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# Existence and Uniqueness of the Solution for Volterra-Fredholm Integro-Differential Equations

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*In this article, modified Adomian decomposition method is successfully applies to find the approximate solutions of Volterra-Fredholm integro-differential equations. Moreover, we prove the existence and uniqueness results and convergence of the solutions. Finally, an example is included to demonstrate the validity and applicability of the proposed technique.*

*Keywords: Modified Adomian decomposition method, Volterra-Fredholm integro-differential equation, existence and uniqueness results, approximate solution.*
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## Introduction

In this paper, our study focuses on a class of Volterra-Fredholm integro-differential equations of the type:

$$\sum_{j=0}^{k} \xi_j(x)u^{(j)}(x) = f(x) + \lambda_1 \int_a^x K_1(x,t)G_1(u(t))dt + \lambda_2 \int_a^b K_2(x,t)G_2(u(t))dt, \qquad (1)$$

with the initial conditions

$$u^{(r)}(a) = b_r, \quad r = 0,1,2,\ldots,(k-1), \qquad (2)$$

where $u^{(j)}(x)$ is the $j^{th}$ derivative of the unknown function $u(x)$ that will be determined, $K_i(x,t), i = 1,2$ are the kernels of the equation, $f(x)$ and $\xi_j(x)$ are an analytic function, $G_1$

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and $G_2$ are nonlinear functions of $u$ and $a, b, \lambda_1, \lambda_2$, and $b_r$ are real finite constants. In recent years, many authors focus on the development of numerical and analytical techniques for integro-differential equations. For instance, we can remember the following works. Abbasbandy and Elyas [2] studied some applications on variational iteration method for solving system of nonlinear volterra integro-differential equations, Alao et al. [4] used Adomian decomposition and variational iteration methods for solving integro-differential equations, Yang and Hou [20] applied the Laplace decomposition method to solve the fractional integro-differential equations, Mittal and Nigam [14] applied the Adomian decomposition method to approximate solutions for fractional integro-differential equations, and Behzadi et al. [6] solved some class of nonlinear Volterra-Fredholm integro-differential equations by homotopy analysis method. Moreover, several authors have applied the Adomian decomposition method and the variational iteration method to find the approximate solutions of various types of integro-differential equations, among these works, see [8–13, 15, 18].

The main objective of the present paper is to study the behavior of the solution that can be formally determined by semi-analytical approximated methods as the Adomian decomposition method and modified Adomian decomposition method. Moreover, we prove the existence, uniqueness results and convergence of the solutions of the Volterra-Fredholm integro-differential equations (1).

# 1. Description of the methods

Some powerful methods have been focusing on the development of more advanced and efficient methods for integro-differential equations such as the Adomian decomposition method [1, 3, 5, 7, 14] and modified Adomian decomposition method [5, 11, 16, 19]. We will describe all these methods in this section:

## 1.1. Adomian decomposition method (ADM)

Now, we can rewrite Eq.(1) in the form

$$\xi_k(x)u^k(x) + \sum_{j=0}^{k-1} \xi_j(x)u^j(x) = f(x) + \lambda_1 \int_a^x K_1(x,t)G_1(u(t))dt + \lambda_2 \int_a^b K_2(x,t)G_2(u(t))dt. \quad (3)$$

Then

$$u^k(x) = \frac{f(x)}{\xi_k(x)} + \lambda_1 \int_a^x \frac{K_1(x,t)}{\xi_k(x)} G_1(u(t))dt + \lambda_2 \int_a^b \frac{K_1(x,t)}{\xi_k(x)} G_2(u(t))dt - \sum_{j=0}^{k-1} \frac{\xi_j(x)}{\xi_k(x)} u^j(x).$$

