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On the Approximation of a Parabolic Inverse Problem by Pseudoparabolic One

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The properties of the solution to the inverse problem on the identification of the leading coefficient of the multi-dimensional pseudoparabolic equation are studied. It is proved that the inverse problem for the pseudoparabolic equation approximates the appropriate inverse problem for the parabolic equation of filtration. The existence and uniqueness of the solution to the parabolic inverse problem is established.

Keywords: filtration, inverse problems for PDE, pseudoparabolic equation, parabolic equation, existence and uniqueness theorems.

Introduction

An inverse problem for the pseudoparabolic equation

$$(u + L_1 u)_t + L_2 u = f \tag{0.1}$$

with the differential operators L_1 and L_2 of the second order in spacial variables is discussed in this paper. We are interested in finding the leading coefficients of L_2 in (0.1) from the additional boundary data. Applications of this problem deal with the recovery of unknown parameters indicating physical properties of a natural stratum which should be determined on the basis of the investigation of its behaviour under the natural non-steady-state conditions (see [1] for details). This leads to the interest in studying the inverse problems for (0.1) and its analogue.

The investigation of inverse problems for pseudoparabolic equations goes back into 1980s. The first result obtained by Rundell in [2] is concerned with the inverse problems of the identification of an unknown source f in the (0.1) with linear elliptic operators L_1 and L_2, $L_1 = L_2$. Rundell proved the global existence and uniqueness theorems in the case that f depends only on x or t. Another kind of inverse problems is considered in [3,4]. These works are devoted to problems of reconstructing the kernels in integral term of (0.1) with the integro-differential operator L_2. As for the determination of unknown coefficients in (0.1) we mention the results of Mamayusupov [5], Lubanova and Tani [6]. Mamayusupov proved the uniqueness theorem and found an algorithm for solving the inverse problem with respect to $u(t, x)$, functions $b(y)$, $c(y)$ and a constant a for the equation

$$u_t - \Delta u_t = a\Delta u + b(y)u_y + c(y) + \delta(t, x, y), \quad \text{for } (x,\, y) \in \mathbf{R}^2,\ t > 0$$

provided that $u(t, x, 0)$, $u_y(t, x, 0)$ and $u(0, x, y)$ are given. Here $\delta(t, x, y)$ is the Dirac delta function.

In [6] an inverse problem of identification of an unknown leading coefficient in the operator L_2 for (0.1) was discussed (see Problem 1 below). The existence, uniqueness and regularity of

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the solution to the inverse problem were established there. The statement of the inverse problem was motivated in [1].

A main goal of this paper is to investigate the behavior of the solution to the inverse problem considered in [6] as $\eta \to 0$. It is well known [7] that when passing to the limit $\eta \to 0$ equation (0.1) formally tends to coincide with the standard linear equation of filtration in a porous medium

$$u_t + L_2 u = f, \tag{0.2}$$

The direct initial boundary value problem for pseudoparabolic equation in a bounded domain $\Omega \subset \mathbf{R}^n$ approximates the appropriate problem for parabolic equation [8]. In particular, under certain assumptions the solution u^η of equation (0.1) with the initial data $u^\eta(0,x) = u_0(x)$ tends to the solution u of (0.2) with the same initial condition in the L^2-norm for all $t \geq 0$ as $\eta \to 0$. It was established in [1] that the inverse problem for the pseudoparabolic equation also approximates weakly the appropriate inverse problem for the parabolic one in the case when $L_1 = \eta\partial^2/\partial x^2$, $L_2 = k(t)\partial^2/\partial x^2$, η is a positive real number and $k(t)$ is an unknown coefficient. In the present paper this result will be extended to the inverse problems for (0.1) and (0.2) with any number of space variables. Such an investigation is also of an interest in studying the inverse problems for evolution equations whose principal terms contain unknown coefficients. The considerable results in this sphere are obtained for parabolic equations (see [9–12] and references given there).

The paper is organized as follows. In Section 1 for the convenience of the reader we repeat the formulation of the inverse problem for (0.1) and the relevant material from [6] without proofs and comments, thus making our exposition self-contained. In Section 2 we discuss the behaivior of the solution to the inverse problem as $\eta \to 0$ and prove the existence and uniqueness theorem for the relevant parabolic inverse problem. Section 3 contains the conclusions and comments to the main results of the paper.

1. Preliminaries

Let Ω be a domain in $\mathbf{R}^n$ with a boundary $\partial\Omega \in C^2$, T an arbitrary real number and $Q_T = \Omega \times (0,T)$. Throughout this paper we use the notation:
$\|\cdot\|$ and $(\cdot,\cdot)$ are the norm and the inner product of $L^2(\Omega)$, respectively;
$\|\cdot\|_j$ and $\langle\cdot,\cdot\rangle_j$ are the norm of $W_2^j(\Omega)$ and the duality relation between $\overset{\circ}{W}{}_2^j(\Omega)$ and $W_2^{-j}(\Omega)$, respectively $(j = 1,2)$; as usual $W_2^0(\Omega) = L^2(\Omega)$.

Let $M : W_2^1(\Omega) \to (W_2^1(\Omega))^*$ be a linear differential operator of the form

$$Mv = -\mathrm{div}(\mathcal{M}(x)\nabla v) + m(x)v, \tag{1.1}$$

where $\mathcal{M}(x) \equiv (m_{ij}(x))$ is a matrix of functions $m_{ij}(x),\ i,j = 1,2,\dots,n$. We assume that the following conditions are fulfilled.

