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On One Condition for the Decomposition of an Entire Function into an Infinite Product

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The aim of this paper is proof a decomposition of entire function of finite order of growth with given zero set into the infinite product.

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Entire functions in $\mathbb{C}^1$ have the well-known decomposition (Weierstrass theorem). The decomposition of functions of finite order of growth is given by the theorem of Hadamard.

In several complex variables there exist the analogs of Hadamard theorem (see [1, 2]), but they not to be the infinite products.

The object of study of this work is the entire functions in several complex variables of finite order of growth with a special type of zeros.

Let Q_j be a polynomials in $\mathbb{C}^n$, and let $[1 - Q_j]$ be the divisor (zero set) of function $1 - Q_j$. If there exist entire function $f(z)$, for which zero set are equal to $\bigcup_{j=1}^{\infty}[1 - Q_j]$, then every ball of $\mathbb{C}^n$ meets with only finite number of sets $[1 - Q_j]$.

Let us assume

$$f(z) = \prod_{j=1}^{\infty} E(Q_j(z), p_j - 1) = \prod_{j=1}^{\infty} (1 - Q_j(z)) e^{Q_j(z) + \frac{Q_j^2(z)}{2} + \ldots + \frac{Q_j^{p_j-1}(z)}{p_j-1}},$$

where the expression

$$E(Q, p) = (1 - Q(z)) e^{Q(z) + \frac{Q^2(z)}{2} + \ldots + \frac{Q^p(z)}{p}}, \quad p = 1, 2, \ldots .$$

We call it a *primary factor.* Every primary factor vanishes on the set $\{Q(z) = 1\}$. Under condition $|Q| < 1$ the Taylor expansion of logarithm is given by

$$\log E(Q, p) = -\frac{Q^{p+1}}{p+1} - \frac{Q^{p+2}}{p+2} - \ldots .$$

If $k > 1$ and $|Q| \leqslant \dfrac{1}{k}$, then

$$|\log E(Q, p)| \leqslant |Q|^{p+1} + |Q|^{p+2} + \ldots \leqslant |Q|^{p+1} \left(1 + \frac{1}{k} + \frac{1}{k^2} + \ldots\right) = \frac{k}{k-1} |Q|^{p+1}.$$

Theorem 1. *Consider the sequence of polynomials Q_j, $j = 1, \ldots, n, \ldots$, for which the degrees of all Q_j are bounded by a number q, and Q_j have the form*

$$Q_j(z) = \sum_{\|\beta| \leqslant q} c_\beta^j z^\beta.$$

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Then under the condition

$$\alpha^j \to 0, \quad j \to \infty$$

there exists the entire function, having zero set $\bigcup_{j=1}^{\infty}[1-Q_j]$. *Here* $\beta = (\beta_1, \dots, \beta_n)$ *is multi-index, and* $\|\beta\| = \beta_1 + \ldots + \beta_n$.

Proof. Put

$$f(z) = \prod_{j=1}^{\infty} E(Q_j(z), p_j - 1).$$

Consider $r = |z| > 1$. Then

$$|Q_j|^{p_j} \leqslant \left(\sum_{\|\alpha\| \leqslant q} |c_\alpha^j||z^\alpha| \right)^{p_j} \leqslant \left(c_j \alpha_j |z|^{\deg Q_j}\right)^{p_j} \leqslant \left(c_j \alpha_j r^{\deg Q_j}\right)^{p_j} \leqslant (c_j \alpha_j r^q)^{p_j} \to 0,$$

if $j \to \infty$. Here c_j are numbers of monomials in Q_j.
Therefore

$$|\log E(Q_j(z), p_j - 1)| \leqslant \frac{k}{k-1}|Q_j|^{p_j} \leqslant \frac{k}{k-1} (c_j \alpha_j r^q)^{p_j} \leqslant 2 (c_j \alpha_j r^q)^{p_j}. \tag{1}$$

Choose p_j so, that $\sum\limits_j (c_j \alpha_j r^q)^{p_j}$ converges for all $r \leqslant R$. Such p_j are always exist. For example, take $p_j = j$ and under condition $\alpha_j \to 0$, $\dfrac{1}{2} > c_j \alpha_j r^q$ we have

$$(c_j \alpha_j r^q)^j < \frac{1}{2^j}$$

Therefore under $|z| \leqslant R$ the series

$$\sum_j \log E(Q_j(z), p_j - 1)$$

uniform and absolutely converges and uniform and absolutely converges the product

$$\prod_j E(Q_j(z), p_j - 1).$$

Therefore, function $f(z)$ is holomorphic in every ball $|z| \leqslant R$, and zero set of f is $\bigcup_{j=1}^{\infty}[1 - Q_j]$. □

Theorem 1 is the analog of Weierstrass theorem for entire functions of one complex variable.
An entire function $f(z)$ has *finite order of growth*, if there exists a positive number A, for which

$$f(z) = O(e^{r^A})$$

under $|z| = r \to \infty$.
Lower boundary ρ of such numbers A is called *order of growth of function*.
So if $f(z)$ is a function of order ρ, then for any positive ε we have

$$f(z) = O(e^{r^{\rho+\varepsilon}}), \quad r \to \infty,$$

but for any negative ε this unequality fails.
We will assume $f(0) = 1$. Let us the series $\sum\limits_j \alpha_j^\beta$ converge for the function $f(z)$ for some $\beta > 0$.