To obtain the approximate solution, we integrating $(k)$-times in the interval $[a, x]$ with respect to $x$ we obtain,

$$u(x) = L^{-1}\left(\frac{f(x)}{\xi_k(x)}\right) + \sum_{r=0}^{k-1} \frac{1}{r!}(x-a)^r b_r + \lambda_1 L^{-1}\left(\int_a^x \frac{K_1(x,t)}{\xi_k(x)} G_1(u(t))dt\right) + \\ + \lambda_2 L^{-1}\left(\int_a^b \frac{K_2(x,t)}{\xi_k(x)} G_1(u(t))dt\right) - \sum_{j=0}^{k-1} L^{-1}\left(\frac{\xi_j(x)}{\xi_k(x)} u_n^{(j)}(x)\right), \quad (4)$$

where $L^{-1}$ is the multiple integration operator given as follows:

$$L^{-1}(\cdot) = \int_a^x \int_a^x \cdots \int_a^x (\cdot) dx dx \cdots dx \quad \text{(k-times)}.$$

Now we apply ADM

$$G_1(u(x)) = \sum_{n=0}^{\infty} A_n, \quad G_2(u(x)) = \sum_{n=0}^{\infty} B_n, \tag{5}$$

where $A_n,\ B_n;\ n \geqslant 0$ are the Adomian polynomials determined formally as follows:

$$A_n = \frac{1}{n!}\Big[\frac{d^n}{d\mu^n} G_1(\sum_{i=0}^{\infty} \mu^i u_i)\Big]\Big|_{\mu=0}, \quad B_n = \frac{1}{n!}\Big[\frac{d^n}{d\mu^n} G_2(\sum_{i=0}^{\infty} \mu^i u_i)\Big]\Big|_{\mu=0}. \tag{6}$$

The Adomian polynomials were introduced in [16, 17, 20] as:

$$\begin{aligned} A_0 &= G_1(u_0), \\ A_1 &= u_1 G_1^{'}(u_0), \\ A_2 &= u_2 G_1^{'}(u_0) + \frac{1}{2!} u_1^2 G_1^{''}(u_0), \\ A_3 &= u_3 G_1^{'}(u_0) + u_1 u_2 G_1^{''}(u_0) + \frac{1}{3!} u_1^3 G_1^{'''}(u_0), \ldots \end{aligned}$$

and

$$\begin{aligned} B_0 &= G_2(u_0), \\ B_1 &= u_1 G_2^{'}(u_0), \\ B_2 &= u_2 G_2^{'}(u_0) + \frac{1}{2!} u_1^2 G_2^{''}(u_0), \\ B_3 &= u_3 G_2^{'}(u_0) + u_1 u_2 G_2^{''}(u_0) + \frac{1}{3!} u_1^3 G_2^{'''}(u_0), \ldots \end{aligned}$$

The standard decomposition technique represents the solution of $u$ as the following series:

$$u = \sum_{i=0}^{\infty} u_i. \tag{7}$$

By substituting (5) and (7) in Eq. (4) we have

$$\begin{aligned} \sum_{i=0}^{\infty} u_i(x) &= L^{-1}\left(\frac{f(x)}{\xi_k(x)}\right) + \sum_{r=0}^{k-1} \frac{1}{r!}(x-a)^r b_r + \lambda_1 \sum_{i=0}^{\infty} L^{-1}\left(\int_a^x \frac{K_1(x,t)}{\xi_k(x)} A_i(t) dt\right) + \\ &+ \lambda_2 \sum_{i=0}^{\infty} L^{-1}\left(\int_a^b \frac{K_2(x,t)}{\xi_k(x)} B_i(t) dt\right) - \sum_{i=0}^{\infty}\sum_{j=0}^{k-1} L^{-1}\left(\frac{\xi_j(x)}{\xi_k(x)} u_i^{(j)}(x)\right). \end{aligned}$$

