# Purely Time-dependent Solutions to the Yang-Mills Equations on a 4-dimensional Manifold with Conformal Torsion-free Connection 

Leonid N. Krivonosov<br>Nizhegorodsky University,<br>Minin, 24, Nizhny Novgorod, 603950,<br>Russia<br>Vyacheslav A.Lukyanov*<br>Zavolzhsky Branch, Nizhegorodsky University, Pavlovsky, 1a, Zavolzh'e, 606520,<br>Russia

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#### Abstract

We deduce the purely time-depending Yang-Mills equations in a space with conformal torsion-free connection. Next we find three series of solutions to these equations and study which of them give the Einstein metric or a metric conformally equivalent to it. Also, various representations of these solutions with and without singularities are presented.


Keywords: Einstein equations, Maxwell's equations, Yang-Mills equations, manifold with conformal connection.

## Introduction

In the title we could omit mentioning the dimension of a manifold as the Yang-Mills equations

$$
d * \Phi+\Omega \wedge * \Phi-* \Phi \wedge \Omega=0
$$

in a space with conformal connection make sense only in dimension 4. It is easily explained: the Hodge operator $*$ is defined only for even dimensions $2 n$ and maps the space of external $n$ forms to itself but the curvature matrix $\Phi$ for any dimension greater than 1 consists of 2 -forms. Therefore the matrix $* \Phi$ can be constructed only for $n=2$. In this case, for a metric with signature $(-+++)$ or $(+---)$ we have $*^{2}=-i d$ while any other signature gives $*^{2}=i d$. This difference is crucial when one tries to find solutions to the Yang-Mills equations. Here we investigate only the case of signature $(-+++)$, it is clear that our results are applicable for the signature ( +--- ) as well.

It was shown in [1] that the Yang-Mills equations in a space with conformal torsion-free connection may be reduced to Maxwell's equations and equations (65)

$$
\begin{equation*}
\eta^{m n}\left(-b_{(i j) \mid m n}+b_{(p m)} R_{(i j) n}^{p}+2 b_{[i m]} b_{[j n]}\right)-2 b b_{(i j)}+b_{\mid(i j)}-2 Q \eta_{i j}=0 \tag{1}
\end{equation*}
$$

Here all indexes run from 1 to 4. Indexes in brackets denote symmetrization with respect to them, that is $b_{(i j)}=b_{i j}+b_{j i}$; in square brackets, skew symmetrization, that is $b_{[i j]}=b_{i j}-b_{j i}$. The indexes that follow a vertical line denote the corresponding covariant derivation. Let also

[^0]$\eta_{i j}$ be the Minkowski metric tensor, $\eta^{m n}$ its dual tensor; $b_{(i j)}$ are defined by the Ricci tensor $R_{i j}$ of the quadratic form of the angular metric
\[

$$
\begin{equation*}
\psi=\eta_{i j} \omega^{i} \omega^{j} \tag{2}
\end{equation*}
$$

\]

according to the Einstein equations

$$
\begin{equation*}
b_{(i j)}=R_{i j}-\frac{1}{6} R \eta_{i j} \tag{3}
\end{equation*}
$$

Here $R_{i j n}^{p}$ is the curvature tensor of the quadratic form (2), $b=\eta^{i j} b_{i j}=\frac{1}{2} \eta^{i j} b_{(i j)}=\frac{1}{6} R$, and

$$
\begin{gathered}
Q \stackrel{\text { def }}{=} \eta^{i j} \eta^{m n} b_{i m} b_{j n}=\left(b_{11}\right)^{2}+\left(b_{22}\right)^{2}+\left(b_{33}\right)^{2}+\left(b_{44}\right)^{2}- \\
-\left(b_{12}\right)^{2}-\left(b_{21}\right)^{2}-\left(b_{13}\right)^{2}-\left(b_{31}\right)^{2}-\left(b_{14}\right)^{2}-\left(b_{41}\right)^{2}+ \\
+\left(b_{23}\right)^{2}+\left(b_{32}\right)^{2}+\left(b_{24}\right)^{2}+\left(b_{42}\right)^{2}+\left(b_{34}\right)^{2}+\left(b_{43}\right)^{2}= \\
=\frac{1}{4} \eta^{i j} \eta^{m n} b_{(i m)} b_{(j n)}+\frac{1}{4} \eta^{i j} \eta^{m n} b_{[i m]} b_{[j n]} .
\end{gathered}
$$

Besides terms depending on the components of quadratic form (2), which is interpreted as the gravitational potential, equations (1) contain terms that depend on the components $b_{[i j]}$ of the electromagnetic field. They appear in the term $2 \eta^{m n} b_{[i m]} b_{[j n]}$ and, indirectly, in the scalar $Q$. Thus in a space with conformal torsion-free connection satisfying the Yang-Mills equations the gravitation has purely electromagnetic nature. This means that given the tensor of electromagnetic field $b_{[i j]}$ we obtain the quadratic form (2) by solving 10 nonlinear differential equations of 4 -th order (1). (Note that the equations have solutions even if the electromagnetic field vanishes.) There are only 9 independent equations in system (1) because convolution of the left part of (1) with $\eta^{i j}$ results in identical zero.

We would like to point out here that the way the electromagnetic field generates gravitation according to equations (1) differs from the one that is absolutely groundlessly postulated in all treatises on general relativity. On the other hand, although equations (1) have strong logical foundation, they are hardly realistic since they are obtained by neglecting weak interaction whose existence is proved by numerous and convincing experiments.

The electromagnetic field tensor $b_{[i j]}$ satisfies Maxwell's equations

$$
\begin{equation*}
d \Phi_{0}^{0}=0, \quad d * \Phi_{0}^{0}=0 \tag{4}
\end{equation*}
$$

where $\Phi_{0}^{0}=\frac{1}{2} b_{[i j]} \omega^{j} \wedge \omega^{i}$. Expanding (4), we get 8 linear differential equations of 1 st order with the coefficients depending on the components of quadratic form (2). Thus, 17 equations (1) and (4) form a closed system of differential equations on 16 unknown functions: 10 coefficients of the quadratic form (2) and 6 components of the tensor $b_{[i j]}$. Equations (1) and (4) give a complete system of the Yang-Mills equations in a space with conformal torsion-free connection.

It is clear that to solve the system of (1) and (4) is much more difficult than the system of the Einstein equations. In a closed form solutions can be obtained only in special cases. In this paper we consider only the case when the coefficients of quadratic form (2) and the components $b_{[i j]}$ of electromagnetic field tensor depend only on one parameter $t$ that we interpret as time.
Remark. The left side of (1) for $b_{[i j]}=0$ is the Bach tensor that was introduced in 1921. In the paper [2] it is shown that if $b_{[i j]}=0$ and the Einstein equations (3) are satisfied then the Yang-Mills current (the left side of the Yang-Mills equations) is expressed only via the Bach tensor.

