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On New Parametric Representations of Analytic Area Nevanlinna Type Classes in a Circular Ring K on a Complex Plane \mathbb{C}

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We define certain new large area Nevanlinna type spaces in circular ring K on a complex plane \mathbb{C} and provide complete parametric representations for these new scales of analytic function spaces. Our results complement certain previously known assertions.

Keywords: analytic function, Nevanlinna characteristic, area Nevanlinna type spaces, circular ring.

Introduction

Assuming that $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ is the unit disk of the finite complex plane \mathbb{C} , T is the boundary of \mathbb{D} and $H(\mathbb{D})$ is the space of all functions holomorphic in \mathbb{D} , introduce the following classes of functions:

$$\tilde{N}_\alpha^\infty = \left\{ f \in H(\mathbb{D}) : T(\tau, f) \leq C_f(1 - \tau)^{-\alpha}, \quad 0 \leq \tau \leq 1 \right\}, \quad \alpha \geq 0,$$

where $T(\tau, f)$ is Nevanlinna's characteristic (see eg. [1, 2]). It is obvious that if $\alpha = 0$, then $\tilde{N}_0^\infty = N$, where N is the well-known Nevanlinna's class. The following statement on parametric representation in spaces in unit disk is known. It holds by Nevanlinna's classical result on the parametric representation of N class (see eg. [1, 2]) and it serves (see [1]) as a base of all theory of area Nevanlinna type spaces. *The N class coincides with the set of functions representable in the following form*

$$f(z) = C_\lambda z^\lambda B(z, \{z_k\}) \exp \left\{ \int_{-\pi}^{\pi} \frac{d\mu(\theta)}{1 - ze^{-i\theta}} \right\}, \quad z \in \mathbb{D},$$

where C_λ is any complex number, λ is any nonnegative integer, $B(z, \{z_k\})$ is the classical Blaschke product with zeros $\{z_k\}_{k=1}^{+\infty} \subset \mathbb{D}$ enumerated according their multiplicities and satisfying the condition $\sum_{k=1}^{+\infty} (1 - |z_k|) < +\infty$ and $\mu(\theta)$ is any function of bounded variation in $[-\pi, \pi]$.

Later this important result was extended to weighted Nevanlinna spaces we defined above and other similar spaces, for example, to so-called Nevanlinna-Djrbashyan spaces (see eg. [1, 3, 4]). The goal of this paper is to continue our earlier investigation started in [5–7] of certain new large analytic area Nevanlinna type spaces in the unit disk and to obtain similar parametric representation for these spaces. Our intention here is to extend some results from [8] and [3] on spaces in the circular ring to larger spaces of area Nevanlinna type in the same circular ring. Note

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in our previous mentioned papers we provided already such an extension procedure in the unit disk and we extended some known classical results about zero sets in certain analytic Nevanlinna spaces from [1] and from [2] to larger spaces, but in case of the unit disk. For formulation of our theorems we need to introduce first these analytic spaces in the unit disk. We define below three different scales of large area Nevanlinna spaces in the unit disk and then based on that their direct analogues in circular ring. These large spaces in unit disk appeared and studied for the first time in [5–7]. Later in [9, 10] meromorphic classes of this type were also introduced and studied by authors from mentioned papers. Through out this paper $dm_2(z)$ denotes the normalized Lebesgue measure in the unit disk on the complex plane. Let further

$$N_{\alpha, \beta}^p = \left\{ f \in H(\mathbb{D}) : \int_0^1 \left[\int_{|z| \leq R} \ln^+ |f(z)| (1 - |z|)^\alpha dm_2(z) \right]^p (1 - R)^\beta dR < +\infty \right\},$$

$$N_{\alpha, \beta_1}^\infty = \left\{ f \in H(\mathbb{D}) : \sup_{0 \leq R < 1} \left[\int_{|z| \leq R} \ln^+ |f(z)| (1 - |z|)^\alpha dm_2(z) \right] (1 - R)^{\beta_1} < +\infty \right\},$$

where it is assumed that $\beta_1 \geq 0$, $\alpha > -1$, $\beta > -1$ and $0 < p < \infty$. Note that various properties of $N_{\alpha, 0}^\infty$ are studied in [2]. In particular, both books [1] and [2] provide complete descriptions of zero sets and parametric representations of $N_{\alpha, 0}^\infty$. Later these assertions were extended in [2] and [5] by us to larger analytic spaces as we defined above and these results were used in [7].