The components $u_0, u_1, u_2, \cdots$ are usually determined recursively by

$$\begin{aligned} u_0 &= L^{-1}\left(\frac{f(x)}{\xi_k(x)}\right) + \sum_{r=0}^{k-1} \frac{1}{r!}(x-a)^r b_r, \\ u_1 &= \lambda_1 L^{-1}\left(\int_a^x \frac{K_1(x,t)}{\xi_k(x)} A_0(t) dt\right) + \lambda_2 L^{-1}\left(\int_a^b \frac{K_2(x,t)}{\xi_k(x)} B_0(t) dt\right) - \\ &- \sum_{j=0}^{k-1} L^{-1}\left(\frac{\xi_j(x)}{\xi_k(x)} u_0^{(j)}(x)\right), \end{aligned}$$

$$u_n = \lambda_1 L^{-1}\left(\int_a^x \frac{K_1(x,t)}{\xi_k(x)} A_{n-1}(t)dt\right) + \lambda_2 L^{-1}\left(\int_a^b \frac{K_2(x,t)}{\xi_k(x)} B_{n-1}(t)dt\right) - \\ - \sum_{j=0}^{k-1} L^{-1}\left(\frac{\xi_j(x)}{\xi_k(x)} u_{n-1}^{(j)}(x)\right), \ n \geqslant 1.$$

Then, $u(x) = \sum_{i=0}^{n} u_i$ as the approximate solution.

## 1.2. Modified adomian decomposition method (MADM)

The modified decomposition method was introduced by Wazwaz [16]. This method is based on the assumption that the function $f(x)$ can be divided into two parts, namely $f_1(x)$ and $f_2(x)$. Under this assumption we set

$$f(x) = f_1(x) + f_2(x). \tag{8}$$

We apply this decomposition when the function $f$ consists of several parts and can be decomposed into two different parts. In this case, $f$ is usually a summation of a polynomial and trigonometric or transcendental functions. A proper choice for the part $f_1(x)$ is important. For the method to be more efficient, we select $f_1(x)$ as one term of $f$ or at least a number of terms if possible and $f_2(x)$ consists of the remaining terms of $f$. By using the MADM, from (8), we can write Eq. (3) in the form

$$\xi_k(x)u^k(x) + \sum_{j=0}^{k-1} \xi_j(x)u^j(x) = f_1(x) + f_2(x) + \lambda_1 \int_a^x K_1(x,t)G_1(u(t))dt + \lambda_2 \int_a^b K_2(x,t)G_2(u(t))dt.$$

Then

$$u^k(x) = f_1(x) + f_2(x) + \lambda_1 \int_a^x \frac{K_1(x,t)}{\xi_k(x)} G_1(u(t))dt + \lambda_2 \int_a^b \frac{K_1(x,t)}{\xi_k(x)} G_2(u(t))dt - \sum_{j=0}^{k-1} \frac{\xi_j(x)}{\xi_k(x)} u^j(x).$$

To obtain the approximate solution, we integrating $(k)$-times in the interval $[a, x]$ with respect to $x$ we obtain,

$$u(x) = L^{-1}\left(\frac{f_1(x)}{\xi_k(x)}\right) + L^{-1}\left(\frac{f_2(x)}{\xi_k(x)}\right) + \sum_{r=0}^{k-1} \frac{1}{r!}(x-a)^r b_r + \lambda_1 L^{-1}\left(\int_a^x \frac{K_1(x,t)}{\xi_k(x)} G_1(u(t))dt\right) + \\ + \lambda_2 L^{-1}\left(\int_a^b \frac{K_2(x,t)}{\xi_k(x)} G_1(u(t))dt\right) - \sum_{j=0}^{k-1} L^{-1}\left(\frac{\xi_j(x)}{\xi_k(x)} u_n^{(j)}(x)\right).$$

The components $u_0, u_1, u_2, \ldots$ are usually determined recursively by

$$u_0 = L^{-1}\left(\frac{f_1(x)}{\xi_k(x)}\right) + \sum_{r=0}^{k-1} \frac{1}{r!}(x-a)^r b_r,$$

$$u_1 = L^{-1}\left(\frac{f_2(x)}{\xi_k(x)}\right) + \lambda_1 L^{-1}\left(\int_a^x \frac{K_1(x,t)}{\xi_k(x)} A_0(t)dt\right) + \lambda_2 L^{-1}\left(\int_a^b \frac{K_2(x,t)}{\xi_k(x)} B_0(t)dt\right) - \\ - \sum_{j=0}^{k-1} L^{-1}\left(\frac{\xi_j(x)}{\xi_k(x)} u_0^{(j)}(x)\right),$$

