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Examples of Groups with the Same Number of Subgroups of Every Index

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In this note we show that certain non-isomorphic free amalgamated products with cyclic amalgamated subgroup and the same type have the same number of subgroups of every index.

Keywords: non-isomorphic free amalgamated products.

Introduction

For a group G and a natural number n, we denote the number of subgroups of G of index n by $a_n(G)$. Suppose G and H are groups. Then, following Lubotzky and Segal [5], G and H are called *isospectral* if and only if $a_n(G) = a_n(H)$ for all natural numbers n. Our main theorem is as follows.

Theorem 0.1. *If $\mathcal{A}$ and $\mathcal{A}'$ are cyclic amalgams of the same type, then their universal completions are isospectral.*

All the unexplained terminology in Theorem 0.1 is introduced in Section Two. For here it suffices to say that a cyclic amalgam is one in which the amalgamated subgroup is cyclic. It is very easy to manufacture examples of cyclic amalgams of the same type which have non-isomorphic universal completions. Indeed, if two amalgams of finite groups have the same type and are not isomorphic then their universal completions are not isomorphic. The smallest example is obtained by taking $A_1 \cong A_2$ isomorphic to the Frobenius group of order 20 and B cyclic of order 4. Then there are exactly two isomorphism classes of amalgams say $\mathcal{A}_1$ and $\mathcal{A}_2$ of type $(\mathrm{F}_{20}, \mathrm{F}_{20}, \mathbb{Z}/4\mathbb{Z})$. Their universal completions have presentations

$$G(\mathcal{A}_1) \cong \langle x, y, z \mid x^5 = y^5 = z^4 = 1, x^z = x^3, y^z = y^2 \rangle$$

and

$$G(\mathcal{A}_2) \cong \langle x, y, z \mid x^5 = y^5 = z^4 = 1, x^z = x^2, y^z = y^2 \rangle.$$

Theorem 0.1 asserts that these two (non-isomorphic) groups are isospectral. It is natural to ask about the subgroup lattice in each group. It turns out the they are not isomorphic. In fact,

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using the computational algebra package MAGMA [1], it has been shown that the intersection of all subgroups of index 5 in the first group has index 1296000000 while in the second it has index 100. It follows easily that these groups have non-isomorphic subgroup lattices. It is also easy to see that $G(\mathcal{A}_1)$ has two normal subgroups of index 20 while $G(\mathcal{A}_2)$ has six. I don't know whether $G(\mathcal{A}_1)$ and $G(\mathcal{A}_2)$ have the same number of conjugacy classes of subgroups at a given index. For more information about subgroups of finite index in free amalgamated products see [8].

Theorem 0.1 has the following corollary.

Corollary 0.2. *For each natural number k, there exist k pairwise non-isomorphic amalgams with isospectral universal completions.*

Other instances of isospectral groups can be found in [3, Theorem 1.3] where du Sautoy, McDermott and Smith prove that the groups $\mathbb{Z} \times \mathbb{Z}$ and $\langle x, y, t \mid [x, y], t^2 = y, x^t = x^{-1} \rangle$ are isospectral. Also, in [6], Mednykh shows that Γ_g and Γ_g^* are isospectral where

$$\Gamma_g = \langle x_1, \ldots, x_g, y_1, \ldots, y_g \mid \prod_{i=1}^{g} [x_i, y_i] = 1 \rangle$$

and

$$\Gamma_g^* = \langle x_1, \ldots, x_g, y_1, \ldots, y_g \mid \prod_{i=1}^{g} (x_i y_i)^2 \rangle$$

are the fundamental groups of a closed orientable surface of genus g, respectively, a closed non-orientable surface of genus $2g$.