## 1. Equations (1) and (4) in the Purely Time-dependent Case

We consider the angular metric

$$
\begin{equation*}
\psi=-d t^{2}+a^{2}\left(d x_{1}\right)^{2}+b^{2}\left(d x_{2}\right)^{2}+c^{2}\left(d x_{3}\right)^{2} \tag{5}
\end{equation*}
$$

where $a, b, c$ depend only on time $t$. We denote the derivative with respect to time with a dot.
Let

$$
\omega^{1}=d t, \quad \omega^{2}=a d x_{1}, \quad \omega^{3}=b d x_{2}, \quad \omega^{4}=c d x_{3} .
$$

Then the Christoffel symbols are

$$
\omega_{1}^{2}=\frac{\dot{a}}{a} \omega^{2}, \quad \omega_{1}^{3}=\frac{\dot{b}}{b} \omega^{3}, \quad \omega_{1}^{4}=\frac{\dot{c}}{c} \omega^{4}, \quad \omega_{2}^{3}=\omega_{2}^{4}=\omega_{3}^{4}=0
$$

Non-vanishing components of the curvature tensor of the quadratic form (5) are

$$
\begin{equation*}
R_{112}^{2}=\frac{\ddot{a}}{a}, \quad R_{113}^{3}=\frac{\ddot{b}}{b}, \quad R_{114}^{4}=\frac{\ddot{c}}{c}, \quad R_{223}^{3}=-\frac{\dot{a} \dot{b}}{a b}, \quad R_{224}^{4}=-\frac{\dot{a} \dot{c}}{a c}, \quad R_{334}^{4}=-\frac{\dot{c} \dot{b}}{c b} . \tag{6}
\end{equation*}
$$

Then

$$
\begin{gather*}
R_{11}=\frac{\ddot{a}}{a}+\frac{\ddot{b}}{b}+\frac{\ddot{c}}{c}, \quad R_{22}=-\frac{\ddot{a}}{a}-\frac{\dot{a} \dot{b}}{a b}-\frac{\ddot{a} \dot{c}}{a c} \\
R_{33}=-\frac{\ddot{b}}{b}-\frac{\dot{c} \dot{b}}{c b}-\frac{\dot{a} \dot{b}}{a b}, \quad R_{44}=-\frac{\ddot{c}}{c}-\frac{\dot{a} \dot{c}}{a c}-\frac{\dot{c} \dot{b}}{c b},  \tag{7}\\
R=\eta^{i j} R_{i j}=-2\left(\frac{\ddot{a}}{a}+\frac{\ddot{b}}{b}+\frac{\ddot{c}}{c}+\frac{\dot{a} \dot{b}}{a b}+\frac{\dot{a} \dot{c}}{a c}+\frac{\dot{c} \dot{b}}{c b}\right) .
\end{gather*}
$$

According to (3), we have $b_{(i j)}=0$ for $i \neq j$, and

$$
\begin{align*}
& b_{11}=\frac{1}{3}\left(\frac{\ddot{a}}{a}+\frac{\ddot{b}}{b}+\frac{\ddot{c}}{c}\right)-\frac{1}{6}\left(\frac{\dot{a}}{a} \frac{\dot{b}}{b}+\frac{\dot{a}}{a} \frac{\dot{c}}{c}+\frac{\dot{c}}{c} \frac{\dot{b}}{b}\right), \\
& b_{22}=-\frac{1}{3}\left(\frac{\ddot{a}}{a}+\frac{\dot{a}}{a} \frac{\dot{b}}{b}+\frac{\dot{a}}{a} \frac{\dot{c}}{c}\right)+\frac{1}{6}\left(\frac{\ddot{b}}{b}+\frac{\ddot{c}}{c}+\frac{\dot{c}}{c} \frac{b}{b}\right),  \tag{8}\\
& b_{33}=-\frac{1}{3}\left(\frac{\ddot{b}}{b}+\frac{\dot{a}}{a} \frac{\dot{b}}{b}+\frac{\dot{c}}{c} \dot{b} \bar{b}\right)+\frac{1}{6}\left(\frac{\ddot{a}}{a}+\frac{\ddot{c}}{c}+\frac{\dot{a}}{a} \frac{\dot{c}}{c}\right), \\
& b_{44}=-\frac{1}{3}\left(\frac{\ddot{c}}{c}+\frac{\dot{a}}{a} \frac{\dot{c}}{c}+\frac{\dot{c}}{c} \dot{b} \bar{b}\right)+\frac{1}{6}\left(\frac{\ddot{a}}{a}+\frac{\ddot{b}}{b}+\frac{\dot{a}}{a} \frac{\dot{b}}{b}\right) .
\end{align*}
$$

Direct substitution gives the following identities

$$
\begin{align*}
& \frac{\ddot{a}}{a}+\frac{\ddot{b}}{b}+\frac{\ddot{c}}{c}=3 b_{11}-b_{22}-b_{33}-b_{44},  \tag{9}\\
& \frac{\ddot{a}}{a}+\frac{\dot{a}}{a} \dot{b} \frac{\dot{b}}{b}+\frac{\dot{a}}{a} \frac{\dot{c}}{c}=b_{11}-3 b_{22}-b_{33}-b_{44},  \tag{10}\\
& \ddot{b}  \tag{11}\\
& \frac{b}{b}+\frac{\dot{a}}{a} \dot{b}+\frac{\dot{c}}{b} \frac{\dot{b}}{c}=b_{11}-b_{22}-3 b_{33}-b_{44},  \tag{12}\\
& \ddot{c}  \tag{13}\\
& \frac{\ddot{c}}{c}+\frac{\dot{a}}{a} \frac{\dot{c}}{c}+\frac{\dot{c}}{c} \dot{b}=b_{11}-b_{22}-b_{33}-3 b_{44}, \\
& \dot{b_{22}}+\dot{b} \dot{b_{33}}+b_{44}=-\left(b_{11}+b_{22}\right) \frac{\dot{a}}{a}-\left(b_{11}+b_{33}\right) \frac{\dot{b}}{b}-\left(b_{11}+b_{44}\right) \frac{\dot{c}}{c}
\end{align*}
$$