$$u_n = \lambda_1 L^{-1}\left(\int_a^x \frac{K_1(x,t)}{\xi_k(x)} A_{n-1}(t)dt\right) + \lambda_2 L^{-1}\left(\int_a^b \frac{K_2(x,t)}{\xi_k(x)} B_{n-1}(t)dt\right) - \\ - \sum_{j=0}^{k-1} L^{-1}\left(\frac{\xi_j(x)}{\xi_k(x)} u_{n-1}^{(j)}(x)\right), \ n \geqslant 1.$$

Then, $u(x) = \sum_{i=0}^{n} u_i$ as the approximate solution.

# 2. Main results

In this section, we shall give an existence and uniqueness results of Eq. (1), with the initial condition (2) and prove it.

We can be written Eq. (1) in the form of:

$$u(x) = L^{-1}\Big[\frac{f(x)}{\xi_k(x)}\Big] + \sum_{r=0}^{k-1} \frac{(x-a)^r}{r!} b_r + \lambda_1 L^{-1}\Big[\int_a^x \frac{1}{\xi_k(x)} K_1(x,t) G_1(u_n(t))dt\Big] + \\ + \lambda_2 L^{-1}\Big[\int_a^b \frac{1}{\xi_k(x)} K_2(x,t) G_2(u_n(t))dt\Big] - L^{-1}\Big[\sum_{j=0}^{k-1} \frac{\xi_j(x)}{\xi_k(x)} u^{(j)}(x)\Big]. \tag{9}$$

Such that,

$$L^{-1}\Big[\int_a^x \frac{1}{\xi_k(x)} K_1(x,t) G_1(u_n(t))dt\Big] = \int_a^x \frac{(x-t)^k}{k!\xi_k(x)} K_1(x,t) G_1(u_n(t))dt \tag{10}$$

$$\sum_{j=0}^{k-1} L^{-1}\Big[\frac{\xi_j(x)}{\xi_k(x)}\Big] u^{(j)}(x) = \sum_{j=0}^{k-1} \int_a^x \frac{(x-t)^{k-1}\xi_j(t)}{k-1!\xi_k(t)} u^{(j)}(t)dt. \tag{11}$$

We set,

$$\Psi(x) = L^{-1}\Big[\frac{f(x)}{\xi_k(x)}\Big] + \sum_{r=0}^{k-1} \frac{(x-a)^r}{r!} b_r.$$

Before starting and proving the main results, we introduce the following hypotheses:

**(H1)** There exist two constants $\alpha, \beta$ and $\gamma_j > 0, j = 0, 1, \ldots, k$ such that, for any $u_1, u_2 \in C(J, \mathbb{R})$

$$|G_1(u_1)) - G_1(u_2))| \leqslant \alpha |u_1 - u_2|,$$
$$|G_2(u_1) - G_2(u_2)| \leqslant \beta |u_1 - u_2|$$

and

$$\left|D^j(u_1) - D^j(u_2)\right| \leqslant \gamma_j |u_1 - u_2|,$$

we suppose that the nonlinear terms $G_1(u(x))), G_2(u(x)))$ and $D^j(u) = \left(\frac{d^j}{dx^j}\right)u(x) = \sum_{i=0}^{\infty} \gamma_{i_j}$, ($D^j$ is a derivative operator), $j = 0, 1, \ldots, k$, are Lipschitz continuous.

**(H2)** We suppose that for all $a \leqslant t \leqslant x \leqslant b$, and $j = 0, 1, \ldots, k$:

$$\left|\frac{\lambda_1 (x-t)^k K_1(x,t)}{k!\xi_k(x)}\right| \leqslant \theta_1, \qquad \left|\frac{\lambda_1 (x-t)^k K_1(x,t)}{k!}\right| \leqslant \theta_2,$$

$$\left|\frac{(x-t)^{k-1}\xi_j(t)}{(k-1)!\xi_k(t)}\right| \leqslant \theta_3, \qquad \left|\frac{(x-t)^{k-1}\xi_j(t)}{(k-1)!}\right| \leqslant \theta_4,$$

$$\left|\lambda_2 L^{-1}\Big[\frac{K_2(x,t)}{\xi_k(x)}\Big]\right| \leqslant \ \theta_5, \qquad \left|\lambda_2 L^{-1}\Big[K_2(x,t)\Big]\right| \leqslant \theta_6.$$

