УДК 512.54

Problems of Bounding the p-Length and Fitting Height of Finite Soluble Groups

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Received 10.08.2013, received in revised form 16.09.2013, accepted 20.10.2013

This paper is a survey of some open problems and recent results about bounding the Fitting height and p-length of finite soluble groups. In many problems of finite group theory, nowadays the classification greatly facilitates reduction to soluble groups. Bounding their Fitting height or p-length can be regarded as further reduction to nilpotent groups. This is usually achieved by methods of representation theory, such as Clifford's theorem or theorems of Hall–Higman type. In some problems, it is the case of nilpotent groups where open questions remain, in spite of great successes achieved, in particular, by using Lie ring methods. But there are also important questions that still require reduction to nilpotent groups; the present survey is focused on reduction problems of this type. As examples, we discuss finite groups with fixed-point-free and almost fixed-point-free automorphisms, as well as generalizations of the Restricted Burnside Problem. We also discuss results on coset identities, which have applications in the study of profinite groups. Finally, we mention the open problem of bounding the Fitting height in the study of the analogue of the Restricted Burnside Problem for Moufang loops.

Keywords: Fitting height, p-length, soluble finite group, nilpotent group, profinite group, automorphism, Restricted Burnside Problem, coset identity.

To the 80th anniversary of Vladimir Petrovich Shunkov

Introduction

One of the most important parameters of a finite soluble group is its Fitting height (often also called nilpotent length). Recall that the *Fitting series* of a finite group G starts with $F_1(G) := F(G)$, its Fitting subgroup, which is the largest normal nilpotent subgroup. Further terms of the Fitting series are defined by induction: $F_{i+1}(G)$ is the pre-image of $F(G/F_i(G))$ in G. The least l such that $F_l(G) = G$ is called the *Fitting height* of a soluble group G; we denote it by $l(G)$.

We also recall the definition of p-soluble groups and their p-length. For a prime p, the largest normal p'-subgroup of a finite group G is denoted by $O_{p'}(G)$, and the largest normal p-subgroup by $O_p(G)$. The *upper p-series* starts with $O_{p'}(G)$; then $O_{p',p}(G)$ is the pre-image of $O_p(G/O_{p'}(G))$ in G; then $O_{p',p,p'}(G)$ is the pre-image of $O_{p'}(G/O_{p',p}(G))$ in G, and so on. A finite group G is said to be *p-soluble* if $O_{p',p,p',p,\dots,p,p'}(G) = G$, and the least number of symbols p in the subscript is its *p-length*, denoted by $l_p(G)$.

A "philosophical principle" consists in the observation that results in a particular area of mathematics are often considered "modulo" other areas. Simple or non-soluble finite groups are

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often studied modulo soluble groups: quite typically a problem is to determine simple composition factors, or the quotient by the soluble radical. Of course, nowadays such problems are often solved by using the classification of finite simple groups (CFSG), albeit possibly quite non-trivially. In turn, soluble finite groups are often studied modulo nilpotent groups: a typical result may be bounding the Fitting height, or p-length, etc. Such results are often achieved by methods of representation theory, such as Clifford's theorem or theorems of Hall–Higman type. However, nilpotent finite groups "have their own problems" : even if the group is already known to be nilpotent, it is important to obtain a bound for the nilpotency class or derived length. Such bounds often make a difference in applications to certain problems about locally finite or residually finite groups. Many most fruitful approaches to studying nilpotent groups are based on Lie ring methods.

Successful reductions to soluble and nilpotent groups have already been achieved in many questions of finite group theory, so that it is the case of nilpotent groups where further questions remain open, in spite of great successes achieved, in particular, by using Lie ring methods. But there are in fact several important open problems where reduction to nilpotent groups is still required. The purpose of this survey is to attract attention to this type of problems on soluble finite groups, where bounds for the Fitting height or p-length still have to be obtained.

As examples, we discuss finite groups with fixed-point-free and almost fixed-point-free automorphism and generalizations of the Restricted Burnside Problem. We also discuss results on coset identities, which have applications in the study of profinite groups. Finally, we mention the open problem of bounding the Fitting height of groups with triality in the study of the analogue of the Restricted Burnside Problem for Moufang loops.

1. Fixed-point-free and almost fixed-point-free automorphisms

An automorphism φ of a group G is said to be *fixed-point-free* (or *regular*) if the fixed-point subgroup is trivial: $C_G(\varphi) = 1$. There are very strong results on finite groups with a fixed-point-free automorphism, which we mention below. Studying finite groups G with an *almost fixed-point-free automorphism* φ means obtaining restrictions on G in terms of φ *and* $C_G(\varphi)$. In other words, if an automorphism $\varphi \in \operatorname{Aut} G$ has a relatively small, in some sense, fixed-point subgroup $C_G(\varphi)$, then it is natural to expect that G is 'almost' as good as in the fixed-point-free case. Classical examples of results for groups with an almost fixed-point-free automorphism of order 2 are the Brauer–Fowler theorem [4] for finite groups and Shunkov's theorem [61] for periodic groups.

Automorphism of prime order. We mention results on finite groups with automorphisms of prime order for completeness, as for them major problems have already been solved. Suppose that φ is an automorphism of prime order p. As one of the corner-stones of CFSG, Thompson's theorem [62, 63] on normal p-complements has a consequence that a finite group G with a fixed-point-free automorphism φ of prime order p is nilpotent. (For soluble groups, nilpotency was known since 1930s due to Clifford's theorem [8].) When G is already nilpotent (and even not necessarily finite) the Higman–Kreknin–Kostrikin theorem [24, 41, 42] further gives a bound $h(p)$ (*Higman's function*) for the nilpotency class is terms of p only. Higman's paper [24] contained only a 'pure existence' theorem, without any upper bound for $h(p)$. An explicit upper bound for $h(p)$ was obtained by a different method by Kreknin and Kostrikin [41, 42]. These results for nilpotent groups were obtained by using a Lie ring method. Thus, the situation here is quite satisfactory, the remaining problems being improving an upper bound for Higman's function.

