,v(\cdot,0))_x dx + \int\limits_{\mathcal{C}_T}(f,v)_x dxdt \tag{8.3}$$

for all sections $v\in C^1(\overline{\mathcal{C}}_T,F^i)$ vanishing both on $\mathcal{X}\times\{T\}$ and $\partial\mathcal{X}\times(0,T)$. By continuity, (8.3) still holds for all $v\in H^1(\mathcal{C}_T,F^i)$ satisfying $v=0$ at $\mathcal{X}\times\{T\}$ and $\partial\mathcal{X}\times(0,T)$.

We use the identity (8.3) to introduce weak solutions to the mixed problem (8.1). In the sequel we assume that $f\in L^2(\mathcal{C}_T,F^i)$ and $u_0\in L^2(\mathcal{X},F^i)$. A section $u\in H^{1,0}(\mathcal{C}_T,F^i)$ is said to be a weak solution of problem (8.1), if $u=0$ on $\partial\mathcal{X}\times(0,T)$ and the identity (8.3) is fulfilled for all $v\in H^1(\mathcal{C}_T,F^i)$ vanishing on the cylinder top $\mathcal{X}\times\{T\}$ and on the lateral surface $\partial\mathcal{X}\times(0,T)$. Along with classical and weak solutions of the first mixed problem one can introduce the concept of 'almost everywhere' solution. A section u is said to be an 'almost everywhere' solution of the mixed problem if it belongs to the space $H^{2,1}(\mathcal{C}_T,F^i)$ and satisfies the differential equation of (8.1) for almost all $(x,t)\in\mathcal{C}_T$, the initial condition for almost all $x\in\mathcal{X}$ and the trace of u on the lateral surface vanishes almost everywhere.

Lemma 8.1 shows that any classical solution of problem (8.1) which belongs to $C^{1,0}(\partial X\times(0,T),F^i)$ is also a weak solution of the first mixed problem. Similarly one proves that any 'almost everywhere' solution of the first mixed problem is a weak solution. It is easily seen that if a weak solution of problem (8.1) belongs to $H^{2,1}(\mathcal{C}_T,F^i)$ then it is an 'almost everywhere' solution. And if a weak solution of the first mixed problem belongs to $C^{2,1}(\mathcal{C}_T,F^i)$ and is continuous up to the lower basis and the lateral surface of the cylinder $\mathcal{C}_T$, then it is a classical solution. For proofs of the corresponding assertions for solutions of the first mixed problem for the Lamé system we refer the reader to [23]. It is worth pointing out that any of the classical, weak or 'almost everywhere' solutions bears the following property: If $u(x,t)$ is a solution of (8.1) in the cylinder $\mathcal{C}_T$, then it is a solution in any cylinder $\mathcal{C}_{t'}$ with $0<t'<T$.

Part 3. Linear Navier-Stokes equations

9. Uniqueness of a weak solution

Our next objective is to establish a uniqueness theorem for solutions of the first mixed problem.

Theorem 9.1. *As defined above, the first mixed problem (8.1) has at most one weak solution.*

Proof. The proof is analogous to that of Theorem 3.2 in [23]. Let $u_1(x,t)$ and $u_2(x,t)$ be two weak solutions of (8.1). Then the difference $u = u_1 - u_2$ is a weak solution of the corresponding homogeneous problem with $f = 0$ and $u_0 = 0$. We have to show that $u = 0$ in $\mathcal{C}_T$.

Let $u \in H^{2,1}(\mathcal{C}_T, F^i)$ be a weak solution of the first mixed problem with $f = 0$ in $\mathcal{C}_T$ and $u_0 = 0$ in $\mathcal{X}$. Consider the function

$$v(x,t) = \int_t^T u(x,\theta)d\theta$$

defined in $\mathcal{C}_T$. It is immediately verified that the function v has generalised derivatives

$$v'_{x^j}(x,t) = \int_t^T u'_{x^j}(x,\theta)d\theta,$$

for $j = 1, \dots, n$, and

$$v'_t(x,t) = -u(x,t)$$

in $\mathcal{C}_T$. Since v and v'_{x^j}, v'_t belong to $L^2(\mathcal{C}_T, F^i)$, we deduce that $v \in H^1(\mathcal{C}_T, F^i)$. Moreover, this section vanishes at the lateral boundary and on the top of the cylinder $\mathcal{C}_T$.