I. $m_{ij}(x)$, $\partial m_{ij}/\partial x_l$, $i,j,l = 1,2,\dots,n$, and $m(x)$ are bounded in Ω. M is an operator of elliptic type, that is, there exist positive constants m_1 and m_2 such that for any $v \in \overset{\circ}{W}{}_2^1(\Omega)$

$$m_1\|v\|_1^2 \leqslant \langle Mv, v\rangle_1 \leqslant m_2\|v\|_1^2. \tag{1.2}$$

II. There exists a positive constant m_3 such that for any $v \in W_2^2(\Omega)$

$$\|Mv\| \leqslant m_3\|v\|_2. \tag{1.3}$$

III. $m_{ij}(x) = m_{ji}(x)$ for $i,j = 1,2,\dots,n$ and $m(x) \geqslant 0$ for $x \in \Omega$.

We proceed to study the following inverse problem [6].

Problem 1. For a given constant η and functions $f(t,x)$, $g(t,x)$, $\beta(t,x)$, $U_0(x)$, $\omega(t,x)$, $\varphi_1(t)$, $\varphi_2(t)$ find the pair of functions $(u(t,x),k(t))$ satisfying the equation

$$u_t+\eta Mu_t+k(t)\,Mu+g(t,x)u=f(t,x),\quad (t,x)\in Q_T, \tag{1.4}$$

and the conditions

$$(u+\eta Mu)\big|_{t=0}=U_0(x),\quad x\in\Omega, \tag{1.5}$$

$$u\big|_{\partial\Omega}=\beta(t,x),\quad t\in[0,T], \tag{1.6}$$

$$\int_{\partial\Omega}\Big\{\eta\frac{\partial u_t}{\partial\nu}\ +\ k(t)\frac{\partial u}{\partial\nu}\Big\}\omega(t,x)\,\mathrm{d}S+\varphi_1(t)k(t)=\varphi_2(t),\quad t\in[0,T]. \tag{1.7}$$

Here $\dfrac{\partial}{\partial\nu}=(\mathbf{n},\mathcal{M}(x)\nabla)$ and $\mathbf{n}$ is the unit outward normal to $\partial\Omega$.

We use functions $a(t,x)$, $h^\eta(t,x)$ and $b(t,x)$ as the solutions of the Dirichlet problems

$$Ma=0\quad\text{in }\Omega,\quad a\big|_{\partial\Omega}=\beta(t,x); \tag{1.8}$$

$$Mb=0\quad\text{in }\Omega,\quad b\big|_{\partial\Omega}=\omega(t,x),$$

$$h^\eta\ +\ \eta Mh^\eta=0\quad\text{in }\Omega,\quad h^\eta\big|_{\partial\Omega}=\omega(t,x), \tag{1.9}$$

$$\begin{aligned}
&\big\langle Mv_1,v_2\big\rangle_{1,M}=(\mathcal{M}(x)\nabla v_1,\nabla v_2)+(m(x)v_1,v_2),\quad v_1,v_2\in W_2^1(\Omega);\\
&\Psi(t)=\big\langle Ma,b\big\rangle_{1,M},\qquad F(t,x)=a_t-f(t,x)+g(t,x)a,\\
&\Phi^\eta(t)=\varphi_2(t)-\frac{\eta}{2}\,\langle Ma_t,h^\eta\rangle_{1,M}+(f(t,x)-a_t,h^\eta),\\
&\overline{\Psi}=\max_{t\in[0,T]}\langle Ma,h^\eta\rangle_{1,M}\,,\quad \overline{\varphi}_1=\max_{t\in[0,T]}\varphi_1(t),\quad \overline{\Phi}^\eta=\max_{t\in[0,T]}\Phi^\eta(t).
\end{aligned} \tag{1.10}$$

By a solution $\{u,k\}$ of Problem 1 we mean that

(1) $k(t)$ is continuous for $0\leqslant t\leqslant T$;

(2) $u\in C^1([0,T];W_2^2(\Omega))$;

(3) the equation (1.4) and the conditions (1.5)–(1.7) are satisfied.

The existence and uniqueness of the solution to Problem 1 is established by the following theorem [6].

Theorem 1.1. *Let the assumptions I–III be fulfilled and η be a positive constant. Assume that*
(i) $f\in C([0,T];L^2(\Omega))$, $\beta\in C^1([0,T];W_2^{3/2}(\partial\Omega))$, $U_0\in L^2(\Omega)$, $g\in C(\overline{Q}_T)$, $\omega\in C^1([0,T];W_2^{3/2}(\partial\Omega))$, $\varphi_1\in C^1([0,T])$, $\varphi_2\in C([0,T])$;
(ii) f, U_0, β, ω, φ_1 *are nonnegative and*

$$\int_\Omega h^\eta\,\mathrm{d}x\geqslant h_0=\text{constant}>0,\quad t\in[0,T]; \tag{1.11}$$

(iii) *there exist constants α_i, $i=0,1,2$, such that $0\leqslant\alpha_0,\ \alpha_1\leqslant 1$, $\alpha_0+\alpha_1<2$,*

$$(1-\alpha_0)\,\varphi_1(t)+(1-\alpha_1)\,\Psi(t)\geqslant\alpha_2=\text{constant}>0,\quad t\in[0,T],$$

$$\chi(0) + a(0,x) - U_0(x) \geqslant 0 \quad \textit{for almost all } x \in \Omega,$$
$$g(t,x)\chi(t) + \chi'(t) + F(t,x) \geqslant 0 \quad \textit{for almost all } (t,x) \in Q_T,$$

where $\chi(t) = \eta\,(\alpha_0\varphi_1(t) + \alpha_1\Psi(t))\left[\int_\Omega h^\eta\,\mathrm{d}x\right]^{-1}$;