**(H3)** There exist three functions $\theta_3^*, \theta_4^*,$ and $\gamma^* \in C(D, \mathbb{R}^+)$, the set of all positive function continuous on $D = \{(x,t) \in \mathbb{R} \times \mathbb{R} : 0 \leqslant t \leqslant x \leqslant 1\}$ such that:

$$\theta_3^* = \max|\theta_3|, \quad \theta_4^* = \max|\theta_4|, \quad \text{and} \quad \gamma^* = \max|\gamma_j|.$$

**(H4)** $\Psi(x)$ is bounded function for all $x$ in $J = [a,b]$.

**Theorem 2.1.** *Assume that (H1)–(H4) hold. If*

$$0 < \psi = (\alpha\theta_1 + \beta\theta_5 + k\gamma^*\theta_3^*)(b-a) < 1, \tag{12}$$

*Then there exists a unique solution $u(x) \in C(J)$ to IVB (1)–(2).*

*Proof.* Let $u_1$ and $u_2$ be two different solutions of IVB (1)–(2 ), then

$$\begin{aligned} \Big|u_1 - u_2\Big| &= \Big|\int_a^x \frac{\lambda_1(x-t)^k K_1(x,t)}{\xi_k(x)k!}\big[G_1(u_1) - G_1(u_2)\big]dt + \\ &+ \int_a^b \lambda_1 L^{-1}\Big[\frac{K_2(x,t)}{\xi_k(x)}\Big]\big[G_2(u_1) - G_2(u_2)\big]dt - \\ &- \sum_{j=0}^{k-1}\int_a^x \frac{(x-t)^{k-1}\xi_j(t)}{\xi_k(t)(k-1)!}\big[D^j(u_1) - D^j(u_2)\big]dt\Big| \leqslant \\ &\leqslant \int_a^x \left|\frac{\lambda_1(x-t)^k K_1(x,t)}{\xi_k(x)k!}\right| \Big|G_1(u_1) - G_1(u_2)\Big|dt + \\ &+ \int_a^b \left|\lambda_1 L^{-1}\Big[\frac{K_2(x,t)}{\xi_k(x)}\right| \Big|G_2(u_1) - G_2(u_2)\Big|dt - \\ &- \sum_{j=0}^{k-1}\int_a^x \left|\frac{(x-t)^{k-1}\xi_j(t)}{\xi_k(t)(k-1)!}\right| \Big|D^j(u_1) - D^j(u_2)\Big|dt \leqslant \\ &\leqslant (\alpha\theta_1 + \beta\theta_5 + k\gamma^*\theta_3^*)(b-a)|u_1 - u_2|, \end{aligned}$$

we get $(1-\psi)|u_1 - u_2| \leqslant 0$. Since $0 < \psi < 1$, so $|u_1 - u_2| = 0$. Therefore, $u_1 = u_2$ and the proof is completed. □

**Theorem 2.2.** *Suppose that (H1)–(H4), and If $0 < \psi < 1$, hold, the series solution $u(x) = \sum_{m=0}^{\infty} u_m(x)$ and $\|u_1\|_\infty < \infty$ obtained by the m-order deformation is convergent, then it converges to the exact solution of the Volterra-Fredholm integro-differential equation (1)–(2).*

*Proof.* Denote as $(C[0,1], \|.\|)$ the Banach space of all continuous functions on $J$, with $|u_1(x)| \leqslant \infty$ for all $x$ in $J$.