Substituting the function v into identity (8.3) yields

$$\int_{\mathcal{C}_T} \Big(|u|^2 + \nu(Au, \int_t^T Au(\cdot,\theta)d\theta)_x + \nu(A^*u, \int_t^T A^*u(\cdot,\theta)d\theta)_x \Big) dxdt = 0.$$

Since

$$\begin{aligned}\int_{\mathcal{C}_T} (Au(x,t), \int_t^T Au(x,\theta)d\theta)_x dxdt &= \int_{\mathcal{X}} \int_0^T (Au(x,t), \int_t^T Au(x,\theta)d\theta)_x dxdt = \\ &= \int_{\mathcal{X}} \int_0^T \left(\int_0^\theta Au(x,t)dt, Au(x,\theta) \right)_x dxd\theta\end{aligned}$$

which transforms to

$$\begin{aligned}\int_{\mathcal{X}} \left(\int_0^T Au(x,t)dt, \int_0^T Au(x,\theta)d\theta \right)_x dx - \int_{\mathcal{X}} \int_0^T \left(\int_\theta^T Au(x,t)dt, Au(x,\theta) \right)_x dxd\theta = \\ = \int_{\mathcal{X}} \Big| \int_0^T Au(x,t)dt \Big|^2 dx - \int_{\mathcal{C}_T} \left(\int_\theta^T Au(x,t)dt, Au(x,\theta) \right)_x dxd\theta,\end{aligned}$$

we get

$$\Re \int_{\mathcal{C}_T} (Au(x,t), \int_t^T Au(x,\theta)d\theta)_x dxdt = \frac{1}{2} \int_{\mathcal{X}} \Big| \int_0^T Au(x,t)dt \Big|^2 dx.$$

Similarly we obtain

$$\Re \int_{\mathcal{C}_T} (A^*u(x,t), \int_t^T A^*u(x,\theta)d\theta)_x dxdt = \frac{1}{2} \int_{\mathcal{X}} \Big| \int_0^T A^*u(x,t)dt \Big|^2 dx$$

whence

$$\int_{\mathcal{C}_T} |u(x,t)|^2 dxdt + \frac{\nu}{2} \int_{\mathcal{X}} \Big| \int_0^T Au(x,t)dt \Big|^2 dx + \frac{\nu}{2} \int_{\mathcal{X}} \Big| \int_0^T A^* u(x,t)dt \Big|^2 dx = 0. \tag{9.1}$$

Since $\nu \geqslant 0$, we conclude from (9.1) that

$$\int_{\mathcal{X}} |u(x,t)|^2 dx = 0$$

for all $t \in (0,T)$, and so $u = 0$ in $\mathcal{C}_T$, as desired. □

Since an 'almost everywhere' solution of problem (8.1) is actually a weak solution to this problem, Theorem 9.1 implies

Corollary 9.2. *As defined above, problem (8.1) has at most one 'almost everywhere' solution.*

On combining Theorem 9.1 and Lemma 8.1 we also deduce that the first mixed problem has at most one classical solution belonging to $C^{1,0}(\partial X \times (0,T), F^i)$.

10. Existence of a weak solution

We now turn to the proof of the existence of solutions to problem (8.1). To this end we use the Fourier method in the same way as for hyperbolic equations, see [23].

Let v be a weak eigenfunction of the first boundary value problem for the $-\nu$ multiple of the Laplace operator

$$\begin{array}{rcll} -\nu\, \Delta v & = & \varkappa v & \text{in} \quad \mathring{\mathcal{X}}, \\ v & = & 0 & \text{at} \quad \partial\mathcal{X}, \end{array} \tag{10.1}$$

where $\varkappa$ is the corresponding eigenvalue. Thus, v belongs to $\mathring{H}^1(\mathcal{X}, F^i)$ and satisfies the integral identity

$$\nu \int_{\mathcal{X}} \left((Av, Ag)_x + (A^*v, A^*g)_x\right) dx + \varkappa \int_{\mathcal{X}} (v,g)_x dx = 0$$

for all $g \in \mathring{H}^1(\mathcal{X}, F^i)$.