This note originated from observations made about symmetric presentations and their accompanying progenitors in the monomial case (see [2]). It turns out that there are two progenitors of shape $7^{*7} :_m \mathrm{F}_{21}$ up to isomorphism (where F_{21} denotes the Frobenius group of order 21). One has presentation $\langle x, y, z \mid x^7 = y^7 = z^3 = 1, x^z = x^2, y^z = y^2 \rangle$ and the other has presentation $\langle x, y, z \mid x^7 = y^7 = z^3 = 1, x^z = x^2, y^z = y^4 \rangle$ (so they are cyclic amalgams of the same type $(\mathrm{F}_{21}, \mathrm{F}_{21}, \mathbb{Z}/3\mathbb{Z})$). Experimenting with MAGMA [1] confirmed that the progenitors are not isomorphic (one has $\mathrm{GL}_3(2)$ as an image the other does not) and, more strikingly, showed that they have an identical number of subgroups at each index up to 15 (with 39791 subgroups of index at most 15). The main result of this paper says that the groups are in fact isospectral.

Our notation is standard, but we mention that $\mathrm{Sym}(n)$ denotes the symmetric group of degree n and that, for groups A and B, $\mathrm{Hom}(A, B)$ is the set of homomorphisms from A to B. All our maps are written on the right.

1. Amalgams and completions

A *group amalgam*, or more simply an *amalgam*, is a quintuple $\mathcal{A} = (A_1, A_2, B, \phi_1, \phi_2)$ where A_1, A_2 and B are groups and, for $i = 1, 2$, $\phi_i : B \to A_i$ are monomorphisms. Let $\mathcal{A} = (A_1, A_2, B, \phi_1, \phi_2)$ and $\mathcal{A}' = (A_1', A_2', B', \phi_1', \phi_2')$ be amalgams. Then $\mathcal{A}$ and $\mathcal{A}'$ have the same *type* provided there are group isomorphisms $\alpha_1 : A_1 \to A_1'$, $\alpha_2 : A_2 \to A_2'$ and $\gamma : B \to B'$ satisfying $\mathrm{Im}(\phi_1\alpha_1) = \mathrm{Im}(\gamma\phi_1')$ and $\mathrm{Im}(\phi_2\alpha_2) = \mathrm{Im}(\gamma\phi_2')$. The amalgams $\mathcal{A}$ and $\mathcal{A}'$ are *isomorphic* if there are group isomorphisms $\alpha_1 : A_1 \to A_1'$, $\alpha_2 : A_2 \to A_2'$ and $\gamma : B \to B'$ satisfying $\phi_1\alpha_1 = \gamma\phi_1'$ and $\phi_2\alpha_2 = \gamma\phi_2'$ (as homomorphisms with domain B). Obviously, isomorphic amalgams have the same type. Suppose that $\gamma \in \mathrm{Aut}(B)$. Then we define an amalgam $\mathcal{A}^\gamma = (A_1, A_2, B, \phi_1, \gamma\phi_2)$. The triple of maps $(\mathrm{Id}_{A_1}, \mathrm{Id}_{A_2}, \mathrm{Id}_B)$ demonstrates that $\mathcal{A}$ and $\mathcal{A}^\gamma$ have the same type. Often the type of an amalgam will be uniquely determined by specifying the groups A_1, A_2 and B. This is for example the case if, for $i = 1, 2$, any two subgroups of A_i isomorphic to B are conjugate in A_i. If this is the case we can denote the type of amalgam simply by (A_1, A_2, B) as we did in the introduction.

It is natural to ask how many isomorphism classes of amalgam there are of a given type. The answer is provided by the Goldschmidt Lemma. To state it we require a definition. Suppose that $H \le K$. Then $\mathrm{Aut}(K,H) = N_{\mathrm{Aut}(K)}(H)/C_{\mathrm{Aut}(K)}(H)$ identified as a subgroup of $\mathrm{Aut}(H)$.