Now we compute covariant derivatives of $b_{(i j)}: b_{(i i) \mid 1}=$

$$
=2 \dot{b_{i i}}, \quad b_{(12) \mid 2}=-2\left(b_{11}+b_{22}\right) \frac{\dot{a}}{a}, \quad b_{(13) \mid 3}=-2\left(b_{11}+b_{33}\right) \frac{\dot{b}}{b}, \quad b_{(14) \mid 4}=-2\left(b_{11}+b_{44}\right) \frac{\dot{c}}{c}
$$

while all the other $b_{(i j) \mid k}=0$. The second covariant derivatives of $b_{(i j)}$ are

$$
\begin{align*}
& b_{(11) \mid 22}=4\left(b_{11}+b_{22}\right)\left(\frac{\dot{a}}{a}\right)^{2}-2 b_{11} \frac{\dot{a}}{a}, \quad b_{(11) \mid 33}=4\left(b_{11}+b_{33}\right)\left(\frac{\dot{b}}{b}\right)^{2}-2 \cdot \dot{b_{11}} \frac{\dot{b}}{b}, \\
& b_{(11) \mid 44}=4\left(b_{11}+b_{44}\right)\left(\frac{\dot{c}}{c}\right)^{2}-2 b_{11} \frac{\dot{c}}{c}, \\
& b_{(22) \mid 11}=2 \ddot{b_{22}}, \quad b_{(22) \mid 33}=-2 \dot{b_{22}} \frac{\dot{b}}{\dot{b}}, \quad b_{(22) \mid 44}=-2 \dot{b_{22}} \frac{\dot{c}}{\frac{c}{c}},  \tag{14}\\
& b_{(33) \mid 11}=2 \ddot{b_{33}}, \quad b_{(33) \mid 22}=-2 \dot{b_{33}} \frac{\dot{a}}{a}, \quad b_{(33) \mid 44}=-2 \dot{b_{33}} \frac{\dot{c}}{\frac{c}{c}}, \\
& b_{(44) \mid 11}=2 \ddot{b_{44}}, \quad b_{(44) \mid 22}=-2 b_{44} \frac{\dot{a}}{a}, \quad b_{(44) \mid 33}=-2 \cdot \stackrel{\dot{b}}{44} \frac{\dot{b}}{b} .
\end{align*}
$$

The components $b_{(i i) \mid i i}$ do not vanish but we do not need them since they do not appear in (1). All the other $b_{(i j) \mid k p}=0$.

It follows from these computations that if $i \neq j$, then only the third summand in (1) does not vanish. Therefore (1) turns into

$$
2 \eta^{m n} b_{[i m]} b_{[j n]}=0
$$

Hence if $i \neq j$, then only the following pairs can be different from zero: either $b_{[12]}$ and $b_{[34]}$, or $b_{[13]}$ and $b_{[24]}$, or $b_{[14]}$ and $b_{[23]}$. Without loss of generality, we may assume that $b_{[12]}$ and $b_{[34]}$ do not vanish, and $b_{[13]}=b_{[24]}=b_{[14]}=b_{[23]}=0$. In this case the first group of Maxwell's equations (4) is reduced to the only equation $\dot{b_{34}}+b_{34}\left(\frac{\dot{b}}{b}+\frac{\dot{c}}{c}\right)=0$, where

$$
\begin{equation*}
b_{34}=\frac{N}{b c}, \quad N=\text { const. } \tag{15}
\end{equation*}
$$

The second group of Maxwell's equations (4) is also reduced to one equation

$$
\dot{b_{12}}+b_{12}\left(\frac{\dot{b}}{b}+\frac{\dot{c}}{c}\right)=0
$$

where

$$
\begin{equation*}
b_{12}=\frac{M}{b c}, \quad M=\text { const. } \tag{16}
\end{equation*}
$$

Now we will consider the case $i=j$. Let first $i=j=1$, then, taking into account (14), (13), and (9), equations (1) lead to

$$
\begin{gather*}
-\frac{\dot{a}}{a}\left(b_{22}+\left(b_{11}+b_{22}\right) \frac{\dot{a}}{a}\right)-\frac{\dot{b}}{b}\left(\dot{b_{33}}+\left(b_{11}+b_{33}\right) \frac{\dot{b}}{b}\right)+2\left(b_{12}\right)^{2}+ \\
+2\left(b_{34}\right)^{2}-\frac{\dot{c}}{c}\left(\dot{b_{44}}+\left(b_{11}+b_{44}\right) \frac{\dot{c}}{c}\right)+b_{22}\left(\frac{\ddot{a}}{a}+b_{22}-b_{11}\right)+  \tag{17}\\
+b_{33}\left(\frac{\ddot{b}}{b}+b_{33}-b_{11}\right)+b_{44}\left(\frac{\ddot{c}}{c}+b_{44}-b_{11}\right)=0 .
\end{gather*}
$$

Similarly, when $i=j=2, i=j=3$ and $i=j=4$ using identities (10)-(13) and expressions (14) we see that

$$
\begin{align*}
& \left(\dot{b_{22}}+\left(b_{11}+b_{22}\right) \frac{\dot{a}}{a}\right)+\left(b_{22}+\left(b_{11}+b_{22}\right) \frac{\dot{a}}{a}\right)\left(\frac{\dot{b}}{b}+\frac{\dot{c}}{c}\right)- \\
& \quad-2\left(b_{12}\right)^{2}-2\left(b_{34}\right)^{2}+b_{11}\left(\frac{\ddot{a}}{a}+b_{22}-b_{11}\right)+  \tag{18}\\
& +b_{33}\left(-\frac{\dot{a} \dot{b}}{a b}-b_{33}-b_{22}\right)+b_{44}\left(-\frac{\dot{a} \dot{c}}{a c}-b_{22}-b_{44}\right)=0, \\
& \left(\dot{b} \dot{b 3}+\left(b_{11}+b_{33}\right) \frac{\dot{b}}{b}\right)+\left(\dot{b}{ }_{33}+\left(b_{11}+b_{33}\right) \frac{\dot{b}}{b}\right)\left(\frac{\dot{a}}{a}+\frac{\dot{c}}{c}\right)+ \\
& \quad+2\left(b_{12}\right)^{2}+2\left(b_{34}\right)^{2}+b_{11}\left(\frac{\ddot{b}}{b}+b_{33}-b_{11}\right)+  \tag{19}\\
& +b_{22}\left(-\frac{\dot{a} \dot{b}}{a b}-b_{33}-b_{22}\right)+b_{44}\left(-\frac{\dot{b} \dot{c}}{b c}-b_{33}-b_{44}\right)=0, \\
& \left(\dot{b_{44}}+\left(b_{11}+b_{44}\right) \frac{\dot{c}}{c}\right)+\left(\dot{b} 44+\left(b_{11}+b_{44}\right) \frac{\dot{c}}{c}\right)\left(\frac{\dot{a}}{a}+\frac{\dot{b}}{b}\right)+ \\
& \quad+2\left(b_{12}\right)^{2}+2\left(b_{34}\right)^{2}+b_{11}\left(\frac{\ddot{c}}{c}+b_{44}-b_{11}\right)+  \tag{20}\\
& +b_{22}\left(-\frac{\dot{a} \dot{c}}{a c}-b_{22}-b_{44}\right)+b_{33}\left(-\frac{\dot{b} \dot{c}}{b c}-b_{33}-b_{44}\right)=0 .
\end{align*}
$$

These equations have been written down in [1] without a proof. Only three of them are independent since summing up the last three and subtracting (17) we obtain identically zero.
Remark. To solve the system of equations (1) and (4) it is necessary to replace all the covariant derivatives with their expressions through usual derivatives, and this is a very labour-consuming procedure. In [1] equations (17)-(20) have been obtained using the algorithm described in [3], which is based on differentiation of external forms. This algorithm is easier and more reliable to use.