Lower boundary of positive numbers β, for which the series $\sum_j \alpha_j^\beta$ converges we call the *index of convergence of series* and denote its ρ_1.

We will prove the important property, if $\rho_1 > 0$, then the function $f(z)$ has finite order of growth. We show that there exists independent of j integer number p, such that the product

$$\prod_{j=1}^{\infty} E(Q_j, p) \tag{2}$$

converges for all z. This product converges if the series

$$\sum_j (\alpha_j r^q)^{p+1} = r^{q(p+1)} \sum_j \alpha_j^{p+1}, \tag{3}$$

converges i.e. the series

$$\sum_n (\alpha_n)^{p+1} \tag{4}$$

converges (here $p_j = p+1$). So the series (3) converges for all values r, if $p+1 > \rho_1$.

Product (2) with the smallest of the entire p, for which series (3) converges, we calle the *canonical product*, constructed by the zeros of $f(z)$, and the smallest p we call it *order* of the series so that $p+1 \geqslant \rho_1$.

Theorem 2. *Consider the sequence of polynomials Q_j, $j = 1, \ldots, n, \ldots$, for which the degrees of all Q_j are bounded by a number q, and Q_j have the form*

$$Q_j(z) = \sum_{\|\beta| \leqslant q} c_\beta^j z^\beta.$$

If the index of convergence of a series (4) $\rho_1 > 0$ then there exists entire function f having a finite order of growth ρ and $\rho \leqslant q\rho_1$. Here $\alpha_j = \max_\alpha |c_\alpha^j|$

Proof. Let k be a constant greater than 1. Write

$$\log|P(z)| = \sum_{\alpha_j^{-1} \leqslant kr^q} \log|E(Q_j, p)| + \sum_{\alpha_j^{-1} > kr^q} \log|E(Q_j, p)| = \sum_1 + \sum_2 .$$

Inequalities (1) imply that

$$\sum_2 = O\left[\sum_{\alpha_j^{-1} > kr^q} (\alpha_j r^q)^{p+1}\right] = O\left[r^{q(p+1)} \sum_{\alpha_j^{-1} > kr^q} \alpha_j^{p+1}\right], \quad r \to \infty.$$

If $p = \rho_1 - 1$ then $O(r^{q(p+1)}) = O(r^{q\rho_1})$. If $p+1 < \rho_1 + \varepsilon$, for some ε, then

$$r^{q(p+1)} \sum_{\alpha_j^{-1} > kr^q} \alpha_j^{p+1} = r^{q(p+1)} \sum_{\alpha_j^{-1} > kr^q} \alpha_j^{-\rho_1 - \varepsilon + p + 1} \alpha_j^{\rho_1 + \varepsilon} <$$

$$< r^{q(p+1)} (kr^q)^{\rho_1 + \varepsilon - p - 1} \sum_j \alpha_j^{\rho_1 + \varepsilon} = O(r^{q(\rho_1 + \varepsilon)}) \quad r \to \infty.$$

The sum $\sum_1$ contains terms $\log|E(Q_j, p)|$ for which $|Q_j| \geqslant \dfrac{1}{k}$ therefore

$$\log|E(Q_j, p)| \leqslant \log(1 - |Q_j|) + |Q_j| + \ldots + \frac{|Q_j|^p}{p} < K|Q_j|^p,$$

where K depends only on k. Therefore

$$\sum_1 = O\left(r^{pq}\sum_{\alpha_j^{-1}\leqslant kr^q}\alpha_j^p\right) = O\left(r^{pq}\sum_{\alpha_j^{-1}\leqslant kr^q}\alpha_j^{-\rho_1-\varepsilon+p}\alpha_j^{\rho_1+\varepsilon}\right) =$$

$$= O\left(r^{pq}(kr)^{q(\rho_1+\varepsilon)-pq}\sum r_j^{\rho_1+\varepsilon}\right) = O(r^{q(\rho_1+\varepsilon)}) \quad r\to\infty.$$

Thus, $\log|P(z)| = O(r^{q(\rho_1+\varepsilon)})$ as desire. □

Theorem 3 (analogue of the Hadamard theorem)**.** *If function $f(z)$ is an entire function with zero set $\bigcup_{j=1}^{\infty}[1-Q_j]$, $f(0)=1$ and $\rho_1>0$, then*

$$f(z) = e^{M(z)}P(z),$$

when $P(z)$ is canonical product, constructed by the zero set of a function $f(z)$ and $M(z)$ is a polynomial with degree not higher then $q\rho_1$.

Proof. Theorem 2 and general form of entire function of finite order of growth (see, for example, [2]) imply this theorem. □

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References

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Об одном условии разложения целой функции в бесконечное произведение

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Целью статьи является доказательство разложения целой функции конечного порядка роста с заданными нулевыми множествами определенного вида в бесконечное произведение.

Ключевые слова: каноническое произведение, порядок роста, показатель сходимости ряда.