Lemma 1.1 (Goldschmidt Lemma). *Suppose $\mathcal{A} = (A_1, A_2, B, \phi_1, \phi_2)$ is an amalgam and define*

$$X_1 = \{\phi_1 \alpha \phi_1^{-1} \mid \alpha \in \mathrm{Aut}(A_1, (B)\phi_1)\} \le \mathrm{Aut}(B)$$

and

$$X_2 = \{\phi_2 \beta \phi_2^{-1} \mid \beta \in \mathrm{Aut}(A_2, (B)\phi_2)\} \le \mathrm{Aut}(B).$$

Then the map

$$X_1 \gamma X_2 \mapsto \mathcal{A}^\gamma$$

defines a bijection between the set of (X_1, X_2)-double cosets in $\mathrm{Aut}(B)$ and isomorphism classes of amalgams of the same type as $\mathcal{A}$.

Proof. See [4, (2.7)]. □

Notice that X_1 and X_2 both contain all the inner automorphisms of B and so the calculation of the double cosets in the Goldschmidt Lemma really takes place in the outer automorphism group of B. For the proof of Theorem 0.1 the important point in Lemma 1.1 is conveyed by the following corollary.

Corollary 1.2. *If $\mathcal{A} = (A_1, A_2, B, \phi_1, \phi_2)$ is an amalgam and $\mathcal{A}'$ has the same type as $\mathcal{A}$, then there exists $\gamma \in \mathrm{Aut}(B)$ such that $\mathcal{A}'$ is isomorphic to $\mathcal{A}^\gamma$.*

Let $\mathcal{A} = (A_1, A_2, B, \phi_1, \phi_2)$ be an amalgam. A *representation* of $\mathcal{A}$ *into* a group G is a pair of homomorphisms (ψ_1, ψ_2) where $\psi_i \in \mathrm{Hom}(A_i, G)$, for $i = 1, 2$, such that $\phi_1\psi_1 = \phi_2\psi_2 \in \mathrm{Hom}(B, G)$. The triple

$$(\langle (A_1)\psi_1, (A_2)\psi_2 \rangle, \psi_1, \psi_2)$$

is called a *completion* of $\mathcal{A}$ (in G). A completion (G, ψ_1, ψ_2) of $\mathcal{A}$ is called *universal* provided that given any completion (H, ρ_1, ρ_2) of $\mathcal{A}$, there exists a unique $\pi :\in \mathrm{Hom}(G, H)$ such that $\rho_i = \psi_i \pi$ for $i = 1, 2$. Universal completions of $\mathcal{A}$ exist, are unique up to isomorphism, and the group G in the universal completion can be identified with the *free amalgamated product*

$$G(\mathcal{A}) \cong (A_1 \star A_2)/\langle (b)\phi_1(b^{-1})\phi_2 \mid b \in B \rangle$$

where $A_1 \star A_2$ is the free product of A_1 and A_2 (see [7]). We note that because $G(\mathcal{A})$ is the universal completion of $\mathcal{A}$, every representation of $\mathcal{A}$ into a group G leads to a unique homomorphism from $G(\mathcal{A})$ into G and *vice versa*.

Suppose that $\mathcal{A} = (A_1, A_2, B, \phi_1, \phi_2)$ is an amalgam. For a group G set

$$\mathrm{Hom}(\mathcal{A}, G) = \{\Psi \mid \Psi \text{ is a represention of } \mathcal{A} \text{ into } G\}.$$

Let $\theta \in \mathrm{Hom}(B, G)$. Then we say $(\psi_1, \psi_2) \in \mathrm{Hom}(\mathcal{A}, G)$ *extends* θ if and only if $\theta = \phi_1\psi_1 = \phi_2\psi_2$. Put

$$\mathrm{Hom}_\theta(\mathcal{A}, G) = \{\Psi \in \mathrm{Hom}(\mathcal{A}, G) \mid \Psi \text{ extends } \theta\}.$$