Consider the orthonormal system $(v_k)_{k=1,2,\ldots}$ in $L^2(\mathcal{X}, F^i)$ consisting of all weak eigenfunction of problem (10.1). Let $(\varkappa_k)_{k=1,2,\ldots}$ be the sequence of corresponding eigenvalues. As usual we think of this sequence as nonincreasing sequence with $\varkappa_1 \leqslant 0$ and each eigenvalue repeats itself in accord with its multiplicity. The system $(v_k)_{k=1,2,\ldots}$ is known to be an orthonormal basis in $L^2(\mathcal{X}, F^i)$ and $\varkappa_k \to -\infty$ when $k \to \infty$. Moreover, the first eigenvalue $\varkappa_1$ is obviously strongly negative, if $\nu > 0$ holds.

Assume that $f \in L^2(\mathcal{C}_T, F^i)$ and $u_0 \in L^2(\mathcal{X}, F^i)$. By the Fubini theorem we deduce that $f(\cdot, t) \in L^2(\mathcal{X}, F^i)$ for almost all $t \in (0,T)$. We expand the sections $f(\cdot,t)$ and u_0 as Fourier series over the system of eigenfunctions $(v_k)_{k=1,2,\ldots}$ in $\mathcal{X}$, namely

$$\begin{array}{rcl} f(x,t) & = & \displaystyle\sum_{k=1}^{\infty} f_k(t) v_k(x), \\ u_0(x) & = & \displaystyle\sum_{k=1}^{\infty} u_{0,k} v_k(x), \end{array}$$

where $f_k(t) = (f(\cdot,t), v_k)_{L^2(\mathcal{X},F^i)}$ and $u_{0,k} = (u_0, v_k)_{L^2(\mathcal{X},F^i)}$ for $k = 1, 2, \ldots$, the functions f_k belonging to $L^2(0,T)$. By the Parseval equality, we get

$$\begin{array}{rcl} \displaystyle\sum_{k=1}^{\infty} |f_k(t)|^2 & = & \|f(\cdot,t)\|^2_{L^2(\mathcal{X},F^i)}, \\ \displaystyle\sum_{k=1}^{\infty} |u_{0,k}|^2 & = & \|u_0\|^2_{L^2(\mathcal{X},F^i)}, \end{array} \tag{10.2}$$

where the first equality is valid for almost all $t \in (0,T)$. On integrating both sides of the first equality of (10.2) in $t \in (0,T)$ and using the theorem of Beppo Levi we see that

$$\sum_{k=1}^{\infty} \int_0^T |f_k(t)|^2 dt = \int_{\mathcal{C}_T} |f(x,t)|^2 dx dt. \tag{10.3}$$

For any $k = 1, 2, \ldots$, we introduce the function

$$w_k(t) = u_{0,k} \exp(\varkappa_k t) + \int_0^t f_k(t') \exp(\varkappa_k (t - t'))\, dt',$$

which obviously belongs to $H^1(0,T)$ and satisfies the initial value problem

$$\begin{array}{rcll} w_k' - \varkappa_k w_k & = & f_k & \text{a.e. on} \quad (0,T), \\ w_k(0) & = & u_{0,k}, & \end{array} \tag{10.4}$$

the initial condition is well defined, for $H^1(0,T) \hookrightarrow C[0,T]$ by the Sobolev embedding theorem. In much the same way as in [23] one verifies that the section $u_k(x,t) = w_k(t) v_k(x)$ is a weak solution of problem (8.1) with the right-hand side $f(x,t) = f_k(t) v_k(x)$ and initial data $u_0(x) = u_{0,k} v_k(x)$. Hence it follows by linearity that the partial sums

$$s_N(x,t) = \sum_{k=1}^{N} w_k(t) v_k(x)$$

are weak solutions of problem (8.1) whose right-hand side and initial data are given by the corresponding partial sums of the Fourier series for f and u_0, respectively. To wit,

$$\begin{array}{l} \displaystyle\int_{\mathcal{C}_T} \left(-(s_N, v_t')_x + \nu((A s_N, A v)_x + (A^* s_N, A^* v)_x)\right) dx dt = \\ \displaystyle = \int_{\mathcal{X}} \sum_{k=1}^{N} u_{0,k} \,(v_k, v(\cdot,0))_x dx + \int_{\mathcal{C}_T} \sum_{k=1}^{N} f_k(t)\,(v_k, v(\cdot,t))_x dx dt \end{array} \tag{10.5}$$

for all sections $v \in H^1(\mathcal{C}_T, F^i)$ vanishing both on $\mathcal{X} \times \{T\}$ and $\partial\mathcal{X} \times (0,T)$, cf. (8.3).