(iv) *for any* $t \in [0,T]$

$$\Phi^\eta(t) \geqslant \Phi_0^\eta = \text{constant} > 0$$

holds and $g(t,x)$ *satisfies the inequality*

$$\max_{\overline{Q}_T} g(t,x) \leqslant \frac{\Phi_0^\eta}{\eta}\left[\overline{\varphi}_1 + \overline{\Psi} + 2\eta^{-1}\max_{[0,T]}(a,h^\eta)\right]^{-1} \equiv \frac{k_0}{\eta}.$$

Then Problem 1 has a unique solution $(u,k) \in C^1([0,T];W_2^2(\Omega)) \times C([0,T])$. *Moreover,* u *and* k *satisfies the estimates*

$$0 \leqslant u(t,x) \leqslant \chi(t) + a(t,x) \quad \textit{for almost all } (t,x) \in Q_T, \tag{1.12}$$

$$\|u(t)\|_1^2 + \|u_t(t)\|^2 + \eta\left(\|u(t)\|_2^2 + \|u_t(t)\|_2^2\right) \leqslant C, \quad t \in [0,T], \tag{1.13}$$

$$k_0 \leqslant k(t) \leqslant k_1 \tag{1.14}$$

with positive constants C *and* $k_1 = \alpha_2^{-1}\,\max_{t\in[0,T]}\left\{\Phi^\eta(t) + (|g|(a+\chi(t)),h^\eta)\right\}$.

2. Approximation of Parabolic Inverse Problem

As mentioned above, when passing to the limit $\eta \to 0$ equation (0.2) formally tends to coincise with the linear parabolic equation and Problem 1 transforms to the following parabolic inverse problem.

Problem 2. Given $f(t,x)$, $g(t,x)$, $\beta(t,x)$, $u_0(x)$, $\phi_1(t)$, $\phi_2(t)$; find the pair of functions $(u(t,x),k(t))$ satisfying the equation

$$u_t \;+\; k(t)\,Mu \;+\; g(t,x)u \;=\; f(t,x), \qquad (t,x) \in Q_T, \tag{2.1}$$

and the conditions

$$u\big|_{t=0} \;=\; u_0(x), \qquad x \in \Omega, \tag{2.2}$$

$$u\big|_{\partial\Omega} \;=\; \beta(t,x), \qquad t \in [0,T], \tag{2.3}$$

$$k(t)\int_{\partial\Omega}\frac{\partial u}{\partial\nu}\,\omega\,\mathrm{d}s \;+\; \phi_1(t)\,k(t) \;=\; \phi_2(t), \qquad t \in (0,T). \tag{2.4}$$

Hereafter, by the solution of Problem 2 we mean a pair $(u(t,x),k(t))$ such that
a) $u \in V = \left\{v|\; v \in L^\infty(0,T;W_2^2(\Omega)),\; v_t \in L^\infty(0,T;L^2(\Omega))\right\}$, $k(t) \in L^\infty(0,T)$;
b) system (2.1)–(2.4) is satisfied.

We shall denote the solutions of Problem 1 with the initial data

$$\left(u^\eta \;+\; \eta M u^\eta\right)\Big|_{t=0} \;=\; u_0 \;+\; \eta M u_0 \equiv U_0 \tag{2.5}$$

and Problem 2 by (u^η,k^η) and (u,k), respectively.

In this section we make use of the inequality

$$\left\|\frac{\partial v}{\partial\nu}\right\|_{L^q(\partial\Omega)} \;\leqslant\; C_2\Big(\;\|v\|_2^\alpha\;\|v\|_1^{1-\alpha} \;+\; \|v\|_1\Big) \tag{2.6}$$

valid for any $v \in W_2^2(\Omega)$ where $\alpha = \dfrac{n}{2} - \dfrac{n-1}{q}$, $q \in \left[\frac{2(n-1)}{n}, \dfrac{2(n-1)}{n-2}\right]$ for $n \leqslant 3$ and $q \in [1, \infty]$ for $n = 2$. (2.6) is easily derived from the multiplicative inequality [13]. The constant C_2 depends on n, q, mesΩ, m_2 and m_3. We also use the property of the function h^η established by the following lemma.

Lemma 2.1. *Let $\omega \in C([0,T]; W_2^{3/2}(\Omega))$. Then the solution of the problem* (1.9) satisfies the estimate

$$\|h^\eta\|^2 + \eta \|h^\eta\|_1^2 \leqslant \eta C_3 \tag{2.7}$$

where a positive constant C_3 depends on m_1, m_2, mesΩ, $\|b\|$ and does not depend on η.

Proof. To obtain the estimate (2.7) we multiply the equation (1.9) by h^η in terms of $L^2(\Omega)$ and integrate by parts in the left-hand side. This gives

$$\|h^\eta\|^2 + \eta \big\langle M h^\eta, h^\eta \big\rangle_1 = \eta \int_{\partial\Omega} \frac{\partial h^\eta}{\partial \nu}\, \omega \, \mathrm{d}s. \tag{2.8}$$

By Hölder's inequality for $n \geqslant 2$

$$\Big| \int_{\partial\Omega} \frac{\partial h^\eta}{\partial \nu}\, \omega \, \mathrm{d}s \Big| \leqslant \Big\| \frac{\partial h^\eta}{\partial \nu} \Big\|_{L^p(\partial\Omega)} \|\omega\|_{L^{p/(p-1)}(\partial\Omega)} \tag{2.9}$$

where $p = 2(n-1)/n$. From (2.6) and the embedding theorem [13] it follows that for any $v \in W_2^2(\Omega)$

$$\Big\| \frac{\partial v}{\partial \nu} \Big\|_{L^p(\partial\Omega)} \leqslant C_4 \|v\|_1, \qquad \|v\|_{L^{p/(p-1)}(\partial\Omega)} \leqslant C_5 \|v\|_2. \tag{2.10}$$

Here constants C_4 and C_5 depend on m_2, m_3, n and mesΩ. Applying (2.10) to (2.9) yields

$$\Big| \int_{\partial\Omega} \frac{\partial h^\eta}{\partial \nu}\, \omega \, \mathrm{d}s \Big| \leqslant C_4 C_5 \|h^\eta\|_1 \|b\|_2 \equiv C_6 \|h^\eta\|_1.$$

Estimating the right-hand side of (2.8) with the help of this inequality, one can obtain the estimate (2.7). The lemma is proved. □

The main result of this section is formulated in the next theorem.