Assume that $\theta \in \mathrm{Hom}(B, G)$. Let $\pi_\theta : B \to B/\ker\theta$ be the projection map, and $\overline{\theta}$ be the canonical isomorphism from $B/\ker\theta$ to $(B)\theta$. Then $\theta = \pi_\theta\overline{\theta}$. Define

$$\widetilde{\theta} \in \mathrm{Hom}(N_G((B)\theta), \mathrm{Aut}(B/\ker\theta))$$

by $(x)\widetilde{\theta} = \overline{\theta} c_x \overline{\theta}^{-1}$ for all $x \in (B)\theta$ where c_x denotes the automorphism of $(B)\theta$ induced by conjugation by x. Finally, for $\gamma \in \mathrm{Aut}(B)$ such that $(\ker\theta)\gamma = \ker\theta$, define γ^* so that the $\pi_\theta\gamma^* = \gamma\pi_\theta$. So $\gamma^* \in \mathrm{Aut}(B/\ker\theta)$.

Lemma 1.3. *Assume that* $\theta \in \mathrm{Hom}(B, G)$ *and* $\gamma \in \mathrm{Aut}(B)$ *with* $(\ker\theta)\gamma = \ker\theta$*. If there exists* $x \in N_G((B)\theta)$ *such that* $(x)\widetilde{\theta} = {\gamma^{-1}}^*$*, then there exists a bijection between* $\mathrm{Hom}_\theta(\mathcal{A}, G)$ *and* $\mathrm{Hom}_\theta(\mathcal{A}^\gamma, G)$*.*

Proof. We define two maps

$$\begin{array}{rcl} \alpha : \mathrm{Hom}_\theta(\mathcal{A}, G) & \to & \mathrm{Hom}_\theta(\mathcal{A}^\gamma, G) \\ (\psi_1, \psi_2) & \mapsto & (\psi_1, \psi_2 c_x) \end{array}$$

and

$$\begin{array}{rcl} \beta : \mathrm{Hom}_\theta(\mathcal{A}^\gamma, G) & \to & \mathrm{Hom}_\theta(\mathcal{A}, G) \\ (\psi_1, \psi_2) & \mapsto & (\psi_1, \psi_2 c_{x^{-1}}). \end{array}$$

Clearly α and β are inverse to each other, so if they are well-defined we are done. So we show that $(\psi_1, \psi_2 c_x) \in \mathrm{Hom}_\theta(\mathcal{A}^\gamma, G)$. We have

$$\begin{array}{rcl} \gamma\phi_2\psi_2 c_x & = & \gamma\theta c_x = \gamma\pi_\theta\overline{\theta}c_x = \gamma\pi_\theta\overline{\theta}c_x\overline{\theta}^{-1}\overline{\theta} \\ & = & \gamma\pi_\theta(x)\widetilde{\theta}\overline{\theta} = \gamma\pi_\theta{\gamma^{-1}}^*\overline{\theta} = \gamma\gamma^{-1}\pi_\theta\overline{\theta} = \pi_\theta\overline{\theta} = \theta. \end{array}$$

Since $\phi_1\psi_1 = \theta$, α is well-defined. The proof that β is well-defined is similar. Thus the lemma holds. $\square$

We say that $\mathcal{A} = (A_1, A_2, B, \phi_1, \phi_2)$ is a *cyclic* amalgam if B is a cyclic group.

Theorem 1.4. *Suppose that* $\mathcal{A} = (A_1, A_2, B, \phi_1, \phi_2)$ *is a cyclic amalgam. Then for all natural numbers* n *and for all* $\gamma \in \mathrm{Aut}(B)$*, there is a bijection between* $\mathrm{Hom}(\mathcal{A}, \mathrm{Sym}(n))$ *and* $\mathrm{Hom}(\mathcal{A}^\gamma, \mathrm{Sym}(n))$*.*