Our next objective is to show that the series

$$u(x,t) = \sum_{k=1}^{\infty} w_k(t) v_k(x) \tag{10.6}$$

converges in $H^{1,0}(\mathcal{C}_T, F^i)$ and its sum gives a weak solution to the first mixed problem (8.1).

Theorem 10.1. *If $f \in L^2(\mathcal{C}_T, F^i)$ and $u_0 \in L^2(\mathcal{X}, F^i)$, then problem (8.1) has a weak solution u. The solution is represented by series (10.6) which converges in $H^{1,0}(\mathcal{C}_T, F^i)$. Moreover, there is a constant $C > 0$ independent of f and u_0, such that*

$$\|u\|_{H^{1,0}(\mathcal{C}_T,F^i)} \leqslant C \left(\|f\|_{L^2(\mathcal{C}_T,F^i)} + \|u_0\|_{L^2(\mathcal{X},F^i)} \right). \tag{10.7}$$

Proof. From the formula for $w_k(t)$ it follows readily by the Cauchy-Schwarz inequality that

$$\begin{aligned} |w_k(t)| &\leqslant |u_{0,k}|\,\exp(\varkappa_k t) + \int_0^t |f_k(t')|\,\exp\left(\varkappa_k(t-t')\right)dt' \leqslant \\ &\leqslant |u_{0,k}|\,\exp(\varkappa_k t) + \|f_k\|_{L^2(0,T)}\,\frac{1}{\sqrt{2|\varkappa_k|}} \end{aligned}$$

whenever $t \in [0,T]$. Therefore,

$$|w_k(t)|^2 \leqslant 2\exp(2\varkappa_k t)\,|u_{0,k}|^2 + \frac{1}{|\varkappa_k|}\,\|f_k\|^2_{L^2(0,T)} \tag{10.8}$$

for all $t \in [0,T]$.

Consider a partial sum $s_N(x,t)$ of series (10.6). For any fixed $t \in [0,T]$ it belongs to the space

$$\mathring{H}^1(\mathcal{X}, F^i).$$

It is convenient to endow this space with the so-called Dirichlet scalar product

$$D(v,g) = \int_{\mathcal{X}} \nu\Big((Av, Ag)_x + (A^*v, A^*g)_x\Big)dx$$

and the Dirichlet norm $D(v) := \sqrt{D(v,v)}$. Since the system

$$\left\{\frac{v_k}{\sqrt{-\varkappa_k}}\right\}_{k=1,2,\ldots}$$

is obviously orthonormal with respect to the scalar product $D(v,g)$, we obtain by (10.8)

$$\begin{aligned} \big\|s_N(\cdot,t) - s_M(\cdot,t)\big\|^2_{H^1(\mathcal{X},F^i)} &= \Big\|\sum_{k=M+1}^{N} w_k(t)v_k\Big\|^2_{H^1(\mathcal{X},F^i)} \leqslant \\ &\leqslant C\sum_{k=M+1}^{N} |w_k(t)|^2\,|\varkappa_k| \leqslant \\ &\leqslant C\sum_{k=M+1}^{N}\Big(2|\varkappa_k|\exp(2\varkappa_k t)\,|u_{0,k}|^2 + \|f_k\|^2_{L^2(0,T)}\Big) \end{aligned}$$

for all M and N satisfying $1 \leqslant M < N$, and all $t \in [0,T]$, with C a constant independent of M, N and t. Along with this inequality we obtain in the same manner

$$\|s_N(\cdot,t)\|^2_{H^1(\mathcal{X},F^i)} \leqslant C\sum_{k=1}^{N}\Big(2|\varkappa_k|\exp(2\varkappa_k t)\,|u_{0,k}|^2 + \|f_k\|^2_{L^2(0,T)}\Big)$$