Likewise, the study of almost fixed-point-free automorphisms of prime order is in a sense complete, modulo possible improvements in the bounds for the indices or co-ranks and for the nilpotency class of a nilpotent normal subgroup of bounded nilpotency class. Namely, using CFSG, Fong [13] proved that a finite group G admitting an automorphism φ of prime order p with $m = |C_G(\varphi)|$ fixed points has a soluble subgroup of (p, m)-bounded index. (Henceforth we use an abbreviation of the type "$(a, b, \dots)$-bounded" for "bounded above in terms of $a, b \dots$ only".) Then Hartley and Meixner [22] and Pettet [54] independently proved that there is a nilpotent subgroup of (p, m)-bounded index. This "almost nilpotency" result was obtained by using a Hall–Higman–type theorem. Khukhro [27, 28] proved that there is a subgroup of p-bounded nilpotency class and of (p, m)-bounded index; this result was obtained by using a Lie ring method and the corresponding Lie ring theorem was also proved.

Another measure of the size of a finite group G is its (Prüfer, or sectional) rank $\mathbf{r}(G)$, which is the minimum number r such every subgroup can be generated by r elements. For automorphisms of prime order that are almost fixed-point-free in the sense of rank similar definitive results have been obtained recently. Khukhro and Mazurov [33] proved that if a finite p'-group G admits an automorphism φ of prime order p with $r = \mathbf{r}(C_G(\varphi))$, then G has a soluble normal subgroup with quotient of (p, r)-bounded rank (examples show that no such result holds in a non-coprime case). Khukhro and Mazurov [33] further proved that if a soluble finite group G admits an automorphism φ of prime order p with $r = \mathbf{r}(C_G(\varphi))$, then G has characteristic subgroups $1 \leqslant R \leqslant N \leqslant G$ such that R and G/N have (p, r)-bounded rank, while N/R is nilpotent (examples show that one cannot get rid of the bit R here, even in the coprime case). This "almost nilpotency" result was obtained by combining Hall–Higman–type theorems with powerful p-groups. Finally, Khukhro [29] proved that if a nilpotent finite group G admits an automorphism φ of prime order p with $r = \mathbf{r}(C_G(\varphi))$, then G has a characteristic subgroup N such that G/N has (p, r)-bounded rank and the nilpotency class of N is p-bounded.

Remark 1. Results giving "almost solubility" and "almost nilpotency" of G when $C_G(\varphi)$ is "small" *cannot* be obtained by finding a "large" subgroup (say, of bounded index) on which φ acts s fixed-point-freely.

All these results on automorphisms of prime order are presented in Table 1. Columns correspond to hypotheses on the automorphism and its fixed-point subgroup, rows — to hypotheses on the group, and the results are described in the intersections. The reader can clearly see how the results are arranged in layers in accordance to the above-mentioned "philosophical principle". The bottom layer corresponds to the underlying Lie rings results. In particular, a special paper by Makarenko [47] was required to produce a nilpotent ideal of p-bounded nilpotency class and (p, m)-bounded codimension, because, in contrast to groups, for Lie algebras having a nilpotent subalgebra of finite codimension does not automatically imply having a nilpotent ideal of bounded codimension.

Automorphism of composite coprime order. For finite groups with automorphisms of arbitrary but coprime orders there are already strong results providing reduction to soluble and nilpotent groups. Here, it is the case of nilpotent groups where open problems still exist, in spite of definitive results for Lie rings with fixed-point-free and almost fixed-point-free automorphisms. Let $\varphi \in \operatorname{Aut} G$ be an automorphism of arbitrary (composite) order. It is known, as a consequence of CFSG, that finite groups with fixed-point-free automorphisms are soluble (see, for example, [67]). For soluble groups with fixed-point-free automorphisms, nice bounds for their Fitting heights have been obtained (in many cases also in more general situations, for groups of

Table 1. Automorphisms of prime order

φ / $C_G(\varphi)$ / G	$\|\varphi\| = p$ prime; $C_G(\varphi) = 1$	$\|\varphi\| = p$ prime; $\|C_G(\varphi)\| = m$	$\|\varphi\| = p$ prime ($p \nmid \|G\|$ for insoluble G); $\mathbf{r}(C_G(\varphi)) = r$
finite	nilpotent Thompson, 1959	$\|G/S(G)\| \leqslant f(p,m)$ Fong+CFSG, 1976	$\mathbf{r}(G/S(G)) \leqslant f(p,r)$ Khukhro+Mazurov +CFSG, 2006
+soluble	nilpotent Clifford, 1930s	$\|G/F(G)\| \leqslant f(p,m)$ Hartley+Meixner, 1981 Pettet, 1981	$G \geqslant N \geqslant R \geqslant 1$, $\mathbf{r}(G/N)$, $\mathbf{r}(R) \leqslant f(p,r)$, N/R nilpotent Khukhro+Mazurov, 2006
+nilpotent	class $\leqslant h(p)$ Higman, 1957 Kostrikin– Kreknin, 1963	$G \geqslant H$, $\|G : H\| \leqslant f(p,m)$, H nilpotent of class $\leqslant g(p)$ Khukhro, 1990	$G \geqslant N$, $\mathbf{r}(G/N) \leqslant f(p,r)$, N nilpotent of class $\leqslant g(p)$ Khukhro, 2008
Lie ring	same, by same	same, Khukhro, 1990; H ideal, Makarenko, 2005	same