## 2. Solving Equations (17)-(20)

The system of equations (17)-(20) is rather difficult. We are able to solve it only in special cases.

### 2.1. Case $b=c$

Since the metric (5) is defined up to a factor, we can divide it by $b^{2}$. Making a change of parameter $\tau=\int \frac{d t}{b}$ and denoting $\frac{a}{b}=\frac{a}{c}$ again as $a$, we rewrite the metric as

$$
\begin{equation*}
\psi=-d t^{2}+a^{2}\left(d x_{1}\right)^{2}+\left(d x_{2}\right)^{2}+\left(d x_{3}\right)^{2} \tag{21}
\end{equation*}
$$

Therefore, without loss of generality, we may assume that $b=c=1$ in this case; then equations

$$
b_{11}=\frac{1}{3} \frac{\ddot{a}}{a} \stackrel{\text { def }}{=} 4 A, \quad b_{22}=-\frac{1}{3} \frac{\ddot{a}}{a}=-4 A, \quad b_{33}=b_{44}=\frac{1}{6} \frac{\ddot{a}}{a}=2 A
$$

follow from (8). Furthermore, the system (17)-(20) is reduced to two equations

$$
\begin{equation*}
\frac{2 \dot{a}}{a} \dot{A}-12 A^{2}+K^{2}=0, \quad 2 \ddot{A}-12 A^{2}+K^{2}=0 \tag{22}
\end{equation*}
$$

where $K^{2}=M^{2}+N^{2}=$ const.
Integrating the difference of these equations, we find that

$$
\begin{equation*}
\dot{A}=P a, \quad P=\text { const. } \tag{23}
\end{equation*}
$$

The first integral of the 2nd equation in (22) is

$$
\begin{equation*}
\dot{A}^{2}=4 A^{3}-K^{2} A-Q, \quad Q=\text { const. } \tag{24}
\end{equation*}
$$

This means that $A$ is given by the elliptic Weierstrass $\wp$-function $A=\wp\left(t, K^{2}, Q\right)$, and according to (23)

$$
\begin{equation*}
a=\frac{1}{P} \dot{\wp}\left(t, K^{2}, Q\right) . \tag{25}
\end{equation*}
$$

Formula (25) makes no sense when $P=0$, and in this case we deduce directly from (23) and (22) that $A=\frac{K}{\sqrt{12}}$. Consequently, by definition of $A, \frac{\ddot{a}}{a}=12 A=K \sqrt{12}$. It depends on the sign of $K$ which of the following three functions solves the equation:

$$
\begin{align*}
a & =C_{1} \sin \left(\sqrt{-K \sqrt{12}} t+C_{2}\right)  \tag{26}\\
a & =C_{1} e^{\sqrt{K \sqrt{12}} t}+C_{2} e^{-\sqrt{K \sqrt{12}}}  \tag{27}\\
a & =C_{1} t+C_{2} \tag{28}
\end{align*}
$$

Formulas (25)-(28) give the complete list of all solutions to (17)-(20) in the case $b=c$. These solutions are elementary; solution (25) is also elementary for some values of $K$ and $Q$. Namely, the function $\wp$ is elementary if the cubic equation

$$
\begin{equation*}
4 \wp^{3}-K^{2} \wp-Q=0 \tag{29}
\end{equation*}
$$

has multiple roots. A number $\alpha$ is a root of multiplicity 2 if it is also a root of the derivative, i.e. the solution to the equation $12 \alpha^{2}-K^{2}=0$. We may always assume that $K \geqslant 0$. Substituting $\alpha= \pm \frac{K}{\sqrt{12}}$ in (29), we find $Q=\mp \frac{K^{3}}{3 \sqrt{3}}$. For these values of $Q$ equation (24) can be solved in elementary functions. Let first $Q=\frac{K^{3}}{3 \sqrt{3}}$ then equation (24) turns into

$$
\dot{A}^{2}=4\left(A+\frac{K}{\sqrt{12}}\right)^{2}\left(A-\frac{K}{\sqrt{3}}\right)
$$

Its solution is $\frac{\sqrt{2}}{\sqrt{K} \sqrt[4]{3}} \arctan \frac{\sqrt{2 A-\frac{2 K}{\sqrt{3}}}}{\sqrt{K} \sqrt[4]{3}}=t+C_{1}$, where $C_{1}=$ const. Therefore

$$
A=\wp=\frac{K \sqrt{3}}{6}\left(3 \tan ^{2} \frac{\sqrt{K} \sqrt[4]{3}\left(t+C_{1}\right)}{\sqrt{2}}+2\right)
$$

Denoting $\frac{\sqrt[4]{3} \sqrt{K}}{\sqrt{2}}=C_{2}$ we find $A$; substituting it in (25) we get

$$
a=\frac{\sin C_{2}\left(t+C_{1}\right)}{P \cos ^{3} C_{2}\left(t+C_{1}\right)}, \quad C_{1}, C_{2}, P=\text { const. }
$$

Similar calculations for $Q=-\frac{K^{3}}{3 \sqrt{3}}$ give

$$
a=\frac{\sinh C_{2}\left(t+C_{1}\right)}{P \cosh ^{3} C_{2}\left(t+C_{1}\right)}, \quad C_{1}, C_{2}, P=\text { const. }
$$

If equation (29) has a root of multiplicity 3 , then $K^{2}=Q=0$. In this case equation (24) implies $A=\wp=\frac{1}{\left(t+C_{2}\right)^{2}}$ and, consequently,

$$
\begin{equation*}
a=\frac{\dot{A}}{P}=\frac{C_{1}}{\left(t+C_{2}\right)^{3}}, \quad C_{1}, C_{2}=\text { const. } \tag{30}
\end{equation*}
$$

Let us find out now which metric among (25)-(28) gives the Einstein metric, i.e. $R_{i j}=\varkappa \eta_{i j}$, or a metric conformally equivalent to it. As it was proved in [1, p. 445], the Einstein metric (and a metric conformally equivalent to it) does not allow an electromagnetic field, therefore such solutions should be among (25)-(28) with $K=0$, i.e. they must be

$$
\begin{equation*}
a=\frac{1}{P} \dot{\wp}(t, 0, Q) \tag{31}
\end{equation*}
$$

or $a=C_{1} t+C_{2}$.
Equalities (7) show that for $b=c=1$ the Ricci tensor components are

$$
R_{11}=\frac{\ddot{a}}{a}, \quad R_{22}=-\frac{\ddot{a}}{a}, \quad R_{33}=R_{44}=0
$$