Hence it is natural to consider the problem *on extension of various known results, for example, to all $N_{\alpha, \beta_1}^\infty$ classes.*

Let

$$(NA)_{p, \gamma, v} = \left\{ f \in H(\mathbb{D}) : \int_0^1 \left[\sup_{0 < \tau < R} T(f, \tau) (1 - \tau)^\gamma \right]^p (1 - R)^v dR < +\infty \right\},$$

where $\gamma \geq 0$, $v > -1$ and $0 < p < \infty$, and

$$N_{\alpha, \beta}^{\infty, p} = \left\{ f \in H(\mathbb{D}) : \sup_{0 \leq R < 1} \int_0^R \left[\int_T \ln^+ |f(|z|\xi)| d\xi \right]^p (1 - |z|)^\alpha d|z| (1 - R)^\beta < +\infty \right\},$$

where $0 < p < \infty$, $\alpha > -1$ and $\beta \geq 0$. Note that the zero sets of $N_{\alpha, \beta}^{\infty, p}$ are described in [4] for $\beta = 0$ and later for all positive values of β .

It is not difficult to verify that all these analytic classes are topological vector spaces with complete invariant metrics.

Throughout the paper, we write C for constant which is independent of the functions or variables being discussed.

The following assertion from [1] is crucial for our investigation and in the study of analytic area Nevanlinna spaces in the unit disk. Let (z_k) be an arbitrary sequence of complex numbers from the unit disk D . so that the following condition holds

$$\sum_{k=1}^{+\infty} ((1 - |z_k|)^{\beta+2}) < +\infty,$$

$\beta > -1$, then the Djrbashian infinite product (see [1])

$$\Pi_\beta(z, z_k)$$

is an analytic function and converges uniformly in D and have zeros only at sequence (z_k) .

Complete parametric representations of classes we defined above in case of unit disk were given in [5] and [6]. The intention of this paper to extend further these results to the case of circular rings. The theory of analytic function spaces in circular rings was developed in [8]. Note that similar problems were considered in [3, 12] and [13] for other analytic area Nevanlinna type spaces in circular rings. However, the classes we introduced in circular rings are larger and we can consider our theorems as direct extensions of previously known results formulated in [3].

For $0 < R_1 < R_2 < +\infty$ we denote by K the usual circular ring on the complex plane $K = K(R_1, R_2) = \{z \in \mathbb{C}, R_1 < |z| < R_2\}$, we also denote by $H(K)$ the space of all analytic functions in this circular ring and for f function so that $f \in H(K)$ we denote as usual the Nevanlinna characteristic of f in K by

$$T(\tau, f).$$

We will also use the following definition. We denote by $Z(f)$ the set of zeros of f function in K . Let further $r_0 = \frac{(R_1 + R_2)}{2}$, by $Z(R_1, r_0, f)$ we denote all those points from circular ring $K(r_0, R_1)$ which belong to $Z(f)$. Similarly we define $Z(R_2, r_0, f)$. Let also further $Z(R_1, r_0, f) = (z_k)$, $Z(R_2, r_0, f) = (w_k)$.

We define new spaces in circular ring K . The idea is keeping the same conditions on parameters is to replace in quasinorms the integration intervals, namely the unit interval $(0, 1)$ by (R_1, R_2) and $(0, R)$ by (R_1, R) . Let

$$(NA)_{p, \gamma_1, \gamma_2, v_1, v_2} = \left\{ f \in H(\mathbb{K}) : \int_{R_1}^{R_2} \left[\sup_{R_1 < \tau < R} T(f, \tau) (R - \tau)^{\gamma_1} (\tau - R_1)^{\gamma_2} \right]^p (R - R_1)^{\beta_1} (R_2 - R)^{\beta_2} dR < +\infty \right\},$$

we will write $f \in (NA)_{p, \gamma_1, v_1}$ meaning the quasinorm we defined without factors with γ_2 and v_2 and in the same way we will use the notation $(NA)_{p, \gamma_2, v_2}$ meaning the quasinorm we just provided, but without factors with γ_1 and v_1 . In other spaces in K we assume the same. We

define $(N)_{\alpha_1, \alpha_2, \beta_1, \beta_2}^{\infty, p}$ as space of analytic functions in K , with finite quasinorm

$$\sup_{R_1 < R < R_2} \left[\int_{R_1}^R T(f, |z|) (R - |z|)^{\alpha_1} (|z| - R_1)^{\alpha_2} d|z| \right]^p (R - R_1)^{\beta_1} (R_2 - R)^{\beta_2}$$

and $(N)_{\alpha_1, \alpha_2, \beta_1, \beta_2}^p$ as space of functions analytic in circular ring K so that

$$\int_{R_1}^{R_2} \left[\int_{R_1}^R T(f, |z|) (|z| - R_1)^{\alpha_2} (R - |z|)^{\alpha_1} d|z| \right]^p (R - R_1)^{\beta_1} (R_2 - R)^{\beta_2} dR < +\infty.$$