automorphisms $A \leqslant \operatorname{Aut} G$ such that $C_G(A) = 1$). For an automorphism of coprime order, the best possible bound $l(G) \leqslant \alpha(|\varphi|)$, where $\alpha(n)$ is the number of not necessarily distinct primes whose product is n, was obtained by Hoffman [26], Shult [58], Gross [18], Berger [2]. Thus, the study of finite groups with a fixed-point-free automorphism of coprime order has been reduced to the case of nilpotent groups. Here it is the case of nilpotent groups where a major open problem remains: is the derived length bounded in terms of $|\varphi|$? It is remarkable that a similar problem for Lie rings has been solved in the affirmative long ago by Kreknin's theorem [41] of 1963: if a Lie ring L has a fixed-point-free automorphism of order n, then L is soluble of derived length at most $2^n - 2$.

Similarly, the study of almost fixed-point-free automorphisms of coprime order is in a sense already reduced to the case of nilpotent groups. Hartley [20] proved that a finite group with an automorphism φ of (not even necessarily coprime) order n with $m = |C_G(\varphi)|$ fixed points has a soluble subgroup of (n,m)-bounded index. Using Turull's theorem [65] Hartley and Isaacs [21] further proved that the index of the $(2\alpha(|\varphi|) + 1)$-st Fitting subgroup $F_{2\alpha(|\varphi|)+1}(G)$ is (n,m)-bounded (even for not necessarily cyclic soluble group of automorphisms of coprime order). Thus, there is a good reduction to nilpotent groups. But, as already mentioned, even the case of a fixed-point-free automorphism remains open. Here, too, a similar problem for Lie algebras has been solved in the affirmative by Khukhro and Makarenko [48]: if a Lie algebra L admits an automorphism φ of finite order n with $m = \dim C_L(\varphi)$, then L has a soluble ideal of n-bounded derived length and (n,m)-bounded codimension.

Given the rank $\mathbf{r}(C_G(\varphi))$ of the fixed-point subgroup of an automorphism of coprime order n, Khukhro and Mazurov [34] proved that G has a soluble normal subgroup with quotient of (n,r)-bounded rank (examples show that no such result holds in a non-coprime case). For soluble G,

Table 2. Automorphisms of coprime order

φ $C_G(\varphi)$ G	$\lvert\varphi\rvert = n$ coprime $C_G(\varphi) = 1$	$\lvert\varphi\rvert = n$ coprime $\lvert C_G(\varphi)\rvert = m$	$\lvert\varphi\rvert = n$ coprime $\mathbf{r}(C_G(\varphi)) = r$
finite	soluble CFSG	$\lvert G/S(G)\rvert \leqslant f(n,m)$ Hartley, 1992 +CFSG	$\mathbf{r}(G/S(G)) \leqslant f(n,r)$ Khukhro+Mazurov, 2006 +CFSG
+soluble	Fitting height $\leqslant \alpha(n)$ Shult, Gross, Berger, 1960s–70s	$\lvert G/F_{2\alpha(n)+1}(G)\rvert \leqslant f(n,m)$ Hartley+Isaacs, 1990 +Turull, 1984	$\mathbf{r}(G/F_{4^{\alpha(n)}}(G)) \leqslant f(n,r)$ Khukhro+Mazurov, 2006 +Thompson, 1964
+nilpotent	**Is derived length bounded?**	?	?
Lie algebra	soluble of derived length $\leqslant k(n)$ Kreknin, 1963	there is soluble ideal of codimension $\leqslant f(n,m)$ of derived length $\leqslant g(n)$ Khukhro+Makarenko, 2004	same as $\longleftarrow$

the Khukhro–Mazurov theorem [33] for the case of φ of prime order can be combined with Thompson's theorem [64] stating that $F(C_G(\psi)) \leqslant F_4(G)$ if ψ is an automorphism of prime order of a finite soluble group G of order coprime to $|\psi|$. Namely, a straightforward induction gives that the rank of the quotient by $F_{4^{\alpha(n)}}(G)$ is (n,r)-bounded. This already gives a reduction to the case of nilpotent groups, where, however, recall, even the fixed-point-free case remains open. It is also worth mentioning the conjecture that the exponential function of $\alpha(n)$ in the index of the Fitting subgroup here can be improved to a linear one, as in the Hartley–Isaacs theorem [21] in the order–index context.

These results on automorphisms of coprime order are presented in Table 2. As the reader can see, for nilpotent groups the table contains open questions instead of results. Apart from the safe case of automorphism of prime order, it is only the case of automorphism of order 4, where nice results have been obtained. Kov'acs [40] proved that a finite group with a fixed-point-free automorphism of order 4 is centre-by-metabelian, and Makarenko and Khukhro [45,46,49] proved that if a finite group G has an automorphism φ of order 4 with $m = |C_G(\varphi)|$ fixed points, then G has a subgroup H of m-bounded index such that $[[H,H],H]$ is nilpotent of class bounded by a certain constant.