Theorem 2.2. *Let $\eta \in (0, \eta_0]$, $n \geqslant 2$, the condition (ii) of Theorem* 1.1 *and the assumptions I–III are fulfilled. Let*

(i″) $f \in L^2(0,T; W_2^1(\Omega)) \cap C(\overline{Q}_T)$, $\beta \in C^1([0,T]; W_2^{3/2}(\partial\Omega))$, $u_0 \in W_2^2(\Omega)$, $g \in C(\overline{Q}_T)$, $\omega \in C^1([0,T]; W_2^{3/2}(\partial\Omega))$, $\varphi_1 \in C^1([0,T])$, $\varphi_2 \in C([0,T])$;

(iii″) *u_0 and β obey the compatibility condition* $u_0(x)\big|_{\partial\Omega} = \beta(0,x)$,

$$a(0,x) - u_0(x) \quad - \quad \eta_0 M u_0 \geqslant 0, \quad x \in \Omega, \tag{2.11}$$

$$F(t,x) \;\geqslant\; 0, \qquad (t,x) \in Q_T,$$

$$\phi_1(t) \;+\; \Psi(t) \;\geqslant\; \alpha_2 = const > 0, \qquad t \in [0,T]. \tag{2.12}$$

(iv″) *there exist positive constants $\overline{\phi}_2$, $\underline{\phi}_2$ such that*

$$\underline{\phi}_2 \;\leqslant\; \phi_2(t) \;\leqslant\; \overline{\phi}_2, \qquad t \in [0,T], \tag{2.13}$$

Then

$$\begin{array}{lllll} u^\eta & \to & u & *-\ \textit{weakly} & \textit{in}\ L^\infty(0,T;W_2^2(\Omega)),\\ u_t^\eta & \to & u_t & *-\ \textit{weakly} & \textit{in}\ L^\infty(0,T;L^2(\Omega))\quad \textit{and}\quad \textit{weakly}\quad \textit{in}\ L^2(0,T;W_2^1(\Omega)),\\ k^\eta & \to & k & *-\ \textit{weakly} & \textit{in}\ L^\infty(0,T) \end{array}$$

as $\eta \to 0$. *Moreover,*

$$0\ <\ r(\eta)\ \leqslant\ k^\eta(t)\ \leqslant\ \alpha_2^{-1}\big(\overline{\phi}_2+\max_{t\in[0,T]}(ga,h^\eta)\big)\equiv k_2 \tag{2.14}$$

where $r(\eta)$ *is a continuous function of* η *on* $[0,\eta_0]$ *and* $r(0)>0$.

Proof. Without loss of generality we can assume η_0 to be chosen so that $\eta_0 \leqslant 1$,

$$0<\Phi_0^{\eta_0}\equiv \underline{\phi_2}-\max_{t\in[0,T]}\Big\{\frac{\eta_0}{2}\|Ma_t\|_1\|b\|+\|a_t\|\|h^{\eta_0}\|\Big\}\leqslant \Phi^\eta(t)\leqslant \overline{\phi}_2, \tag{2.15}$$

$$\max_{\overline{Q_T}} g(t,x)\leqslant \Phi_0^{\eta_0}\Big[\eta_0\big(\phi_1(t)+\Psi(t)\big)+2C_3\eta_0^{1/2}\max_{t\in[0,T]}\|a\|\Big]^{-1}$$

because of (2.7). Therefore the hypotheses of the theorem imply that all assumptions of Theorem 1.1 are fulfilled with $\alpha_0=\alpha_1=0$. This shows that Problem 1 has a unique solution $(u^\eta(t,x),\ k^\eta(t))\in C^1([0,T];W_2^2(\Omega))\times C([0,T])$ and the estimates (1.12)–(1.14) hold for any η, $0<\eta\leqslant\eta_0$. Our next step is to get a uniform lower bound (2.14) for k^η and then uniform estimates for the derivatives of u^η.

Let us set

$$w^\eta(t,x)=a(t,x)-u^\eta(t,x). \tag{2.16}$$

The function $w(t,x)$ satisfies the equation

$$w_t^\eta+\eta M w_t^\eta+k^\eta(t)\,Mw^\eta+g(t,x)w^\eta=F(t,x),\quad (t,x)\in Q_T, \tag{2.17}$$

and the conditions

$$(w^\eta+\eta M w^\eta)\big|_{t=0}\ =\ a(0,x)-U_0(x),\quad x\in\Omega, \tag{2.18}$$

$$w^\eta\big|_{\partial\Omega}\ =\ 0,\quad t\in[0,T], \tag{2.19}$$

$$\int_{\partial\Omega}\Big\{\eta\frac{\partial w_t^\eta}{\partial\nu}+k^\eta\frac{\partial w^\eta}{\partial\nu}\Big\}\omega \mathrm{d}S\ =\ (\varphi_1+\Psi)k^\eta+\eta\langle Ma_t,h^\eta\rangle_{1,M}-\varphi_2,\ t\in[0,T]. \tag{2.20}$$

As was shown in [6], multiplying (2.17) by $h^\eta(t,x)$ in terms of $L^2(\Omega)$, the integration by parts in the left side and substituting (2.20) into the resulting equation leads to the equation

$$k^\eta(t)\Big(\varphi_1(t)+\Psi(t)+\frac{1}{\eta}(w^\eta,h^\eta)\Big)=\Phi^\eta(t)-(g(t,x)(a-w^\eta),h^\eta) \tag{2.21}$$

by virtue of (1.8), (1.9), (1.10), (1.11).