for all $N = 1,2,\ldots$ and $t \in [0,T]$. On integrating the last two inequalities in $t \in [0,T]$ we obtain

$$\begin{aligned} \|s_N - s_M\|^2_{H^{1,0}(\mathcal{C}_T,F^i)} &\leqslant C'\sum_{k=M+1}^{N}\Big(|u_{0,k}|^2 + \|f_k\|^2_{L^2(0,T)}\Big), \\ \|s_N\|^2_{H^{1,0}(\mathcal{C}_T,F^i)} &\leqslant C'\sum_{k=1}^{N}\Big(|u_{0,k}|^2 + \|f_k\|^2_{L^2(0,T)}\Big), \end{aligned} \tag{10.9}$$

where the constant C' is independent of N and M.

By (10.2) the series with general term $|u_{0,k}|^2 + \|f_k\|^2_{L^2(0,T)}$ converges. Hence, from the first estimate of (10.9) we deduce that series (10.6) converges in $H^{1,0}(\mathcal{C}_T, F^i)$, and so its sum $u(x,t)$ belongs to $H^{1,0}(\mathcal{C}_T, F^i)$ and satisfies $u = 0$ on the lateral boundary $\partial\mathcal{X} \times (0,T)$ of the cylinder. Letting $N \to \infty$ in identity (10.5) we see that the section u is a weak solution of problem (8.1). Estimate (10.7) follows immediately from the second inequality of (10.9), if we let $N \to \infty$ in (10.9) and use equalities (10.2). □

Note that similarly to the hyperbolic case [23] one can prove the existence of a weak solutions to problem (8.1) by means of the Galerkin method.

11. Regularity of weak solutions

We now discuss briefly the regularity of weak solutions. Assume that the boundary $\partial\mathcal{X}$ of $\mathcal{X}$ is of class C^{2s} for some integer $s \geqslant 1$. Then the eigenfunctions $(v_k)_{k=1,2,\ldots}$ of problem (10.1) belong to $H^{2s}(\mathcal{X}, F^i)$ and satisfy the boundary conditions

$$(-\nu\Delta)^i v_k \;=\; 0 \quad \text{on} \quad \partial\mathcal{X} \tag{11.1}$$

for $i = 0, 1, \ldots, s-1$.

Let $H^{2s}_{\mathcal{D}}(\mathcal{X}, F^i)$ stand for the subspace of $H^{2s}(\mathcal{X}, F^i)$ consisting of all functions v satisfying (11.1). We put additional restrictions on the data of the problem to attain to a classical solution. More precisely, we require that $u_0 \in H^{2s-1}_{\mathcal{D}}(\mathcal{X}, F^i)$ and f belongs to the subspace of $H^{2(s-1),s-1}(\mathcal{C}_T, F^i)$ that consists of all functions satisfying

$$(-\nu\Delta)^i f \;=\; 0 \quad \text{at} \quad \partial\mathcal{X} \times (0,T) \tag{11.2}$$

for $i = 0, 1, \ldots, s-2$.

For $s = 1$, the latter equations are empty and we arrive at $f \in L^2(\mathcal{X}, F^i)$, as above.

Theorem 11.1. *Under the above hypotheses, series (10.6) converges to the weak solution u of problem (8.1) in $H^{2s,s}(\mathcal{C}_T, F^i)$. Moreover, there is a constant $C > 0$ independent of f and u_0, such that*

$$\|u\|_{H^{2s,s}(\mathcal{C}_T,F^i)} \leqslant C \left(\|f\|_{H^{2(s-1),s-1}(\mathcal{C}_T,F^i)} + \|u_0\|_{H^{2s-1}(\mathcal{X},F^i)} \right). \tag{11.3}$$

Proof. The proof of this theorem runs similarly to the proof of Theorem 4 of [24, p. 372], if one exploits the techniques developed above. □

Since a weak solution of problem (8.1) which belongs to $H^{2,1}(\mathcal{C}_T, F^i)$ is an 'almost everywhere' solution, Theorem 11.1 for $s = 1$ implies