For that reason, solution (28) provides the Einstein metric, while solutions (31) do not.
It is known that the metrics

$$
\begin{align*}
& \psi_{1}=-d t^{2}+\cos ^{\frac{4}{3}} \omega \cdot\left(\tan ^{2} \omega \cdot\left(d x_{1}\right)^{2}+\left(d x_{2}\right)^{2}+\left(d x_{3}\right)^{2}\right)  \tag{32}\\
& \psi_{2}=-d t^{2}+\sin ^{\frac{4}{3}} \omega \cdot\left(\cot ^{2} \omega \cdot\left(d x_{1}\right)^{2}+\left(d x_{2}\right)^{2}+\left(d x_{3}\right)^{2}\right)
\end{align*}
$$

where $\omega=\frac{3}{2} \sqrt{\frac{\varkappa}{3}} t, \varkappa=$ const, $\varkappa>0$, and

$$
\begin{align*}
& \psi_{3}=-d t^{2}+\cosh ^{\frac{4}{3}} \omega \cdot\left(\tanh ^{2} \omega \cdot\left(d x_{1}\right)^{2}+\left(d x_{2}\right)^{2}+\left(d x_{3}\right)^{2}\right), \\
& \psi_{4}=-d t^{2}+\sinh ^{\frac{4}{3}} \omega \cdot\left(\operatorname{coth}^{2} \omega \cdot\left(d x_{1}\right)^{2}+\left(d x_{2}\right)^{2}+\left(d x_{3}\right)^{2}\right), \tag{33}
\end{align*}
$$

where $\omega=\frac{3}{2} \sqrt{-\frac{\varkappa}{3}} t, \varkappa=$ const, $\varkappa<0$, are the Einstein metrics [2, p. 230]. Here $\varkappa$ is the Einstein curvature $R_{i j}=\varkappa \eta_{i j}$. The coefficients at $\left(d x_{2}\right)^{2}$ and $\left(d x_{3}\right)^{2}$ in these metrics are equal, therefore they are of the form (5) with $b=c$, which is the case we consider in this section. As it was noted in [3, p. 73], the Einstein metrics automatically satisfy all the Yang-Mills equations. Hence these metrics are conformally equivalent to some of the solutions (31). It remains now to establish the parameters $P$ and $Q$.

For this purpose we will rewrite the metric $\psi_{1}$ in the form (5) with $b=c=1$. To do that, we divide the metric by $\cos ^{\frac{4}{3}} \omega$, change the time parameter $\tau=\int \frac{d t}{\cos ^{\frac{2}{3}}\left(\frac{3}{2} \sqrt{\frac{\varkappa}{3}} t\right)}$ and let $a$ be the composition of functions $a=\tan \omega$ and $\omega=\frac{3}{2} \sqrt{\frac{\varkappa}{3}} t(\tau)$. Let the function $\dot{\wp}(\tau, 0, Q)=P a$ satisfy the differential equation

$$
(\ddot{\wp})^{3}=\frac{27}{2}\left((\dot{\wp})^{2}+Q\right)^{2},
$$

which follows from the equation $(\dot{\wp})^{2}=4 \wp^{3}-Q$ for the Weierstrass function $\wp(\tau, 0, Q)$. This brings us to

$$
\begin{equation*}
P=\frac{1}{4}\left(\frac{\varkappa}{3}\right)^{\frac{3}{2}}, \quad Q=P^{2}=\frac{1}{16}\left(\frac{\varkappa}{3}\right)^{3} . \tag{34}
\end{equation*}
$$

The same result holds for the metric $\psi_{2}$.
Analogously, for the metrics $\psi_{3}$ and $\psi_{4}$ we come to the similar result

$$
\begin{equation*}
P=\frac{1}{4}\left(-\frac{\varkappa}{3}\right)^{\frac{3}{2}}, \quad Q=-P^{2}=\frac{1}{16}\left(\frac{\varkappa}{3}\right)^{3} . \tag{35}
\end{equation*}
$$

From this it follows that the metrics $\psi_{3}$ and $\psi_{4}$ are conformally equivalent. The metrics $\psi_{1}$ and $\psi_{2}$ are not just conformally equivalent but isometric because we may obtain one from another by the time parameter change $t \rightarrow \frac{\pi}{2 \lambda}-t, \quad \lambda=\frac{3}{2} \sqrt{\frac{\varkappa}{3}}$.

Up to an isometry, there are no other Einstein metrics of the form (5) with $b=c$ and $\varkappa \neq 0$, except for (32) and (33).

Let us summarize this in other words. A metric in a space with conformal connection is given up to a factor, therefore a numerical value of the parameter $\varkappa$ is unimportant, only its sign matters. It follows that all the Einstein metrics of the form (5) with $b=c$ and $\varkappa \neq 0$ are conformally equivalent to two specific (not Einstein) metrics of the form (5) with $b=c=1$, and $a=\dot{\wp}(t, 0,1)$ or $a=\dot{\wp}(t, 0,-1)$.

There are two series of the Einstein metrics of the form (5) with $b=c$ and $\varkappa=0$

$$
\begin{array}{ll}
a=\alpha(\beta t+\gamma)^{-\frac{1}{3}}, & b=c=(\beta t+\gamma)^{\frac{2}{3}}  \tag{36}\\
a=\beta t+\gamma, & b=c=\text { const }
\end{array}
$$

where $\alpha, \beta, \gamma$ are const ant. All metrics in the first of these series are conformally equivalent to the non-Einstein metric with the coefficients $a=\frac{1}{t^{3}}, b=c=1$.

Metrics in the second series are isometric to the Einstein metric with the coefficients $a=$ $t, b=c=1$.
Remark. According to (15) and (16), for $b=c=1$ we have $b_{12}=M=\mathrm{const}, b_{34}=N=\mathrm{const}$, but this does not mean that the electromagnetic field is constant, since (4) implies that

$$
\begin{equation*}
\Phi_{0}^{0}=-2 M \omega^{1} \wedge \omega^{2}-2 N \omega^{3} \wedge \omega^{4}=-2 M a d t \wedge d x_{1}-2 N d x_{2} \wedge d x_{3} \tag{37}
\end{equation*}
$$

and we see that the coefficient at $d t \wedge d x_{1}$ depends on time.

### 2.2. Case $a=b$ (or $a=c$ )

The same reasoning as above allows us to assume $a=b=1$ (or $a=c=1$ ). It turns out that such a metric is incompatible with the electromagnetic field with $b_{[13]}=b_{[24]}=b_{[14]}=b_{[23]}=0$. In this case we may have only solutions of the form (31) or (28) with a replaced by the function $c$ (or b).