Automorphism of non-coprime order. In the study of finite groups with automorphisms of arbitrary, not necessarily coprime, order there are already reductions to soluble groups based on CFSG. Rowley [55] proved the solubility of a finite group with a fixed-point-free automorphism. Hartley [20] proved a 'generalized Brauer–Fowler theorem': if a finite group G admits an automorphism of order n with $m = |C_G(\varphi)|$ fixed points, then it has a soluble subgroup of (n,m)-bounded index. Of course, open problems for nilpotent groups with (almost) fixed-point-free automorphisms of composite order that we saw in the coprime case remain open in the non-coprime case. But in the non-coprime case there are also open questions that still require reduction to nilpotent groups. The only exception is the case of a fixed-point-free automorphism:

Table 3. Automorphisms of non-coprime order

φ	$\lvert\varphi\rvert = n$ non-coprime		
G	$C_G(\varphi) = 1$	$\lvert C_G(\varphi)\rvert = m$	$\mathbf{r}(C_G(\varphi)) = r$
finite	soluble Rowley, 1995 +CFSG	$\lvert G/S(G)\rvert \leqslant f(n,m)$ Hartley, 1992 +CFSG	$\mathbf{r}(G/S(G)) \to \infty$ even n prime
+soluble	Fitting height $\leqslant 10{\cdot}2^{\alpha(n)}$ Dade, 1969 **Is there a bound that is polynomial in $\alpha(n)$?** **linear in $\alpha(n)$?**	**Are there functions g and f such that $\lvert G/F_{f(\alpha(n))}(G)\rvert \leqslant f(n,m)$?** **At least, is Fitting height $\leqslant f(n,m)$?**	?
+nilpotent	**Is derived length bounded?**	?	?
Lie algebra	soluble of derived length $\leqslant k(n)$ Kreknin, 1963	there is soluble ideal of codimension $\leqslant f(n,m)$ of derived length $\leqslant g(n)$ Khukhro+Makarenko, 2004	same as $\longleftarrow$

if a finite group G admits a nilpotent group of automorphisms A such that $C_G(A) = 1$, then G is soluble by a theorem of Belyaev and Hartley [1] based on CFSG, and the Fitting height of G is at most $10 \cdot 2^{\alpha(|A|)}$ by a special case of Dade's theorem [11]. It is conjectured that there must be a polynomial, or even a linear bound for the Fitting height, but even in the case of A cyclic this has been proved only in some special cases by Turull [66], Ercan and Güloğlu [10], and Ercan [9].

The whole situation is broadly reflected in Table 3.

We state separately one of the open problems in Table 3 on bounding the Fitting height, namely, its weaker version.

Problem 1 (Belyaev–Hartley, Kourovka Notebook, 13.8(a))**.** Suppose that φ is an automorphism of a soluble finite group G. Is the Fitting height of G bounded in terms of $|\varphi|$ and $|C_G(\varphi)|$?

Obviously, this is equivalent to the following: given an element $g \in G$ in a finite soluble group G, is the Fitting height of G bounded in terms of $|C_G(g)|$? An affirmative answer is known only in the case where the order $|\varphi| = p^k$ is a prime-power: Hartley and Turau [23] proved that then $|G : F_k(G)|$ is $(k, |C_G(g)|)$-bounded, the proof being basically a reduction to the coprime case. However, even the case $|\varphi| = 6$ of Problem 1 is open.

Characteristic subgroups of bounded index. In all the results giving "almost solubility" and "almost nilpotency" of a group G it is possible to claim that there is even a characteristic subgroup of bounded index (or in some cases of bounded 'co-rank') which is soluble or nilpotent with *the same* bound for the derived length or nilpotency class. More precisely, if an (arbitrary!) group G has a subgroup of index n satisfying a multilinear commutator identity $\varkappa = 1$, then G has a characteristic subgroup of $(n, \varkappa)$-bounded index satisfying the same identity $\varkappa = 1$. (By definition, a *multilinear commutator* $\varkappa$ is any commutator of weight m in m distinct variables; examples include commutators defining nilpotency, solubilty, etc.) This result was proved by Bruno and Napolitani [5] for nilpotent subgroups of finite index, and by Khukhro and Makarenko [31] for subgroups of finite index satisfying an arbitrary multilinear identity. Khukhro and Makarenko [32] also proved a similar result for subgroups of finite 'co-rank' (under some additional unavoidable conditions) satisfying an arbitrary multilinear identity. Moreover, this information on the existence of characteristic subgroups was crucial for proving some of the above-mentioned theorems on almost fixed-point-free automorphisms.

The anove-mentioned results on large characteristic subgroups were also further developed by Klyachko and Mel'nikova [36], Khukhro, Klyachko, Makarenko, and Mel'nikova [30], Makarenko and Shumyatsky [50], and Klyachko and Milent'eva [37].

2. Restricted Burnside problem and its generalizations

Recall that the affirmative solution of the Restricted Burnside Problem means the following:

If a finite group G is d-generated and has exponent n, then $|G| \leqslant f(d,n)$.

One can distinguish stages in this solution in accordance with the 'philosophical principle'. Reduction to soluble groups is provided by CFSG. The Hall–Higman paper [19] gave a reduction to (nilpotent) p-groups, by bounding the p-length (for odd primes). For nilpotent groups Zelmanov [69] performed a reduction to a problem on Engel Lie algebras. (Earlier connections with Lie rings were developed by Magnus [44] and Sanov [56]). Finally, the Restricted Burnside Problem was solved in the affirmative by Zelmanov's theorem [70, 71] on Engel Lie algebras. (For prime exponent, the result was obtained earlier by Kostrikin [38].)

In this section we discuss some generalizations of the Restricted Burnside Problem, focusing on related problems of bounding the Fitting height or p-length of finite groups. As in the original RBP, such bounds are intended to provide reductions to (pro-) p-groups, and their proofs require results similar to the Hall–Higman theorems.

Generalizations of Hall–Higman theorems. First we present theorems of Hall and Higman together with certain further improvements.

Theorem 1 (Exponent Theorem)**.** *If a finite group G is p-soluble and p^e is the exponent of its Sylow p-subgroup, then the p-length of G is at most $2e+1$. Furthermore, the p-length is at most e if p is not a Fermat prime or a Sylow 2-subgroup of G is abelian.*

The Hall–Higman paper [19] contained only the case of odd p (which was enough for the reduction theorem in the Restricted Burnside Problem). The case $p = 2$ was done later by Hoare [25] with the bound $l_2 \leqslant 3e - 1$, Gross [17] with the bound $l_2 \leqslant 2e - 1$, and finally Bryukhanova [6] with the best-possible bound $l_2 \leqslant e$.