We will use later the following expression. It provides a parametric representation of the f function analytic in K . We will call it standard parametric representation of f function in $K = K(R_1, R_2)$. Let further

$$f(z) = c_m z^m \Pi_{t_1}(v_1, v_2) \Pi_{t_2}(v_3, v_4) \exp h_1(v_1) \exp h_2(v_3),$$

$v_1 = \frac{R_1}{z}$, $v_2 = \frac{R_1}{z_k}$, $v_3 = \frac{z}{R_2}$, $v_4 = \frac{w_k}{R_2}$, $z \in K(R_1, R_2)$, where c_m is an arbitrary complex number, m in a nonnegative integer, h_1, h_2 are analytic functions in the unit disk D . In this case we say that the f function in K allows standard parametric representation in K . If X is a certain class of analytic functions in K and each function f from X admits such a parametric representation then we will say that the X class admits standard parametric representation. Various results on parametric representations are well-known (see, for example, [1] and [2]).

1. Theorems on Parametric Representations of $N_{\alpha_1, \alpha_2, \beta_1, \beta_2}^p$, $(NA)_{p, \gamma_1, \gamma_2, v_1, v_2}$ and $N_{\alpha_1, \alpha_2, \beta_1, \beta_2}^{\infty, p}$ Classes in Circular Rings

In this section we formulate our main results for spaces in circular ring. These are three theorems providing parametric representation for each space in circular ring we defined. We consider these results for analytic spaces in circular ring as direct extensions of our earlier theorems in the unit disk from [5–7].

Theorem 1. Let $p \in (0, \infty)$, $t_j > \frac{\beta_j + 1}{p} + \alpha_j$, $\alpha_j > -1$, $\beta_j > -1$, $j = 1, 2$.

Then let

$$n_k = \{\text{card} z_k : |z_k| > R_1 \tau_k\},$$

$$m_k = \{\text{card} w_k : |w_k| < R_2 s_k\},$$

where $s_k = 1 - 2^{-k}$, $\tau_k = 1 + 2^{-k}$ and where (z_k) and (w_k) are arbitrary sequences from halfrings of K , and

$$\Pi_{t_1}(z, z_k) \quad \text{and} \quad \Pi_{t_2}(z, w_k)$$

are mentioned Djrbashyan products constructed via (z_k) or (w_k) sequences.

We also assume that the following sum is finite

$$\sum_{k=1}^{\infty} \frac{n_k^p}{p_k}, \quad p_k = 2^{k(\beta_1 + \alpha_1 p + 2p + 1)}$$

and we also assume that the same condition but for (m_k) instead of (n_k) with α_2, β_2 instead of α_1 and β_1 is also true.

Under these conditions a f function from the

$$N_{\alpha_1, \alpha_2, \beta_1, \beta_2}^p$$

space allows standard parametric representation in circular ring K . In such a parametric representation $h_1(\frac{R_1}{z})$ belongs to N_{α_1, β_1}^p and $h_2(\frac{z}{R_2})$ belongs to N_{α_2, β_2}^p .

And moreover each f function from the mentioned class in circular ring admits such a standard parametric representation.

In the following two assertions we again deal with two sequences (z_k) and (w_k) in circular halfring of K and use again notations (n_k) and (m_k) as they were defined in theorem 1. We assume also again that two infinite products

$$\Pi_{t_1}(z, z_k) \quad \text{and} \quad \Pi_{t_2}(z, w_k)$$

are constructed via these given sequences.

Theorem 2. Let $p \in (0, \infty)$, $v_j > 0$, $\gamma_j > 0$, and also $t_j > \frac{v_j + 1}{p} + \gamma_j - 1$, $j = 1, 2$. Let further (z_k) and (w_k) be two sequences in the unit circular ring K , we also assume that the following two sums are finite

$$\sum_{k=1}^{+\infty} \frac{n_k^p}{l_k}, \quad \sum_{k=1}^{+\infty} \frac{m_k^p}{d_k},$$

$$l_k = 2^{k(p+1+v_1+p\gamma_1)}, \quad d_k = 2^{k(p+1+v_2+p\gamma_2)}.$$

Then there is a f function from

$$(NA)_{p, \gamma_1, \gamma_2, v_1, v_2}$$

space which admits standard parametric representation in circular ring K . Moreover in such a parametric representation h_1 and h_2 are analytic in the unit disk, and $h_1(\frac{R_1}{z})$ belongs to $(NA)_{p,\gamma_1,v_1}$, and $h_2(\frac{z}{R_2})$ belongs to $(NA)_{p,\gamma_2,v_2}$. And each f function from these classes in circular ring K admits such a standard parametric representation.