According to Theorem 1.1 the pair $(w^\eta,k^\eta)\in C^1([0,T];\ W_2^2(\Omega))$ for every $0<T<+\infty$. Since the problems (2.17)–(2.20) and (2.17)–(2.19),(2.21) are equivalent, the pair (w,k) also solutions the problem (2.17)–(2.19),(2.21).

Let us set

$$k_0^\eta\ =\ \min_{t\in[0,T]} k^\eta(t). \tag{2.22}$$

We multiply (2.17) by Mw^η in terms of the inner product of $L^2(\Omega)$ and integrate by parts in the following way:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|w\|_{1,M}^2 &+ \frac{\eta}{2}\frac{d}{dt}\|Mw^\eta\|^2 + k_0^\eta\|Mw^\eta\|^2 \\ &= -\big\langle gw^\eta, Mw^\eta\big\rangle_{1,M} - \int_{\partial\Omega} F\,\frac{\partial w^\eta}{\partial\nu}\,\mathrm{d}s + \big\langle F, Mw^\eta\big\rangle_{1,M}.\end{aligned} \tag{2.23}$$

By (1.2),(1.3),(2.6) and the Young inequality,

$$\Big|\int_{\partial\Omega} F\,\frac{\partial w^\eta}{\partial\nu}\,\mathrm{d}s\Big| \leqslant C_7\Big(\Big(\frac{1}{(k_0^\eta)^{1/2}}+1\Big)\|F\|_1^2 + \|w^\eta\|_{1,M}^2\Big) + \frac{k_0^\eta}{4}\|Mw^\eta\|^2 \tag{2.24}$$

where $C_7 = \text{const} > 0$ depends on C_2, mesΩ, m_i, $i = 1, 2, 3$. Then (1.1),(2.23), (2.24) give

$$\|w\|_{1,M}^2 + \eta\|Mw^\eta\|^2 + k_0^\eta\int_0^t \|Mw^\eta\|^2\mathrm{d}\tau \leqslant \frac{C_8}{(k_0^\eta)^{1/2}} + C_9\int_0^t \|w\|_{1,M}^2\mathrm{d}\tau. \tag{2.25}$$

The positive constants C_8 and C_9 depends on C_7, $\|g\|_{C^1(\overline{Q}_T)}$, m_i, $i = 1, 2, 3$. In accordance with Gronwall's lemma, it follows from (2.25) that

$$\|w\|_{1,M}^2 + \eta\|Mw^\eta\|^2 + k_0^\eta\int_0^t \|Mw^\eta\|^2\mathrm{d}\tau \leqslant \frac{C_{10}}{(k_0^\eta)^{1/2}}. \tag{2.26}$$

Here C_{10} =const > 0 depends on C_8, C_9 and does not depend on η and k_0^η.

Let us come back to the equation (2.21). We first note that the numerator of (2.21) is bounded below by a positive constant independent of k_0^η when η_0 is small enough. Indeed, by (2.7) and (2.26),

$$\big|\big(gw^\eta, h^\eta\big)\big| \leqslant C_{11}\eta^{1/2} \tag{2.27}$$

The constant $C_{11} > 0$ depends on C_2, C_{10} and does not depend on η and k_0^η. Thus, (2.13), (2.15) and (2.27) give

$$\phi_2 - \eta\big\langle Ma_t, h^\eta\big\rangle_{1,M} - \big(F, h^\eta\big) + \big(gw^\eta, h^\eta\big) \geqslant \underline{\phi}_2 - C_{12}\eta^{1/2}. \tag{2.28}$$

Here the positive constant C_{12} depends on C_{11}, $\|g\|_{C(\overline{Q}_T)}$, $\max_{t\in[0,T]}\|a\|$ and does not depend on η and k_0^η. If we choose $\eta_0 < (\underline{\phi}_2 C_{12}^{-1})^2$, then $\underline{\phi}_2 - C_{12}\eta^{1/2} > 0$. Furthermore, by (2.10),(2.26),

$$\frac{1}{\eta}\big(w^\eta, h^\eta\big) \leqslant \Big|\int_{\partial\Omega}\frac{\partial w^\eta}{\partial\nu}\,\omega\,\mathrm{d}s\Big| + \big|\big(Mw^\eta, h^\eta\big)\big| \leqslant \frac{C_{13}}{(k_0^\eta)^{1/4}} \tag{2.29}$$

where $C_{13} =$ const > 0 depends on C_4, C_5, C_{10}, η_0, m_i, $i = 1, 2, 3$, and does not depend on η and k_0^η. Thus, by (2.12), (2.21), (2.22), (2.28), (2.29), we have

$$k_0^\eta \geqslant C_{14}\,(k_0^\eta)^{1/4}\big[\alpha_2(k_0^\eta)^{1/4} + C_{13}\big]^{-1},$$