### 2.3. Second Representation in the Case $b=c$

If we divide metric (2) not by $b^{2}$, as in Section 2.1, but $a^{2}$, change the parameter $\tau=\int \frac{d t}{a}$, and denote $\frac{b}{a}=\frac{c}{a}$ by $b$, then we obtain $a=1, b=c$. Writing $B=\frac{\dot{b}}{b}=\frac{\dot{c}}{c}$, we rewrite system
(17)-(20) as follows

$$
\begin{gather*}
\frac{1}{3} B \ddot{B}-\frac{1}{6} \dot{B}^{2}+\frac{2}{3} B^{2} \dot{B}+\frac{2 K^{2}}{b^{4}}=0  \tag{38}\\
\frac{1}{3} \dddot{B}+\frac{5}{3} B \ddot{B}+\frac{5}{6} \dot{B}^{2}+2 B^{2} \dot{B}-\frac{2 K^{2}}{b^{4}}=0
\end{gather*}
$$

where $K^{2}=M^{2}+N^{2}=$ const as above. This system is much more difficult to solve than (22). It would not be clear how to proceed with solving it if we did not know that the system was obtained from (22) by change of the unknown function $b=\frac{1}{a}$ and the time parameter $\tau=\int \frac{d t}{a}$. For us, there are two reasons to write down this system of equations. First, to illustrate how much the choice of the gauge transformation (renormalization of the quadratic form (5) in this case) is important to find a solution to the Yang-Mills system of equations. Second, the form of the solution may have advantages or disadvantages, and this depends on a choice of the solution representation. In particular, solutions (26), (28), and (30) attain the form

$$
\begin{align*}
b & =c=\frac{1}{C_{1}} \cosh \left(C_{1} \sqrt{-K \sqrt{12}} t+C_{2}\right)  \tag{39}\\
b & =c=C_{2} e^{-C_{1} t}  \tag{40}\\
b & =c=C_{1}\left(t+C_{2}\right)^{\frac{3}{4}} \tag{41}
\end{align*}
$$

In order to write down solutions (27) we first rewrite them

$$
\begin{array}{ll}
a=C_{3} \cosh \left(\sqrt{K \sqrt{12}} t+C_{4}\right), & \left(\text { if } C_{1} C_{2}>0\right) \\
a=C_{3} \sinh \left(\sqrt{K \sqrt{12}} t+C_{4}\right), & \left(\text { if } C_{1} C_{2}<0\right) \\
a=C_{1} e^{\sqrt{K \sqrt{12}} t}, \quad\left(\text { if } C_{2}=0\right) \\
a=C_{2} e^{\sqrt{K \sqrt{12}},} \quad\left(\text { if } C_{1}=0\right) \tag{45}
\end{array}
$$

After the gauge transformation these solutions become

$$
\begin{align*}
b & =c=C_{1} \cos \left(\frac{\sqrt{K \sqrt{12}}}{C_{1}} t+C_{2}\right)  \tag{46}\\
b & =c=C_{1} \sinh \left(\frac{\sqrt{K \sqrt{12}}}{C_{1}} t+C_{2}\right)  \tag{47}\\
b & =c=C_{1}-\sqrt{K \sqrt{12}} t  \tag{48}\\
b & =c=\sqrt{K \sqrt{12}} t+C_{2} \tag{49}
\end{align*}
$$

A direct substitution easily shows that (46)-(49) and (39)-(41) are solutions to (38) (the last two are for $K=0$ ). The metrics of solutions (26) and (28) have singularities but the corresponding metrics for (39) and (40) do not. On the contrary, the metrics of solutions (42), (44), and (45) have no singularities but acquire them in the new gauge, as we see in (46), (48), and (49). Moreover, metrics (32) and (33) are given by elementary functions but the same metrics after the gauge transformation are given by formula (31) with conditions (34) and (35). These functions are not elementary.

## 3. Solutions of the Monomial Type

Let us now study solutions to system (17)-(20) of the form

$$
\begin{equation*}
a=t^{\alpha_{1}}, \quad b=t^{\alpha_{2}}, \quad c=t^{\alpha_{3}} \tag{50}
\end{equation*}
$$

where $\alpha_{1}, \alpha_{2}, \alpha_{3}$ are constant; i.e.

$$
\begin{equation*}
\psi=-d t^{2}+t^{2 \alpha_{1}}\left(d x_{1}\right)^{2}+t^{2 \alpha_{2}}\left(d x_{2}\right)^{2}+t^{2 \alpha_{3}}\left(d x_{3}\right)^{2} \tag{51}
\end{equation*}
$$

Solutions (28), (30), and (41) for $C_{1}=1, C_{2}=0$ and (36) for $\alpha=\beta=1, \gamma=0$ are of this form. Moreover, Kasner showed that if

$$
\begin{equation*}
\alpha_{1}+\alpha_{2}+\alpha_{3}=\left(\alpha_{1}\right)^{2}+\left(\alpha_{2}\right)^{2}+\left(\alpha_{3}\right)^{2}=1 \tag{52}
\end{equation*}
$$

then (51) is the Einstein metric [4, p. 491]. Hence all functions (50) with condition (52) are solutions to system (17)-(20). In particular, solutions (28) and (36) satisfy this condition. It is natural to try to find all the solutions of form (50) but it is a difficult problem. A straightforward substitution of (50) in equations (17)-(20) leads to severe technical difficulties because the algebraic equations obtained in the result are too difficult, and it is impossible to see what algebraic variety they define. We may overcome these difficulties introducing auxiliary parameters

$$
\begin{equation*}
u=\left(\alpha_{1}\right)^{2}+\left(\alpha_{2}\right)^{2}+\left(\alpha_{3}\right)^{2}-1, \quad v=\alpha_{1}+\alpha_{2}+\alpha_{3}-1 . \tag{53}
\end{equation*}
$$

This allows to obtain elegant and unexpected solutions, which include all the solutions mentioned above.

Using formulas (8) we express $b_{i i}$ via introduced parameters

$$
\begin{aligned}
b_{11}=\frac{1}{t^{2}}\left(\frac{5}{12} u-\frac{1}{2} v-\frac{1}{12} v^{2}\right), & b_{22}=\frac{1}{t^{2}}\left(-\frac{1}{2} \alpha_{1} v+\frac{1}{12} u+\frac{1}{12} v^{2}\right), \\
b_{33}=\frac{1}{t^{2}}\left(-\frac{1}{2} \alpha_{2} v+\frac{1}{12} u+\frac{1}{12} v^{2}\right), & b_{44}=\frac{1}{t^{2}}\left(-\frac{1}{2} \alpha_{3} v+\frac{1}{12} u+\frac{1}{12} v^{2}\right) .
\end{aligned}
$$

Then equations (17)-(20) turn into

$$
\begin{align*}
& \left.A_{0}+2 K^{2} t^{4-2\left(\alpha_{2}+\alpha_{3}\right)}=0, \quad\right] A_{1}-2 K^{2} t^{4-2\left(\alpha_{2}+\alpha_{3}\right)}=0 \\
& A_{2}+2 K^{2} t^{4-2\left(\alpha_{2}+\alpha_{3}\right)}=0, \quad A_{3}+2 K^{2} t^{4-2\left(\alpha_{2}+\alpha_{3}\right)}=0 \tag{54}
\end{align*}
$$

where $K^{2}=M^{2}+N^{2}=$ const, and $A_{0}, A_{i}$ are defined by

$$
\begin{aligned}
& A_{0} \stackrel{\text { def }}{=}-\frac{1}{12} v^{4}+\frac{5}{12} u v^{2}-\frac{1}{2} u^{2}-\frac{1}{6} v^{3}+\frac{1}{3} u v+\frac{1}{6} v^{2}-\frac{1}{3} u, \\
& A_{i} \stackrel{\text { def }}{=}-\frac{1}{6} u^{2}+\frac{1}{12} v^{4}-\frac{1}{12} u v^{2}+\frac{2}{3} \alpha_{i} u v-\frac{1}{3} \alpha_{i} v^{3}-\frac{1}{6} v^{3}+ \\
& \quad+\frac{1}{3} u v-\frac{4}{3} \alpha_{i} u+\frac{2}{3} \alpha_{i} v^{2}-\frac{1}{6} v^{2}+\frac{1}{3} u, \quad i=1,2,3 .
\end{aligned}
$$