Theorem 3. Let $p \in (0, +\infty)$, $\alpha_j > 0$, $\beta_j > -1$, $t_j > \alpha_j + \beta_j + 1 + p$, $j = 1, 2$.

Let further

$$(n_1)^p(\tau) < c(\tau - R_1)^{-\alpha_1 - \beta_1 - p - 1},$$

where c is a constant.

We assume the same condition holds for (n_2) instead of (n_1) and with α_2 and β_2 instead of α_1 and β_1 and $R_2 - \tau$ instead of $\tau - R_1$, where

$$n_1(\tau) = \{\text{card} z_k : |z_k| > \tau\}, \tau \in (R_1, R_2), \quad n_2(\tau) = \{\text{card} w_k : |w_k| < \tau\}, \tau \in (R_1, R_2)$$

Then there is a f function from $(N)_{\alpha_1, \alpha_2, \beta_1, \beta_2}^{\infty, p}$ space which admits standard parametric representation in circular ring K .

Moreover in such a representation h_2 and h_1 are analytic in the unit disk and $h_1(\frac{R_1}{z})$ belongs to $(N)_{\alpha_1, \beta_1}^{\infty, p}$, and $h_2(\frac{z}{R_2})$ belongs to $(N)_{\alpha_2, \beta_2}^{\infty, p}$.

And moreover each f function from this class in circular ring K admits standard parametric representation.

Proofs of these assertions will be presented elsewhere. Results we formulated in case of the unit disk were already known (as we indicated above). In case of similar type but narrower spaces these results were obtained previously in [3]. Our arguments follow similar strategy as in [3], but with more technically involved estimates in all proofs.

Proofs of our assertions are based on a simple idea to replace functions analytic in circular ring by functions analytic in the unit disk via factorization. Note the same procedure was applied in [3]. We formulate this procedure in the following assertions.

Lemma 1. Each f function analytic in K admits representation

$$f = f_1 f_2,$$

where each f_i is an analytic function in the unit disk and $f_1 = f_1\left(\frac{R_1}{z}\right)$, $f_2 = f_2\left(\frac{z}{R_2}\right)$, $z \in K(R_1, R_2)$.

Moreover

$$T(f, r) = T\left(f_1, \frac{R_1}{r}\right) + O(1), \quad r \in (R_1, r_0),$$

$$T(f, r) = T\left(f_2, \frac{r}{R_2}\right) + O(1), \quad r \in (r_0, R_2),$$

$r_0 \in (R_1, R_2)$. In addition in proofs we constantly use the fact that both $T(f, r)$ and $T(f_i, r)$ functions are growing functions.

We formulate one more assertion which is a base of proofs.

Lemma 2. Let X be one of the spaces in our theorems (for example $(NA)_{p,\gamma_1,\gamma_2,v_1,v_2}$ space which admits standard parametric representation for some restrictions on parameters- see formulations of theorems).

Then the following are equivalent $f \in X(K)$ and

$$f = f_1 f_2,$$

where $f_1 = f_1\left(\frac{R_1}{z}\right)$, $f_2 = f_2\left(\frac{z}{R_2}\right)$, $z \in K(R_1, R_2)$ and f_i belongs to $X_i(D)$ where $X_i(D)$ is a parallel to $X(K)$ analytic class (without two factors in quasinorm of $X(K)$ as it was indicated above).

For example, if $X = N_{\alpha_1, \alpha_2, \beta_1, \beta_2}^{\infty, p}$ then X_1 is equal to $(N)_{\alpha_1, \beta_1}^{\infty, p}$ and X_2 is equal to $(N)_{\alpha_2, \beta_2}^{\infty, p}$.

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О некоторых новых параметрических представлениях аналитических классов типа Неванлинны в круговом кольце K на комплексной плоскости C

Роми Ф. Шамоян

Мы определяем новые пространства типа Неванлинны в круговом кольце и даем их параметрические представления. Эти результаты дополняют ранее известные теоремы о параметрических представлениях аналитических классов типа Неванлинны.

Ключевые слова: аналитическая функция, характеристика Неванлинны, пространства типа Неванлинны, круговое кольцо.