Theorem 2 (Derived Length Theorem)**.** *If a finite group G is p-soluble and d is the derived length of its Sylow p-subgroup, then the p-length of G is at most d.*

Here, too, the Hall–Higman paper [19] contained only the case of odd p. The case $p = 2$ was done later by Berger and Gross [3] with the bound $l_2 \leqslant 2d - 2$, which was improved by Bryukhanova [7] to the best-possible bound $l_2 \leqslant d$.

The following problem is a natural generalization of the Hall–Higman–type theorems.

Problem 2 (Wilson, Kourovka Notebook, 9.68)**.** Let w be a nontrivial group word. Is there a bound for the p-lengths of finite p-soluble groups whose Sylow p-subgroups satisfy the identity $w \equiv 1$?

So far, the answer is known to be affirmative only in some special cases. The Exponent and Derived Length Hall–Higman–type Theorems 1 and 2 give affirmative answers for the identities $x^n \equiv 1$ and $\delta_d(x_1, \dots, x_{2^d}) \equiv 1$, where, by induction, $\delta_1(x_1, x_2) = [x_1, x_2]$ and $\delta_{k+1}(x_1, \dots, x_{2^{k+1}}) = [\delta_k(x_1, \dots, x_{2^k}, \delta_k(x_1, \dots, x_{2^k}]$ (and of course for any other identities that imply solubility or a bound for the exponent). It is also clear from the proof in the Hall–Higman paper [19] that the answer to Problem 2 is affirmative for the n-Engel identity $[...[[x, \underbrace{y], y], \dots, y}_{n}] \equiv 1$.

In this connection it is appropriate to mention the following problem in the Hall–Higman paper about wreath products, which we present in a somewhat weaker form.

Problem 3 (Hall and Higman, [19, p. 8])**.** Does there exist a function $s(l)$ that is unboundedly increasing as l increases such that any finite p-soluble group G of p-length l_p necessarily has a section isomorphic to the $s(l_p)$-fold wreath product $C_p \wr (\cdots \wr (C_p \wr (C_p \wr C_p)) \dots)$ of the cyclic groups C_p of order p?

If this problem were to be solved in the affirmative, then Wilson's problem would be solved in the affirmative, since no non-trivial identity can hold on the class of such wreath products.
Shumyatsky's problems. The affirmative solution of the Restricted Burnside Problem can also be stated as follows:

If a group G is residually finite and satisfies the identity $x^n \equiv 1$, then it is locally finite.

Problem 4 (Shumyatsky)**.** If a group G is residually finite and satisfies the identity $w^n \equiv 1$ for some group word w, must the verbal subgroup $w(G)$ be locally finite?

Note that for $w = x_1$ this is precisely the Restricted Burnside Problem.

Remark 2. In contrast to the Restricted Burnside Problem, it is unclear if an affirmative answer to Problem 4 for a given word w would also imply the existence of a function $f(d)$ bounding the order of d-generator subgroups of $w(G)$. This additional property can be regarded as a separate further question.

So far, the answer to Problem 4 is known to be affirmative only in some special cases. For example, the following holds.

Theorem 3 (Shumyatsky, [59])**.** *If a group G is residually finite and satisfies the identity $[x, y]^{p^k} \equiv 1$ for a prime p, then the derived subgroup $[G, G]$ is locally finite.*

The proof is essentially about finite groups. The first step is to show that $[G, G]$ is a p-group — this provides reduction to nilpotent groups. Then the following theorem of Zelmanov is applied to the Zassenhaus–Lazard Lie algebra $L_p([G, G])$, which is constructed on the direct sum of the so-called lower p-central series by defining the Lie products via groups commutators. We do not go into details of this construction, as our primary purpose is discussion of problems of bounding the Fitting height and p-length, which will appear in this context below.

Theorem 4 (Zelmanov, [73])**.** *If a Lie algebra over a field k satisfies a polynomial identity $f \equiv 0$ and is generated by d elements such that all (multiple) commutators in the generators are n-Engelian, then the Lie algebra is nilpotent of class bounded in terms of k, f, d, and n.*

The Lie algebra technique using Zelmanov's Theorem 4 that was developed by Shumyatsky also works in more general situations, including the nilpotent case of the following problem.

Problem 5 (Shumyatsky, Kourovka Notebook, 17.126)**.** If a group G is residually finite and satisfies the identity $[x, y]^n \equiv 1$, must $[G, G]$ be locally finite?

Thus, to solve Problem 5 in the affirmative, it remains only to provide reduction to nilpotent groups. (Reduction to soluble groups has already been obtained by using CFSG.)

Problem 6 (Shumyatsky, Kourovka Notebook, 17.126)**.** Suppose that a finite soluble group G satisfies the identity $[x, y]^n \equiv 1$. Is it true that the Fitting height of G is bounded in terms of n?

Of course, an affirmative solution of Wilson's Problem 2 for the identity $[x, y]^{p^k} \equiv 1$ would give an affirmative answer to Problem 6. But in Problem 6 the condition is stronger, on all commutators in the group, not just in a Sylow p-subgroup, so Problem 6 may be easier than Wilson's Problem 2 for the identity $[x, y]^{p^k} \equiv 1$. For the moment, an affirmative solution could only be achieved under stronger conditions. By definition, a multilinear commutator word is a commutator of weight n in n variables.