Proof. Let $G = \mathrm{Sym}(n)$. Suppose that $\theta \in \mathrm{Hom}(B, G)$. Since B is a cyclic group, $(\ker\theta)\gamma = \ker\theta$. Furthermore, the generators for $(B)\theta$ in G all have the same cycle type and so are conjugate in G. Thus $\widetilde{\theta} \in \mathrm{Hom}(N_G((B)\theta), \mathrm{Aut}(B/\ker\theta))$ is an isomorphism. In particular, for all $\theta \in \mathrm{Hom}(B, G)$, there exists $x \in N_G((B)\theta)$ such that $\widetilde{(x)\theta} = {\gamma^{-1}}^*$. It follows from Lemma 1.3 that for all $\theta \in \mathrm{Hom}(B, G)$, there is a bijection between $\mathrm{Hom}_\theta(\mathcal{A}, G)$ and $\mathrm{Hom}_\theta(\mathcal{A}^\gamma, G)$. Since $\mathrm{Hom}(\mathcal{A}, G) = \coprod_{\theta\in\mathrm{Hom}(B,G)} \mathrm{Hom}_\theta(\mathcal{A}, G)$ and $\mathrm{Hom}(\mathcal{A}^\gamma, G) = \coprod_{\theta\in\mathrm{Hom}(B,G)} \mathrm{Hom}_\theta(\mathcal{A}^\gamma, G)$ the theorem is true. $\square$

Since there is a bijection between $\mathrm{Hom}(\mathcal{A}, G)$ and $\mathrm{Hom}(G(\mathcal{A}), G)$, we have the following corollary.

Corollary 1.5. *For all natural numbers* n *and all* $\gamma \in \mathrm{Aut}(B)$*, there is a bijection between* $\mathrm{Hom}(G(\mathcal{A}), \mathrm{Sym}(n))$ *and* $\mathrm{Hom}(G(\mathcal{A}^\gamma), \mathrm{Sym}(n))$*.*

For a group G, let $h_n(G) = |\mathrm{Hom}(G, \mathrm{Sym}(n))|$. The following result intertwines $h_n(G)$ and $a_n(G)$.

Lemma 1.6. *Suppose that* G *is a group. Then*

$$a_n(G) = \frac{1}{(n-1)!}h_n(G) - \sum_{k=1}^{n-1} \frac{1}{(n-k)!}h_{n-k}(G)a_k(G).$$

Proof. See [5, Corollary 1.1.4]. $\square$

Proof of the Main Theorem. Suppose that $\mathcal{A}$ and $\mathcal{A}'$ are cyclic amalgams of the same type. Then by Corollary 1.2 there exists $\gamma \in \mathrm{Aut}(B)$ such that $\mathcal{A}'$ is isomorphic to $\mathcal{A}^{\gamma}$. It follows from Corollary 1.5 that $h_n(G(\mathcal{A})) = h_n(G(\mathcal{A}'))$ for all n. We have that $a_1(G(\mathcal{A})) = a_1(G(\mathcal{A}'))$ and so using Lemma 1.6 and induction gives us that $a_n(G(\mathcal{A})) = a_n(G(\mathcal{A}'))$ for all n. □

We now prove Corollary 0.2. Suppose that k is a natural number. Let p be a prime such that $p-1 \geq k^2$, then $\phi(p-1) \geq k$ where ϕ is the Euler totient function. Now let A_1 and A_2 be Frobenius groups of order $p(p-1)$ with kernel of order p and let B be the cyclic group of order $p-1$. Then, with X_1 and X_2 as in the Goldschmidt Lemma, $X_1 = X_2 = 1$ and so there are exactly $|\mathrm{Aut}(B)| = \phi(p-1) \geq k$ pairwise non-isomorphic amalgams of type (A_1, A_2, B). Now the corollary follows from Theorem 0.1.

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Примеры групп с одинаковым числом подгрупп любого индекса

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Мы показываем, что при определённых условиях неизоморфные свободные произведения с циклической амальгамой имеют и одинаковое число подгрупп каждого индекса.

Ключевые слова: неизоморфные свободные смешанные произведения.