Frist we define the sequence of partial sums $s_n$, let $s_n$ and $s_m$ be arbitrary partial sums with $n \geqslant m$. We are going to prove that $s_n = \sum_{i=0}^{n} u_i(x)$ is a Cauchy sequence in this Banach space:

$$\|s_n - s_m\|_\infty = \max_{\forall x \in J}|s_n - s_m| = \max_{\forall x \in J}\Big|\sum_{i=0}^{n} u_i(x) - \sum_{i=0}^{m} u_i(x)\Big| =$$

$$
\begin{aligned}
&= \max_{\forall x\in J}\Big|\sum_{i=m+1}^{n} u_i(x)\Big| = \\
&= \max_{\forall x\in J}\Big|\sum_{i=m+1}^{n}\int_a^x \frac{\lambda_1(x-t)^k K_1(x,t)}{k!\xi_k(x)}A_{i-1}dt + \\
&\quad + \sum_{i=m+1}^{n}\int_a^b \lambda_2 L^{-1}\Big[\frac{K_2(x,t)}{\xi_k(x)}\Big]B_{i-1}dt - \sum_{j=0}^{k-1}\int_a^x \frac{\xi_j(t)(x-t)^{k-1}}{(k-1)!\xi_k(t)}L_{(i-1)J}dt\Big| = \\
&= \max_{\forall x\in J}\Big|\int_a^x \frac{\lambda_1(x-t)^k K_1(x,t)}{k!\xi_k(x)}\Big(\sum_{i=m}^{n-1}A_i\Big)dt + \int_a^b \lambda_2 L^{-1}\Big[\frac{K_2(x,t)}{\xi_k(x)}\Big]\Big(\sum_{i=m}^{n-1}B_i\Big)dt - \\
&\quad - \sum_{j=0}^{k-1}\int_a^x \frac{\xi_j(t)(x-t)^{k-1}}{(k-1)!\xi_k(t)}\Big(\sum_{i=m}^{n-1}L_{iJ}dt\Big)\Big|.
\end{aligned}
$$

From (6), we have

$$
\begin{aligned}
&\sum_{i=m}^{n-1}A_i = G_1(s_{n-1}) - G_1(s_{m-1}),\ \sum_{i=m}^{n-1}B_i = G_2(s_{n-1}) - G_2(s_{m-1}),\\
&\sum_{i=m}^{n-1}L_i = D^j(s_{n-1}) - D^j(s_{m-1}).
\end{aligned}
$$

So,

$$
\begin{aligned}
\|s_n - s_m\|_\infty &= \max_{\forall x\in J}\Big|\int_0^x \frac{\lambda_1(x-t)^k K_1(x,t)}{k!\xi_k(x)}[G_1(s_{n-1}) - G_1(s_{m-1})]dt + \\
&\quad + \int_a^b \lambda_2 L^{-1}\Big[\frac{K_2(x,t)}{\xi_k(x)}\Big][G_2(s_{n-1}) - G_2(s_{m-1})]dt - \\
&\quad - \sum_{j=0}^{k-1}\int_a^x \frac{\xi_j(t)(x-t)^{k-1}}{(k-1)!\xi_k(t)}[D^j(s_{n-1}) - D^j(s_{m-1})]dt\Big| \leqslant \\
&\leqslant \max_{\forall x\in J}\int_0^x \Big|\frac{\lambda_1(x-t)^k K_1(x,t)}{k!\xi_k(x)}\Big|\Big|G_1(s_{n-1}) - G_1(s_{m-1})\Big|dt \\
&\quad + \int_a^b \Big|\lambda_2 L^{-1}\Big[\frac{K_2(x,t)}{\xi_k(x)}\Big]\Big|\Big|G_2(s_{n-1}) - G_2(s_{m-1})\Big|dt + \\
&\quad + \sum_{j=0}^{k-1}\int_a^x \Big|\frac{\xi_j(t)(x-t)^{k-1}}{(k-1)!\xi_k(t)}\Big|\Big|D^j(s_{n-1}) - D^j(s_{m-1})\Big|dt
\end{aligned}
$$