Theorem 5 (Shumyatsky, [60])**.** *Let w be a multilinear commutator. Suppose that G is a residually finite group in which any product of at most 896 values of the word w has order dividing a given positive integer n. Then the verbal subgroup $w(G)$ is locally finite. Furthermore, for $w = [x, y]$ one can replace 896 by 68.*

In the proof, which is essentially about finite groups, reduction to the soluble case is using CFSG. In the case of soluble finite groups the following results are used. By a theorem of Flavell, Guest, and Guralnick [12], an element a of a finite group G belongs to $F_l(G)$ if and only if every 4 conjugates of a generate a soluble subgroup of Fitting height at most l. Segal [57] proved that every element of the derived subgroup of a finite soluble d-generated group is a product of at most $72d^2+46d$ commutators, and a better bound was obtained by Nikolov and Segal [53], which for $d = 4$ gives 448. This result is applied in the situation where each commutator is a product of two w-values, which gives rise to 896. Thus a bound for the Fitting height is obtained, providing reduction to nilpotent groups. Then Shumyatsky's technique of using Zelmanov's Theorem 4 is applied similarly to the proof of Theorem 3.

Using information on soluble and p-soluble subgroups.

Information on the p-length of p-soluble subgroups or on the Fitting height of soluble subgroups can be used (together with CFSG) for providing reduction to p-soluble or soluble groups. Here is an example of such a reduction.

Theorem 6. *Let p be an odd prime. Suppose that the p-length of all p-soluble subgroups of a finite group G is at most h. Then G has a series of length $2h + 1$*

$$1 \leqslant H_0 \leqslant H_1 \leqslant \cdots \leqslant H_{2h+1} = G$$

such that H_i/H_{i-1} is p-soluble for i odd, and is a direct product (possibly empty) of non-abelian simple groups of order divisible by p for i even.

Proof. Let F be the pre-image in G of the generalized Fitting subgroup $F^*(G/R_p(G))$ of the quotient by the maximal normal p-soluble subgroup $R_p(G)$ (so-called p-soluble radical), and let $E = F/R_p(G)$. Then E contains its centralizer in $G/R_p(G)$ and $E = S_1 \times \cdots \times S_n$ is a direct product of subnormal non-abelian simple groups S_i of order divisible by p. Action by conjugation induces a permutational action of G on $\{S_1, \dots, S_n\}$; let K be the kernel of this action. We know from the classification that K/F is soluble. It is sufficient to prove that the maximum p-length of p-soluble subgroups for G/K is strictly smaller than for G. This will enable easy induction, so that G will have a required series of length $2h+1$ (with at most h non-p-soluble layers). We can of course assume that $R_p(G) = 1$. Thus, the result will follow from the following proposition.

Proposition. *Let p be an odd prime. Suppose that a finite group G has trivial p-soluble radical $R_p(G) = 1$, and let $E = S_1 \times \cdots \times S_n = F^*(G)$ be the generalized Fitting subgroup equal to a direct product of subnormal non-abelian simple groups S_i (of order divisible by p). Let $K = \bigcap_i N_G(S_i)$ be the kernel of the permutational action of G on $\{S_1, \ldots, S_n\}$ induced by conjugation. Then the maximum p-length of p-soluble subgroups for G/K is strictly smaller than for G.*

Proof. We use induction on the order of G. The argument proving the basis of this induction is incorporated in the step of induction.

Let M be a subgroup of G containing K such that M/K is a p-soluble subgroup of G/K with maximum possible p-length k. Our task is to find a p-soluble subgroup of G with p-length higher than k. Note that $R_p(M) = 1$ because $C_G(E) \leqslant E \leqslant M$. By induction we may assume that $G = M$, so that G/K is p-soluble of p-length k. Let T be a Sylow 2-subgroup of K and let $H = N_G(T)$. Then H is obviously a p-soluble subgroup of G such that $KH = G$ by the Frattini argument.

Let, without loss of generality, $\{S_1, \ldots, S_r\}$ be an orbit in the permutational action of G on $\{S_1, \ldots, S_n\}$ such that the image of G in the action on this orbit also has p-length k; this image coincides with the image of H. Let $D = S_1 \times \cdots \times S_r$; then $DH/R_p(DH)$ satisfies the hypothesis of the proposition with $F^*(DH/R_p(DH)) = DR_p(DH)/R_p(DH) \cong D$. Since $R_p(DH)$ is in the kernel of the permutational action of DH on $\{S_1, \ldots, S_r\}$, the p-length of the quotient of $DH/R_p(DH)$ by the kernel of the action on the set of the images of $S_1, \ldots, S_r$ is also equal to k. Therefore by induction we may assume that $R_p(DH) = 1$ and $G = DH$, so that $E = S_1 \times \cdots \times S_r$.

Since $p \neq 2$ by hypothesis and p divides $|S_1|$, we can use Thompson's theorem on normal p-complements to choose a characteristic subgroup C_1 of a Sylow p-subgroup P_1 of S_1 such that $N_{S_1}(C_1)/C_{S_1}(C_1)$ is not a p-group. Choose a p'-subgroup Q_1 in $N_{S_1}(C_1)$ that acts non-trivially on C_1. Choose elements $a_i \in G$ such that $S_i = S_1^{a_i}$ for $i = 1, \ldots, r$. Let $P_i = P_1^{a_i}$ and $C_i = C_1^{a_i}$. Let $P = \prod P_i$ and $C = \prod C_i$. Then P is a Sylow p-subgroup of E.

We claim that $N_G(C) \geqslant N_G(P)$. Indeed, if $x \in N_G(P)$ fixes S_i, then it fixes P_i, and therefore fixes C_i, which is characteristic in P_i. Now let $S_i^x = S_j$. Then we must have $P_i^x = P_j$, since $x \in N_G(P)$. Then $P_1^{a_i x} = P_1^{a_j}$, that is, $P_1^{a_i x a_j^{-1}} = P_1$, whence $C_1^{a_i x a_j^{-1}} = C_1$, since this is a characteristic subgroup of P_1. As a result, $C_i^x = C_1^{a_i x} = C_1^{a_j} = C_j$, which completes the proof of the claim.