whence

$$\alpha_2 k_0^\eta + C_{13}(k_0^\eta)^{3/4} - C_{14} \geqslant 0. \tag{2.30}$$

Here $C_{14} = \underline{\phi}_2 - C_{12}\eta^{1/2}$, Since there exists a unique positive real root y_0 of the equation

$$G(y) \equiv \alpha_2 y^4 + C_{13}y^3 - C_{14} = 0,$$

(2.30) implies $(k_0^\eta)^{3/4} \geqslant y_0 > 0$. From the obvious inequality

$$G(y) \leqslant \begin{cases} (\alpha_2 + C_{13})y^3 - C_{14}, & 0 \leqslant y < 1, \\ (\alpha_2 + C_{13})y^4 - C_{14}, & 1 \leqslant y < +\infty, \end{cases}$$

we conclude that $y_0 \geqslant y^*$ if

$$y^* = C_{14}^{1/3}(\alpha_2 + C_{13})^{-1/3} < 1;$$

otherwise, $y_0 \geqslant (y^*)^{3/4}$. Thus we get

$$k_0^\eta \geqslant \min\left\{y^*, (y^*)^{4/3}\right\} \equiv r(\eta) > 0. \tag{2.31}$$

It is clear that $r(\eta)$ is continuous function of η on $[0, \eta_0]$ and $r(0) > 0$.

Now the uniform estimates of u^η and Mu^η become evident. By (2.26) and (2.31),

$$\|u^\eta\|_1^2 + \eta\|Mu^\eta\|^2 + r(\eta)\int_0^t \|Mu^\eta\|^2 \mathrm{d}\tau \leqslant C_{15}(r(\eta))^{-1/2} + C_{16}. \tag{2.32}$$

The constants C_{15}, C_{16} depends on C_{10}, $\|a\|_{C([0,T];W_2^1(\Omega))}$ and does not depend on η. The uniform estimate of u_t^η can be derived from (2.17)–(2.19), (2.21), (2.31) and (2.32). Multiplying (2.17) by Mw_t^η in terms of the inner product of $L^2(\Omega)$ and integrating by parts in the resulting equation we obtain

$$\begin{aligned} \frac{1}{k^\eta(t)}\|w_t^\eta\|_{1,M}^2 &+ \frac{\eta}{k^\eta(t)}\|Mw_t^\eta\|^2 + \frac{1}{2}\frac{d}{dt}\|Mw^\eta\|^2 \\ &= -\frac{1}{k^\eta(t)}\int_{\partial\Omega} F\,\frac{\partial w_t^\eta}{\partial\nu}\,\mathrm{d}s - \frac{1}{k^\eta(t)}\langle F + gw^\eta,\, Mw_t^\eta\rangle_{1,M}. \end{aligned} \tag{2.33}$$

By the smoothness of f, the embedding theorem and (2.10)?

$$\left|\int_{\partial\Omega} F\,\frac{\partial w_t^\eta}{\partial\nu}\,\mathrm{d}s\right| \leqslant C_{17}\big(\|f\|_{C(\overline{\Omega})} + \|a\|_{w_2^2(\Omega)}\big)\ \|w_t^\eta\|_1 \tag{2.34}$$

The constant C_{17} depends on C_4, n and mesΩ. Therefore, taking into account (2.31), (2.32), (2.34) we can readily derive the estimate

$$\int_0^t \|w_\tau^\eta\|_{1,M}^2\ \mathrm{d}\tau + \eta\int_0^t \|Mw_\tau^\eta\|^2\ \mathrm{d}\tau + \|Mw^\eta\|^2 \leqslant \frac{C_{18}}{(r(\eta))^{1/2}} \tag{2.35}$$

from (2.33). Here the constant C_{18} depends on k_1, c, $\|Mu_0\|$, $\|F\|_{C(\overline{Q}_T)}$ and does not depend on η. (2.17), (2.19) and (2.35) lead to the estimate

$$\|w_t\|^2 + \eta\|w_t\|_{1,M}^2 \leqslant \frac{C_{19}}{(r(\eta))^{1/2}} + C_{20} \tag{2.36}$$

where the constants C_{19} and C_{20} depend on C_{18}, k_2, $\|F\|_{C([0,T];L^2(\Omega))}$, $\|g\|_{C(\overline{Q}_T)}$ and does not depend on η. Thus, from (2.14), (2.32), (2.35), (2.36) it follows that there exists a subsequence (u^{η_l}, k^{η_l}) of (u^η, k^η) and a pair of functions (u, k) such that

$$u^{\eta_l} \to u \quad * - \text{weakly} \quad \text{in } L^\infty(0,T;W_2^2(\Omega)), \tag{2.37}$$

$$u_t^{\eta_l} \to u_t \quad * - \text{weakly} \quad \text{in } L^\infty(0,T;L^2(\Omega)) \ \text{ and } \ \text{weakly} \ \text{ in } L^2(0,T;W_2^1(\Omega)), \tag{2.38}$$

$$k^{\eta_l} \to k \quad * - \text{weakly} \quad \text{in } L^\infty(0,T) \tag{2.39}$$

as $\eta_l \to 0$. By the compactness theorem [14], (2.37)–(2.39) implies

$$u^{\eta_l} \quad \to \quad u \qquad \text{in} \quad L^4(0,T;W_2^1(\Omega)), \tag{2.40}$$

$$k^{\eta_l} \quad \to \quad k \qquad \text{weakly} \quad \text{in} \quad L^4(0,T) \quad \text{as } \eta_l \to 0. \tag{2.41}$$