Subtracting in (54) the 3rd equation from the 2 nd and the 4 th from the 3 rd, we get

$$
\begin{gather*}
\left(2 u-v^{2}\right)(v-2)\left(\alpha_{1}-\alpha_{2}\right)=12 K^{2} t^{4-2\left(\alpha_{2}+\alpha_{3}\right)}  \tag{55}\\
\left(2 u-v^{2}\right)(v-2)\left(\alpha_{2}-\alpha_{3}\right)=0
\end{gather*}
$$

The 2 nd of these equations presents three possibilities. If the 1 st factor is equal to zero, that is

$$
\begin{equation*}
u=\frac{1}{2} v^{2} \tag{56}
\end{equation*}
$$

then it is easy to check that all $A_{i}$ vanish. The 1st equation of (55) yields $K=0$, i.e. $M=N=0$. From (37) we deduce that $\Phi_{0}^{0}=0$, hence (56) gives a purely gravitational solution to the YangMills equations. We see that if $A_{i}=0$ and $K=0$, then all equations (54) become identities. Equation (56) in parameters $\alpha_{i}$ assumes form

$$
\begin{equation*}
\left(\alpha_{1}\right)^{2}+\left(\alpha_{2}\right)^{2}+\left(\alpha_{3}\right)^{2}-2 \alpha_{1} \alpha_{2}-2 \alpha_{1} \alpha_{3}-2 \alpha_{2} \alpha_{3}+2 \alpha_{1}+2 \alpha_{2}+2 \alpha_{3}-3=0 \tag{57}
\end{equation*}
$$

In the space of parameters $\alpha_{1}, \alpha_{2}, \alpha_{3}$ this surface of 2 nd order is a cone with its apex in the point ( $1,1,1$ ). The circle (52) lays on this cone because its equation can be written down as $u=v=0$, which, obviously, satisfies equation (56).

If in equations (55) $v=2$, then again $K=0$, and the first of equations (54) becomes

$$
A_{0}=-\frac{1}{2} u^{2}+2 u-2=0
$$

which is equivalent to $u=2$. But the pair $u=2, v=2$ satisfies equality (56), therefore we get a special case of solution (57).

Let us now put $\alpha_{2}=\alpha_{3}$ in the 2 nd equation of (55). From (54), for $K \neq 0$ it obviously follows that $\alpha_{2}+\alpha_{3}=2$ because all $A_{i}$ and $K$ are constant. Consequently, $\alpha_{2}=\alpha_{3}=1$. From (53) we have $u=\left(\alpha_{1}\right)^{2}+1, v=\alpha_{1}+1$. For these values of $u$ and $v$ all equations (54) turn into $\left(\alpha_{1}-1\right)^{4}-12 K^{2}=0$, hence $\alpha_{1}=1 \pm \sqrt{K \sqrt{12}}$.

Thus, we get the solution

$$
\begin{equation*}
a=t^{1 \pm \sqrt{K \sqrt{12}}}, \quad b=c=t \tag{58}
\end{equation*}
$$

The last possibility for the 2 nd equation of (55) is $\alpha_{2}=\alpha_{3}$ and $K=0$. The first equation then becomes $\left(2 u-v^{2}\right)(v-2)\left(\alpha_{1}-\alpha_{2}\right)=0$.

To derive a result different from (57), we put $\alpha_{1}=\alpha_{2}$, so that $\alpha_{1}=\alpha_{2}=\alpha_{3}$. We have $u=3\left(\alpha_{1}\right)^{2}-1, v=3 \alpha_{1}+1$.

With such $u$ and $v$ equations (54) are satisfied identically. Thus,

$$
\begin{equation*}
\alpha_{1}=\alpha_{2}=\alpha_{3} \tag{59}
\end{equation*}
$$

and, with (50) in mind, $a=b=c$. This gives a new solution to the system of the Yang-Mills equations, but it is of no interest to us since dividing the metric (5) by $a^{2}$ and changing the parameter $\tau=\int \frac{d t}{a}$ brings us to the Minkowski metric.
Conclusion. We obtain two series of solutions of monomial type (50): one is purely gravitational (57) and another with non-zero electromagnetic field (58). The last solution is of the form (5) with $b=c$ and not a new one. It is conformally equivalent to a special case of solution (27).

## 4. Solutions of Exponential Type

Let us study a metric of the form

$$
\begin{equation*}
\psi=-d t^{2}+e^{2 \alpha_{1} t}\left(d x_{1}\right)^{2}+e^{2 \alpha_{2} t}\left(d x_{2}\right)^{2}+e^{2 \alpha_{3} t}\left(d x_{3}\right)^{2} \tag{60}
\end{equation*}
$$

where $\alpha_{1}, \alpha_{2}, \alpha_{3}$ are constant, that is $a=e^{\alpha_{1} t}, b=e^{\alpha_{2} t}, q c=e^{\alpha_{3} t}$.
Solutions to equations (17)-(20) can be found after a suitable gauge transformation of the given metric to a metric of monomial type. Namely, we will multiply metric (60) by $e^{2 t}$ and introduce the new time parameter $\tau=e^{t}$. We get

$$
\psi=-d \tau^{2}+\tau^{2\left(\alpha_{1}+1\right)}\left(d x_{1}\right)^{2}+\tau^{2\left(\alpha_{2}+1\right)}\left(d x_{2}\right)^{2}+\tau^{2\left(\alpha_{3}+1\right)}\left(d x_{3}\right)^{2}
$$

precisely like we have already examined. As proved above, this metric satisfies equations (17)-(20) either if indexes $\alpha_{1}, \alpha_{2}, \alpha_{3}$ are related by the equation similar to (57)

$$
\left(\alpha_{1}\right)^{2}+\left(\alpha_{2}\right)^{2}+\left(\alpha_{3}\right)^{2}-2 \alpha_{1} \alpha_{2}-2 \alpha_{1} \alpha_{3}-2 \alpha_{2} \alpha_{3}=0
$$

or $\alpha_{1}=\alpha_{2}=\alpha_{3}$, or if $\alpha_{1}, \alpha_{2}, \alpha_{3}$ give a solution equivalent to (58), i.e.

$$
\alpha_{1}= \pm \sqrt{K \sqrt{12}}, \quad \alpha_{2}=\alpha_{3}=0
$$

All metrics of the monomial type (51) have a singularity at $t=0$ that can not be removed within the framework of Riemannian geometry. However, the metrics of exponential type (60) conformally equivalent to them have no singularities. That is why there is no sense in relating singularities of a metric to singular states in the Universe development as it is done in numerous cosmological models, for example, in the Kasner or Friedmann models. That is, the authors do not agree with Landau and Lifshitz's point of view asserting that "...the very fact of the appearance of a singularity in the solutions of the Einstein equations (both in their cosmological aspect and for the collapse of finite bodies) has a profound physical meaning" [4, p. 423].