Now, by the Frattini argument we have $EN_G(C) \geqslant EN_G(P) = G$, so that $EN_G(C) = G$.

We can assume that $N = N_G(C)$ is p-soluble. Indeed, any non-p-soluble section of G is inside E, and $N_E(C) = \prod N_{S_i}(C_i)$. If the $N_{S_i}(C_i)$ are not p-soluble, then we can use the induction hypothesis as follows. In the quotient $\bar{N} = N/R_p(N)$ by the p-soluble radical, the simple factors of the generalized Fitting subgroup $F^*(\bar{N})$ are images of some subgroups of the $N_{S_i}(C_i)$ (possibly, several subgroups in the same $N_{S_i}(C_i)$). The subgroups $N_{S_i}(C_i)$ are permuted by N, since N permutes the S_i and therefore must permute the C_i being $N = N_G(C_1 \times \cdots \times C_r)$. Hence the kernel L of the permutational action of $\bar{N}$ on the set of factors of $F^*(\bar{N})$ is contained in the image of $K \cap N$. Since $G/K = NK/K \cong N/(K \cap N)$, it follows that the maximum p-length of p-soluble subgroups for $\bar{N}/L$ is at least the same as for G/K. Obviously, N is a proper subgroup of G, so if N was not p-soluble, we could use induction for $\bar{N}$, which would finish the proof for G.

Thus, we can assume that $N = N_G(C)$ is p-soluble. (This case also accounts for the basis of induction on $|G|$.) We claim that $O_{p',p}(N) \leqslant K$. Indeed, if $x \in O_{p',p}(N)$ but $x \notin K$, then there is $i \in \{1, \dots, r\}$ such that $S_i^x \neq S_i$. Then the S_i-projection of $[x, C_iQ_i]$ contains C_iQ_i, which does not have a normal p-complement. This contradicts the fact that $[x, C_iQ_i] \leqslant O_{p',p}(N)$, since, obviously, $C_iQ_i \leqslant N_G(C)$.

Since $N/(N \cap K) \cong KN/K = EN/K = G/K$ and $O_{p',p}(N) \leqslant K \cap N$ as shown above, N is a p-soluble subgroup of p-length greater than k. □

It is conjectured that a result of the same type as Theorem 6 must also hold for $p = 2$.

Problem 7. Suppose that the 2-length of all soluble subgroups of a finite group G is at most h. Must G have a normal series of h-bounded length $f = f(h)$

$$1 \leqslant H_0 \leqslant H_1 \leqslant \cdots \leqslant H_f = G$$

such that H_i/H_{i-1} is soluble for i odd, and is a direct product of non-abelian simple groups for i even?

This is known to be true when the 2-length of all soluble subgroups is 1, due to a theorem by Mazurov [51], which, moreover, gives a complete description of finite groups with this condition. The obvious reason why the same kind of arguments as in the proof of Theorem 6 cannot be used in the case of $p = 2$, that is, why Problem 7 is open, is the absence of an analogue of Thompson's normal p-complement theorem for $p = 2$.

Information on the Fitting height of soluble subgroups can also be used for providing reduction to soluble groups, as in the following general theorem in the forthcoming paper by Khukhro and Shumyatsky [35]. In particular, this result can be used as an alternative way of providing reduction to the soluble case in the proof of Theorem 5, because it was proved in [60] that the Fitting height is bounded in the soluble case.

Theorem 7. *Suppose that the Fitting height of all soluble subgroups of a finite group G is at most h. Then G has a series of length $2h + 1$*

$$1 \leqslant H_0 \leqslant H_1 \leqslant \cdots \leqslant H_{2h+1} = G$$

such that H_i/H_{i-1} is soluble for i odd, and is a direct product (possibly empty) of non-abelian simple groups for i even.

The proof of Theorem 7 is rather similar to the proof of Theorem 6.

Coset identities. In the study of profinite groups it is usual to have a so-called coset identity, that is, a group identity that holds on elements of a coset of a subgroup (usually of finite index). For example, if a profinite group G is periodic (of not necessarily finite exponent), which is equivalent to saying that G is a periodic compact topological group, then G has cosets of subgroups of finite index satisfying an identity $x^n \equiv 1$, that is, cosets consisting of elements of orders dividing n.

As a consequence of his work on the Restricted Burnside Problem, Zelmanov [72] proved that periodic compact groups are locally finite. Reduction to the case of periodic pro-p-groups was done earlier by Wilson [68] using CFSG and Hall–Higman theorems. Here we focus on this reduction aspect in further studies of analogous problems on profinite groups. Moreover, we confine ourselves to the corresponding results on finite groups, to which the study of profinite groups reduces.

The following recent new result is designed to provide reduction to pro-p-soluble groups in the forthcoming paper of Khukhro and Shumyatsky [35]. Recall that δ_d is the 'solubility' commutator of weight 2^d such that the identity $\delta_d \equiv 1$ defines the variety of soluble groups of derived length d.

Theorem 8. *Let H be a normal subgroup of a finite group G, and P a Sylow p-subgroup of H. Let $W = W_G(P)$ be the subgroup generated by all elements $x \in P$ that are conjugates of δ_d-values on elements of P. Suppose that there exists an element $t \in G$ such that the coset tW is of exponent dividing p^a. Then H has a normal series of length bounded in terms of a and d only each of whose factors either is p-soluble or is a direct product of non-abelian simple groups of order divisible by p.*

The proof uses CFSG. Earlier a special case of this theorem was proved by Wilson [68, Theorem 2*] in the case where $W = P$, and only for $p \neq 2$.