We are now in a position to show that the pair (u,k) is a solution of Problem 2. In fact, the pair (u^{η_l},k^{η_l}) satisfies the identity

$$\int_0^T \{(u_t^{\eta_l} + gu^{\eta_l}, v) + \eta_l\langle Mu_t^{\eta_l}, v\rangle_1 + k^{\eta_l}\langle Mu^{\eta_l}, v\rangle_1\}\mathrm{d}t \;=\; \int_0^T (f,v)\,\mathrm{d}t \tag{2.42}$$

for every $v \in L^2(0,T;\overset{\circ}{W}{}_2^1(\Omega))$. In view of (2.37)–(2.41) we can pass to the limit in (2.42). Since

$$\eta_l \int_0^T \langle Mu_t^{\eta_l}, v\rangle_1 \,\mathrm{d}t \;\to\; 0$$

as $\eta_l \to 0$ (because of (2.36)), we have

$$\int_0^T \Big\{(u_t, v) \;+\; k(t)\,\langle Mu, v\rangle_1 \;+\; (gu, v)\Big\}\,\mathrm{d}t \;=\; \int_0^T (f,v)\,\mathrm{d}t \tag{2.43}$$

for every $v \in L^2(0,T;\overset{\circ}{W}{}_2^1(\Omega))$. Moreover, by (1.12), (2.14), (2.16), (2.31), (2.32), (2.35) and (2.36), the estimates

$$r(0) \;\leqslant\; k(t) \;\leqslant\; \overline{\phi}_2\alpha_2^{-1}, \tag{2.44}$$

$$0 \;\leqslant\; u(t,x) \;\leqslant\; a(t,x), \tag{2.45}$$

$$\int_0^T \|u_t\|_1^2\,\mathrm{d}\tau + \|Mu\|^2_{L^\infty(0,T;L^2(\Omega))} \;\leqslant\; \frac{C_{18}}{m_1(r(0))^{3/2}} \;+ \int_0^T \|a_t\|_1^2\,\mathrm{d}\tau, \tag{2.46}$$

$$\|u_t\|_{L^\infty(0,T;L^2(\Omega))} \leqslant \Big(\frac{C_{19}}{(r(0))^{1/2}} \;+ C_{20} + \|a_t\|^2_{L^\infty(0,T;L^2(\Omega))}\Big)^{1/2} \tag{2.47}$$

are valid. From (2.43)–(2.47) it follows that the pair (u,k) satisfies equation (2.1) for almost all $(t,x) \in Q_T$. Furthermore, by (1.6), (2.5), (2.37), (2.38) $u(t,x)$ obeys (2.2), (2.3).

It remains to prove that the condition (2.4) is also fulfilled. Let $v(t,x) = \bar{v}(t,x)h(t)$ where $\bar{v}(t,x)$ and $h(t)$ are arbitrary functions of classes $L^\infty(0,T;W_2^1(\Omega))$ and $L^2(0,T)$, respectively, $\bar{v}\big|_{\partial\Omega} = \omega$. Then the identity

$$\int_0^T \Big\{(u_t^{\eta_l} + gu^{\eta_l}, \bar{v}) + \eta_l\langle Mu_t^{\eta_l}, \bar{v}\rangle_{1,M} \;+\; k^{\eta_l}(t)\big(\langle Mu^{\eta_l}, \bar{v}\rangle_{1,M} + \phi_1(t)\big)\Big\}h\,\mathrm{d}t$$

$$= \int_0^T ((f,\bar{v}) + \phi_2)h\,\mathrm{d}t \tag{2.48}$$

holds because of (1.7). A passage to the limit in (2.48) similar to the above yields

$$\int_0^T \Big\{\big(u_t + gu, \bar{v}\big)_0 + k(t)\big(\langle Mu, \bar{v}\rangle_{1,M} + \phi_1\big)\Big\}h\,\mathrm{d}t = \int_0^T ((f,\bar{v}) + \phi_2)h\,\mathrm{d}t. \tag{2.49}$$

By virtue of (2.1), integrating by parts in the second term of the left-hand side of (2.49) gives

$$\int_0^T \Big\{k(t)\int_{\partial\Omega} \frac{\partial u}{\partial\nu}\omega\,\mathrm{d}s + \phi_1(t)k(t) - \phi_2(t)\Big\}\,h(t)\,\mathrm{d}t \;=\; 0$$

for any $h(t) \in L^2(0,T)$, which implies that the pair (u,k) satisfies (2.4) for almost all $t \in (0,T)$. The theorem is proved. $\square$

Under the hypotheses of Theorem 2.2 the solution to Problem 2 is unique in the class $V \times L^\infty(0,T)$.

Theorem 2.3. *Let the conditions of Theorem* 2.2 *be fulfilled. Then Problem* 2 *has a unique solution* $(u(t,x), k(t))$. *The pair* $(u(t,x), k(t))$ *satisfies the estimates* (2.44)–(2.47) *and* $u_t \in L^2(0,T;W_2^1(\Omega))$.

Proof. The existence of the solution to Problem 2 and the estimates (2.44)–2.47) were proved in Theorem 2.2. It remains to establish the uniqueness.