Corollary 11.2. *Suppose $\partial\mathcal{X} \in C^2$ and $f \in L^2(\mathcal{C}_T, F^i)$, $u_0 \in \overset{\circ}{H}{}^1(\mathcal{X}, F^i)$. Then series (10.6) converges in $H^{2,1}(\mathcal{C}_T, F^i)$ and its sum is an 'almost everywhere' solution of problem (8.1). Moreover, there is a constant C independent of f and u_0, such that*

$$\|u\|_{H^{2,1}(\mathcal{C}_T,F^i)} \leqslant C \left(\|f\|_{L^2(\mathcal{C}_T,F^i)} + \|u_0\|_{H^1(\mathcal{X},F^i)} \right).$$

If the boundary of $\mathcal{X}$ is of class $C^{[n/2]+3}$, then the eigenfunctions $v_k(x)$ of problem (10.1) belong to the space $H^{[n/2]+3}(\mathcal{X}, F^i)$, and so to the space $C^2(\mathcal{X}, F^i)$, which is due to the Sobolev embedding theorem. Therefore, the partial sums s_N of series (10.6 are in $C^{2,1}(\overline{\mathcal{C}}_T, F^i)$

Corollary 11.3. *Let $\partial\mathcal{X} \in C^{2s+1}$, where $2s+1 \geqslant [n/2]+3$, and $f \in H^{2s,s}_{\mathcal{D}}(\mathcal{C}_T, F^i)$, $u_0 \in H^{2s+1}_{\mathcal{D}}(\mathcal{X}, F^i)$. Then series (10.6) converges in $C^{2,1}(\overline{\mathcal{C}}_T, F^i)$ and its sum u is a classical solution of problem (8.1). Moreover, there is a constant $C > 0$ independent of f and u_0, such that*

$$\|u\|_{C(\overline{\mathcal{C}}_T,F^i)} \leqslant C \left(\|f\|_{H^{2(s-1),s-1}(\mathcal{C}_T,F^i)} + \|u_0\|_{H^{2s-1}(\mathcal{X},F^i)} \right).$$

Proof. The proof of this corollary runs in much the same way as the proof of Theorem 5 of [24, p. 381]. □

12. Generalised Navier-Stokes equations revisited

The arguments of this chapter still apply if we replace the Laplacian Δ by the composition $P\Delta$, where P is the Helmholtz projector introduced in Lemma 1.2. The only difference is in substituting Pf for f, i.e., in the choice of data $f(t,\cdot)$ and u_0 in the subspace of $L^2(\mathcal{X},F^i)$ consisting of those sections which belong to the kernel of the adjoint operator for A^i in the sense of Hilbert spaces. More precisely, assume that the boundary of $\mathcal{X}$ is of class C^2 and $f \in L^2(\mathcal{C}_T,F^i)$, $u_0 \in H^1(\mathcal{X},F^i)$ satisfies $A^* u_0 = 0$ in $\mathcal{X}$ and $u_0 = 0$ at $\partial\mathcal{X}$. Then there is a unique section $u \in H^{2,1}(\mathcal{C}_T,F^i)$ such that

$$\begin{array}{rcll} u'_t + \nu\, P\Delta u & = & Pf & \text{in} \quad \mathcal{C}_T, \\ u & = & u_0 & \text{at} \quad \mathcal{X}\times\{0\}, \\ u & = & 0 & \text{at} \quad \partial\mathcal{X}\times(0,T) \end{array} \tag{12.1}$$

in a weak sense or, what is equivalent, almost everywhere on the corresponding strata. Moreover,

$$\|u\|_{H^{2,1}(\mathcal{C}_T,F^i)} \leqslant C\left(\|Pf\|_{L^2(\mathcal{C}_T,F^i)} + \|u_0\|_{H^1(\mathcal{X},F^i)}\right)$$

with $C > 0$ a constant independent of f and u_0.

We complete the work by describing those nonlinear perturbations $N(u)$ of the equation $u'_t + \nu\,\Delta u = f$ which can be handled within the Leray-Schauder continuation method. Without restriction of generality we can assume that the initial data u_0 is zero.