The following theorem is designed to provide reduction to pro-p-groups in [35].

Theorem 9. *Let H be a normal p-soluble subgroup of a finite group G, and P a Sylow p-subgroup of H. Let $W = W_G(P)$ be the subgroup generated by all elements $x \in P$ that are conjugates of δ_d-values on elements of P. Suppose that there exists an element $t \in G$ such that the coset tW is of exponent dividing p^a. Then the p-length of H is at most $d+e+1$ if p is an odd non-Fermat prime, at most $d+2e+1$ if p is an odd prime, and $d+3e+1$ if $p = 2$.*

Earlier a special case of this theorem was proved by Wilson [68, Theorem 3*] in the case where $W = P$, and only for $p \neq 2$.

The proof of Theorem 9 uses the Hall–Higman theorems in the case $p \neq 2$. Recall that for $p = 2$ the Hall–Higman paper did not give any bound for the 2-length of soluble groups with Sylow 2-subgroups of given exponent 2^e. The case $p = 2$ was done later by Hoare [25] with the bound $l_2 \leqslant 3e - 1$, Gross [17] with the bound $l_2 \leqslant 2e - 1$, and finally Bryukhanova [6] with the best-possible bound $l_2 \leqslant e$. Interestingly, it is the forgotten paper by Hoare [25] that proved to be useful in the proof of Theorem 9 for $p = 2$!

Restricted Burnside Problem for Moufang loops. In the study of Moufang loops there is a programme (pursued by Grishkov, Zavarnitsine, and Zelmanov) of solving for them, hopefully in the affirmative, an analogue of the Restricted Burnside Problem. Without even giving the definition of a Moufang loop, here we describe a question on finite groups that is one of possible crucial steps in this programme.

Just like for groups, one can distinguish three stages in the study of the analogue of the Restricted Burnside Problem for Moufang loops:

1) reduction to soluble objects;

2) reduction to nilpotent objects;

3) studying nilpotent objects (by using Lie algebras).

Progress has already been achieved for stages 1 and 3. Simple Moufang loops have been classified by Liebeck [43]. However, there is still no reduction in RBP from the general case to soluble. Stage 3 seems to be near to completion. Grishkov [16] settled the case of prime exponent, except for exponent 3, which was done later by Nagy [52] (and exponent 2 is an easy exercise). Reduction from nilpotent to prime-power exponent follows from the results of Glauberman [14] and Glauberman and Wright [15]. Finally the case of prime-power exponent was announced by Grishkov and Zelmanov (unpublished).

But so far the question of reduction to the nilpotent case is wide open. This question can be stated in purely group-theoretic terms using groups with triality. A finite soluble group G is said to be a *group with triality* if G admits a group of automorphisms $S \leqslant \operatorname{Aut} G$ isomorphic to the symmetric group of degree 3 given by the presentation $S = \langle \sigma, \rho \mid \sigma^2 = \rho^3 = 1;\ \sigma\rho\sigma = \rho^2 \rangle \cong \mathbb{S}_3$ such that $m \cdot m^{\rho} \cdot m^{\rho^2} = 1$ for all $m \in M(G) = \{[g, \sigma] \mid g \in G\}$. This notation for σ, ρ, S, and $M(G)$ is usually fixed when groups with triality are considered. Note that in this definition $M(G)$ is not necessarily a subgroup but the *set* of all elements of the form $[g, \sigma] = g^{-1}g^{\sigma}$. This circumstance causes one of the major difficulties in working with groups with triality.

We can now state a question in terms of groups with triality, which is equivalent to the problem of reduction of the analogue of the Restricted Burnside Problem for Moufang loops to the nilpotent case.

Problem 8. Let G be a finite soluble group with triality as defined above. Suppose in addition that $G = [G, S]$, the group G is generated by d elements of $M(G)$ and their images under S, and $x^n = 1$ for all $x \in M(G)$. Is it true that the Fitting height of G is bounded in terms of d and n?

In spite of quite a small group of automorphisms $\mathbb{S}_3$ acting on G, so far this Hall–Higman–type problem remains unsolved.

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Задачи об ограничении p-длины и высоты Фиттинга конечных разрешимых групп

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Данная статья является обзором некоторых открытых проблем и недавних результатов об ограничении высоты Фиттинга и p-длины конечных разрешимых групп. Во многих задачах теории конечных групп в настоящее время классификация в большой степени облегчает сведение к разрешимым группам. Ограничение их высоты Фиттинга или p-длины можно рассматривать как дальнейшее сведение к нильпотентным группам. Это обычно достигается методами теории представлений, такими как теорема Клиффорда или теоремы типа Холла–Хигмэна. В некоторых задачах открытые вопросы остаются именно в случае нильпотентных групп, несмотря на значительные успехи, достигнутые, в частности, с помощью методов колец Ли. Но имеются также важные вопросы, которые всё ещё требуют сведения к нильпотентным группам; настоящий обзор нацелен на задачи сведения этого типа. В качестве примеров обсуждаются конечные группы с автоморфизмами без неподвижных точек и почти без неподвижных точек, а также обобщения ослабленной проблемы Бернсайда. Также обсуждаются результаты о тождествах смежных классов, которые применяются в изучении проконечных групп. Наконец, упоминается открытая проблема ограничения высоты Фиттинга в изучении аналога ослабленной проблемы Бернсайда для луп Муфанг.

Ключевые слова: высота Фиттинга, p-длина, разрешимая конечная группа, нильпотентная группа, проконечная группа, автоморфизм, ослабленная проблема Бернсайда, тождество смежного класса.