Let $(u_1(t,x),\, k_1(t))$ and $(u_2(t,x),\, k_2(t))$ be two solutions of Problem 2. Then the pair $(w(t,x),\, p(t)) = (u_1 - u_2,\; k_1(t) - k_2(t))$ solutions the problem

$$w_t - k_1(t)Mw = -p(t)\, Mu_2, \qquad (t,x) \in Q_T, \tag{2.50}$$

$$w\big|_{t=0} = w\big|_{\partial\Omega} = 0, \tag{2.51}$$

$$k_1(t) \int_{\partial\Omega} \frac{\partial w}{\partial \nu}\, \omega \, \mathrm{d}s = -\frac{\phi_2(t)}{k_2(t)}\, p(t), \qquad t \in [0,T]. \tag{2.52}$$

Multiplying (2.50) by Mw in terms of the inner product of $L^2(\Omega)$ and integrating by parts, we can easily obtain

$$\frac{1}{2}\frac{d}{dt}\|w\|_{1,M}^2 + k_1(t)\|Mw\|^2 \leqslant |p(t)|\; \|Mu_2\|\; \|Mw\|. \tag{2.53}$$

From (2.6) with $q = 2$, (2.14), (2.52) and the Young inequality it follows that

$$\begin{aligned} |p(t)|\; \|Mu_2\|\, \|Mw\| &\leqslant C_{21}\, \|Mu_2\|\; \|w\|_{1,M}^{1/2}\; \|Mw\|^{3/2} \\ &\leqslant \frac{r(0)}{2}\, \|Mw\|^2 + \frac{C_{21}^4}{2r^3(0)}\, \|Mu_2\|^4\; \|w\|_{1,M}^2 \end{aligned} \tag{2.54}$$

where $C_{21} = \text{const} > 0$ depends on C_2, $\underline{\phi_2}$, $r(0)$, α_3, m_i, $i = 1,2,3$. Since $u_2 \in L^\infty(0,T;W_2^2(\Omega))$, according to Gronwall's lemma, (2.51),(2.53) and (2.54) implies that $w = 0$ for almost all $(t,x) \in Q_T$ and $p = 0$ for almost all $t \in (0,T)$. The theorem is proved. $\square$

Conclusions

In this paper we discussed the behaivior of the solution to the Problem 1 as $\eta \to 0$. It was established that Problem 1 for the pseudoparabolic equation approximates weakly Problem 2 for the parabolic one under the hypotheses of Theorem 2.2 when $\eta \to 0$. Theorems 1.1 and 2.2 remains true if $\omega \in C([0,T]; W_2^{3/2}(\partial\Omega))$ and $\varphi_1 \in C([0,T])$.

In general Problem 1 does not approximate Problem 2. As was shown in [1], if the initial and boundary data do not satisfy (2.11), then Problem 1 may be unsolvable.

Theorem 2.2 implies that Problem 2 for the relevant parabolic equation is solved relying on the results on Problem 1. The uniqueness of the solution to Problem 2 is provided by Theorem 2.3.

References

[1] A.Sh.Lyubanova, A.Tani On Inverse Problems for Pseudoparabolic and Parabolic Equations of Filtration, *Inverse Problems in science and enjineering*, **19**(2011), 1023–1042.

[2] W.Rundell, Determination of an unknown nonhomogeneous term in a linear partial differential equation from overspecified boundary data, *Appl. Anal.*, **10**(1980), 231–242.

[3] A.A.Asanov, E.R.Atamanov, An inverse problem for an operator integro-differential pseudoparabolic equation, *Siberian Math. J.*, **36**(1995), 752–762 (Russian).

[4] A.Lorenzi, E.Paparoni, Identification problems for pseudoparabolic integrodifferential operator equations, *J. Inver. Ill-Posed Probl.*, **5**(1997), 235–253.

[5] M.Sh.Mamayusupov, The problem of determining coefficients of a pseudoparabolic equation, *Studies in integro-differential equations*, "Ilim", Frunze, 1983, no. 16, 290–297. (Russian)

[6] A.Sh.Lyubanova, A.Tani An inverse problem for pseudoparabolic equation of filtration. The existence, uniqueness and regularity, *Appl. Anal.*, **90**(2011), 1557–1571.

[7] G.I.Barenblatt, Iu.P.Zheltov, I.N.Kochina, Basic concepts in the theory of seepage of homogeneous liquids in fissured blocks [strata], *J. Appl. Math. Mech.*, **24**(1960), 1286–1303 (Russian).

[8] T.W.Ting, Parabolic and pseudo-parabolic partial differential equations, *J. Math. Soc. Japan.* **21**(1969), 440–453

[9] J.R.Cannon, W.Rundell, Recovering a time dependent coefficient in a parabolic differential equation, *J. Math. Anal. Appl.* **160** (1991), 572–582.

[10] P.DuChateau, Monotonicity and invertibility of coefficient-to-data mappings for parabolic inverse problems, *SIAM J. Math. Anal.* **26** (1995), 1473–1487.

[11] A.Hasanov, P.DuChateau, B.Pektas, An adjoint problem approach and coarse-fine mesh method for identification of the diffusion coefficient in a linear parabolic equation, *Journal of Inverse and Ill-Posed Problems*, **14**(2006), 435–463.

[12] M.V.Klibanov, Inverse problems and Carleman estimates, *Inverse Problems*, **8**(1992), 575–596.

[13] O.A.Ladyzhenskaya, V.A.Solonnikov, N.N.Ural'ceva, *Linear and Quasilinear Equations of Parabolic Type*, Nauka, Moskow, 1967 (Russian).

[14] R.Temam, *Navier-Stokes Equations. Theory and numerical analysis*, Studies in Mathematics and its Applications, vol. 2, North-Holland Publishing Company, Amsterdam, 1974.

Об аппроксимации параболической обратной задачи псевдопараболической задачей

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Исследуется обратная задача идентификации одного из старших коэффициентов псевдопараболического уравнения. Доказывается, что обратная задача для псевдопараболического уравнения аппроксимирует соответствующую обратную задачу для параболического уравнения. Устанавливается также существование и единственность решения параболической обратной задачи.

Ключевые слова: фильтрация, обратные задачи для уравнений в частных производных, псевдопараболическое уравнение, параболическое уравнение, теоремы существования и единственности.