Let $n = m+1$, then

$$
\|s_n - s_m\|_\infty \leqslant \psi\|s_m - s_{m-1}\|_\infty \leqslant \psi^2\|s_{m-1} - s_{m-2}\|_\infty \leqslant \ldots \leqslant \psi^m\|s_1 - s_0\|_\infty,
$$

so,

$$
\begin{aligned}
\|s_n - s_m\|_\infty &\leqslant \|s_{m+1} - s_m\|_\infty + \|s_{m+2} - s_{m+1}\|_\infty + \cdots + \|s_n - s_{n-1}\|_\infty \leqslant \\
&\leqslant [\psi^m + \psi^{m+1} + \cdots + \psi^{n-1}]\|s_1 - s_0\|_\infty \leqslant \\
&\leqslant \psi^m[1 + \psi + \psi^2 + \cdots + \psi^{n-m-1}]\|s_1 - s_0\|_\infty \leqslant \\
&\leqslant \psi^m(\frac{1-\psi^{n-m}}{1-\psi})\|u_1\|_\infty.
\end{aligned}
$$

Since $0 < \psi < 1$, we have $(1 - \psi^{n-m}) < 1$, then

$$\|s_n - s_m\|_\infty \leqslant \frac{\psi^m}{1-\psi}\|u_1\|_\infty.$$

But $|u_1(x)| < \infty$, so, as $m \longrightarrow \infty$, then $\|s_n - s_m\|_\infty \longrightarrow 0$.

We conclude that $s_n$ is a Cauchy sequence in $C[0,1]$, therefore $u = \lim_{n\to\infty} u_n$.
Then, the series is convergence and the proof is complete. □

## 3. Illustrative example

In this section, we present the semi-analytical techniques based on ADM and MADM to solve Volterra-Fredholm integro-differential equations.

**Example 1.** Consider the following Volterra-Fredholm integro-differential equation.

$$u'(x) + xu(x) = 2x + x^3 - \frac{x^5}{5} - \frac{0.9^7}{7}x + \int_0^x u^2(t)dt + \int_0^{0.9} xu^3(t)dt, \tag{13}$$

with the initial conditions

$$u(0) = 0, \quad u'(0) = 0, \tag{14}$$

and the exact solution is $u(x) = x^2$. The numerical results of Example 1 are given in Tab. 1.

Table 1. Numerical results of the Example 1

| x | Exact solution | MADM | ADM |
|---|---|---|---|
| 0.1 | 0.010000 | 0.016377 | 0.010397 |
| 0.2 | 0.040000 | 0.046990 | 0.043354 |
| 0.3 | 0.090000 | 0.094713 | 0.097463 |
| 0.4 | 0.160000 | 0.148751 | 0.148954 |
| 0.5 | 0.250000 | 0.236624 | 0.240548 |
| 0.6 | 0.360000 | 0.342563 | 0.348973 |
| 0.7 | 0.490000 | 0.478846 | 0.473681 |
| 0.8 | 0.640000 | 0.635372 | 0.627596 |
| 0.9 | 0.810000 | 0.790145 | 0.764797 |

## Conclusion

We discussed different methods for solving nonlinear Volterra-Fredholm integro-differential equations, namely, Adomian decomposition method and modified decomposition method. To assess the accuracy of each method, the test example with known exact solution is used. The study outlines important features of these methods as well as sheds some light on advantages of one method over the other. In this work, the above methods have been successfully employed to obtain the approximate solution of a nonlinear Volterra-Fredholm integro-differential equation. The results show that these methods are very efficient, convenient and can be adapted to fit a larger class of problems, MADM is the easiest, the most efficient and convenient.

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## Существование и единственность решения для интегрально-дифференциальных уравнений Вольтерра-Фредгольма

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*В этой статье применяется модифицированный метод разложения Адомиана для нахождения приближенных решений интегрально-дифференциальных уравнений Вольтерра-Фредгольма. Мы доказываем существование и единственность результатов и сходимость решений. Наконец, приведен пример для демонстрации обоснованности и применимости предлагаемого метода.*

*Ключевые слова: интегрально-дифференциальное уравнение Вольтерра-Фредгольма, модифицированный метод разложения Адомиана, результаты существования и единственности, приближенное решение.*