Denote by U the subspace of $H^{2,1}(\mathcal{C}_T,F^i)$ consisting of those sections u which satisfy $Pu = u$ and vanish on the basis $\mathcal{X}\times\{0\}$ and on the lateral surface $\partial\mathcal{X}\times(0,T)$ of $\mathcal{C}_T$. When endowed with the scalar product induced from $H^{2,1}(\mathcal{C}_T,F^i)$, the space U is Hilbert. By the above, the mapping $Lu = u'_t + \nu\,P\Delta u$ is an isomorphism of U onto the range of the projector P in $L^2(\mathcal{C}_T,F^i)$. Let N be a compact continuous mapping of $H^{2,1}(\mathcal{C}_T,F^i)$ into $L^2(\mathcal{C}_T,F^i)$. Given any $f \in L^2(\mathcal{C}_T,F^i)$, we look for $u \in U$ satisfying

$$Lu + PN(u) = Pf \tag{12.2}$$

in $\mathcal{C}_T$.

On applying L^{-1} to both sides of this equation we transform it to the form $u = c_0 + K(u)$, where $c_0 = L^{-1}Pf$ and

$$K(u) := -L^{-1}PN(u)$$

for $u \in U$. Since both L^{-1} and P are bounded linear operators, we conclude readily that K is a compact continuous self-mapping of U. If $u \in U$ is a solution of the equation

$$u = c_0 + \vartheta K(u),$$

for some $\vartheta \in [0,1]$, then

$$\begin{array}{rcl} \|u - c_0\|_U & = & \vartheta\|K(u)\|_U \leqslant \\ & \leqslant & \|L^{-1}\|\,\|N(u)\|_{L^2(\mathcal{C}_T,F^i)} \end{array}$$

for all $u \in U$.

Assume that

$$\|N(u)\|_{L^2(\mathcal{C}_T,F^i)} = o\left(\|u\|_U\right) \tag{12.3}$$

for $\|u\|_U \to \infty$. Then from the above inequality it follows that there is a number $R > 0$ independent of u, such that $\|u - c_0\|_U < R$. In other words, any solution to the equation

$u = c_0 + \vartheta K(u)$ with some $\vartheta \in [0,1]$ belongs to the ball $B(c_0, R)$ in U. On applying Lemma 5.2 we see that the equation $u = c_0 + K(u)$ possesses at least one solution in U. We can now return to the perturbed equation (12.2) and conclude that under condition (12.3) it has at least one solution $u \in U$ for each right-hand side $f \in L^2(\mathcal{C}_T, F^i)$.

Condition (12.3) gives rise to a broad class of nonlinear parabolic equations for which the first mixed problem is solvable in the space $H^{2,1}(\mathcal{C}_T, F^i)$. Still, as is mentioned in Section 5, the nonlinearity in the classical Navier-Stokes equations does not satisfy (12.3).

Conclusion

We formulate the Navier-Stokes equations, which describe the dynamics of incompressible viscous fluid in the Euclidean space $\mathbb{R}^2$ or $\mathbb{R}^3$, within the general framework of elliptic complexes on compact smooth manifolds with boundary. The nonlinearity in the Navier-Stokes equations is specified as a nonlinear cochain self-mapping of the complex. We find the corresponding potential equations, which in the case of classical Navier-Stokes equations reduce to the well-known Burgers equation. We develop the theory of the Neumann problem after Spencer for elliptic complexes on compact smooth manifolds with boundary which enables one to eliminate the so-called continuity equation from the system. Finally, we consider an extensive class of nonlinear perturbations and prove the uniqueness and existence of the classical solution of the first mixed boundary problem for finite times. However, the classical Navier-Stokes equations go beyond this class and require further efforts.

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Уравнения Навье-Стокса для эллиптических комплексов

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Мы продолжаем изучение инвариантной формы классических уравнений математической физики, таких как уравнения Максвелла или система Ламе, на многообразии с краем. С этой целью мы формулируем их в терминах комплекса де Рама на определенном шаге. Используя структуру комплекса, мы получаем возможность увидеть вырождение, глубоко заложенное в уравнениях. В настоящей работе мы развиваем инвариантный подход к классическим уравнениям Навье-Стокса.

Ключевые слова: уравнения Навье-Стокса, классическое решение.