Among the solutions of exponential type the Einstein metric occurs only if $\alpha_{1}=\alpha_{2}=\alpha_{3}$, and the metric conformally equivalent to the Einstein metric is obtained under the conditions that are equivalent to (52):

$$
\alpha_{1}+\alpha_{2}+\alpha_{3}=-2, \quad\left(\alpha_{1}+1\right)^{2}+\left(\alpha_{2}+1\right)^{2}+\left(\alpha_{3}+1\right)^{2}=1
$$

## 5. Conformally-flat Solutions

Any conformally-flat metric automatically satisfies the Yang-Mills equations. As shown in [1, p. 444], a conformally-flat metric does not permit an electromagnetic field. We shall now establish conditions for the metric (5) to be conformally-flat. Dividing the metric (5) by $a$ and changing the time parameter, we can achieve $a=1$. Hence from (8) we get

$$
\begin{array}{cl}
b_{11}=\frac{1}{3}\left(\frac{\ddot{b}}{b}+\frac{\ddot{c}}{c}\right)-\frac{1}{6} \frac{\dot{c}}{c} \frac{\dot{b}}{c}, & b_{22}=\frac{1}{6}\left(\frac{\ddot{b}}{b}+\frac{\ddot{c}}{c}+\frac{\dot{c}}{c} \dot{b} \frac{b}{b}\right), \\
b_{33}=-\frac{1}{3}\left(\frac{\ddot{b}}{b}+\frac{\dot{c}}{c} \frac{b}{b}\right)+\frac{1}{6} \frac{\ddot{c}}{c}, & b_{44}=-\frac{1}{3}\left(\frac{\ddot{c}}{c}+\frac{\dot{c}}{c} \dot{b} \bar{b}\right)+\frac{1}{6} \frac{\ddot{b}}{b} . \tag{61}
\end{array}
$$

Since there is no electromagnetic field in this case, then all nonzero components of Pfaffian forms $\omega_{j}(j=1,2,3,4)$ consist of ( 61 ).

Let us calculate the external forms $\Phi_{j}^{i}$, the components of conformal curvature matrix, using the formula $\Phi_{j}^{i}=d \omega_{j}^{i}+\omega_{k}^{i} \wedge \omega_{j}^{k}+\omega^{i} \wedge \omega_{j}+\eta^{i m} \eta_{j n} \omega_{m} \wedge \omega^{n}$ :
$\Phi_{1}^{2}=\left(b_{22}-b_{11}\right) \omega^{1} \wedge \omega^{2}, \quad \Phi_{1}^{3}=\left(\frac{\ddot{b}}{b}+b_{33}-b_{11}\right) \omega^{1} \wedge \omega^{3}, \quad \Phi_{1}^{4}=\left(\frac{\ddot{c}}{c}+b_{44}-b_{11}\right) \omega^{1} \wedge \omega^{4}$,
$\Phi_{2}^{3}=\left(-b_{22}-b_{33}\right) \omega^{2} \wedge \omega^{3}, \quad \Phi_{2}^{4}=\left(-b_{22}-b_{44}\right) \omega^{2} \wedge \omega^{4}, \quad \Phi_{3}^{4}=\left(-b_{33}-b_{44}-\frac{\dot{b}}{\bar{b}} \frac{\dot{c}}{c}\right) \omega^{3} \wedge \omega^{4}$.
The components of $\Phi_{j}^{i}$ form the Weyl tensor of conformal curvature of metric (5). In view of (61), there are only three different coefficients. Setting them to zero, we get three equations $\frac{1}{3} \frac{\ddot{b}}{b}-\frac{1}{6} \frac{\ddot{c}}{c}-\frac{1}{6} \frac{\dot{b} \dot{c}}{b c}=0, \frac{1}{3} \frac{\ddot{c}}{c}-\frac{1}{6} \frac{\ddot{b}}{b}-\frac{1}{6} \frac{\dot{b} \dot{c}}{b c}=0, \frac{1}{3} \frac{\dot{b}}{b c}-\frac{1}{6} \frac{\ddot{b}}{b}-\frac{1}{6} \frac{\ddot{c}}{c}=0$. Subtracting the third
equations from the first two, we obtain two equalities $\frac{\ddot{b}}{b}=\frac{\dot{b} \dot{c}}{b c}, \frac{\ddot{c}}{c}=\frac{\dot{b} \dot{c}}{b c}$, which make all three previous equations into identical zeroes. The last system can easily be solved, and we get two conformally-flat solutions: $a=1, b=C_{1} \sinh \left(C_{4} t+C_{2}\right), c=C_{3} \cosh \left(C_{4} t+C_{2}\right)$, and $a=1, b=C_{1} \sin \left(C_{4} t+C_{2}\right), c=C_{3} \cos \left(C_{4} t+C_{2}\right)$.

## 6. The Solution Obtained in [1]

At first sight, the purely time-dependent solution obtained in [1, p. 448]

$$
a=\frac{2 P^{2}}{T-L t}, \quad b=c=\frac{T-L t}{2 P}, \quad b_{12}=\frac{M}{b^{2}}, \quad b_{34}=\frac{N}{b^{2}}, \quad M^{2}+N^{2}=\frac{L^{4}}{12 P^{4}},
$$

is different from all the solutions of the present paper. However, the substitution $\frac{T}{L}-t \rightarrow t$, $\frac{2 P^{2}}{L} x_{1} \rightarrow x_{1}, \frac{L}{2 P} x_{2} \rightarrow x_{2}, \frac{L}{2 P} x_{3} \rightarrow x_{3}$ makes it into a special case of solution (58) for $K=\frac{2}{\sqrt{3}}, a=t^{-1}, b=c=t$.

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## Чисто временное решение уравнений Янга-Миллса на четырехмерном многообразии конформной связности без кручения

Леонид Н. Кривоносов<br>Вячеслав А. Лукьянов

Приведен вывод чисто временных уравнений Янга-Миллса в пространстве конформной связности без кручения. Найдены три серии решений этих уравнений, выяснено, какие из этих решений дают метрику эйнштейнову или конформно эквивалентную эйнштейновой. Указань различные представления этих решений, отличающиеся наличием или отсутствием сингулярностей у соответствующих метрик.

Ключевые слова: уравнения Янга-Миллса, уравнения Эйнштейна, уравнения Максвелла, пространство конформной связности.


[^0]:    *oxyzt@ya.ru
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