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Kostant Partition Function for $\mathfrak{sp}_4(\mathbb{C})$

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In this note, we obtain exact values of the partition function of Kostant for the simple Lie algebra $\mathfrak{sp}_4(\mathbb{C})$. Using the values of the partition function, we can find the weight multiplicities of irreducible representations of $\mathfrak{sp}_4(\mathbb{C})$ by a simple computation.

Keywords: symplectic Lie algebra, Kostant partition function, Weyl group, weight multiplicity.

Introduction

Let L be a finite dimensional complex semi-simple Lie algebra with a Cartan subalgebra H and root system Φ and suppose Φ^+ denotes the set of positive roots. Recall that the Weyl vector is defined by

$$\rho = \sum_{\alpha\in\Phi^+} \alpha.$$

Suppose λ is an integral dominant weight of L and $V(\lambda)$ is the corresponding irreducible L-module. For any other integral dominant weight μ, we define the multiplicity of μ in λ to be the dimension of

$$V(\lambda)_\mu = \{v \in V(\lambda) : \forall h \in H \quad h.v = \mu(h)v\}.$$

There exists a compact formula for computing weight multiplicities, known as *Kostant's multiplicity formula.* It can be stated as follows,

$$\dim V(\lambda)_\mu = \sum_{\omega\in\mathcal{W}} \epsilon(\omega)\mathfrak{P}(\omega(\lambda+\rho) - (\mu+\rho)).$$

Here $\mathcal{W}$ is the Weyl group of L, $\epsilon(\omega)$ is the sign of ω, and $\mathfrak{P}$ is the *Kostant partition function.* By definition, for any weight γ, $\mathfrak{P}(\gamma)$ is the number of ways to write γ as a linear combination of positive roots with non-negative coefficients, (see [1] for details). Although the Kostant partition function is a well-known classical notion in Lie algebra, an explicit expressions for it, might not be easy to find in the literature. In this note, we give an explicit formula for the values of the partition function, in the case $L = \mathfrak{sp}_4(\mathbb{C})$.

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1. Generalities

In this elementary section, we give a review of theories concerning the Lie algebra $\mathfrak{sp}_4(\mathbb{C})$. We also fix a set of notations we will use in the next sections.

The symplectic Lie algebra, $\mathfrak{sp}_4(\mathbb{C})$, is a 10-dimensional simple Lie algebra defined by

$$\mathfrak{sp}_4(\mathbb{C}) = \{x \in Mat_4(\mathbb{C}) : sx = -x^T s\},$$

where

$$s = \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix}$$

A Cartan subalgebra for $\mathfrak{sp}_4(\mathbb{C})$ is H which consists of diagonal matrices

$$h = diag(a_1, a_2, -a_1, -a_2),$$

where $a_1, a_2 \in \mathbb{C}$. For $i = 1, 2$, define a functional $\mu_i : H \to \mathbb{C}$ by $\mu_i(h) = a_i$. So the set

$$\Phi = \{\pm\mu_i \pm \mu_j : 1 \leqslant i, j \leqslant 2\} - \{0\}$$

is a root system for $\mathfrak{sp}_4(\mathbb{C})$. Also, the set

$$\Pi = \{R_1 = \mu_1 - \mu_2, R_2 = 2\mu_2\}$$

is a basis for Φ. Finally

$$\lambda_1 = \mu_1, \quad \lambda_2 = \mu_1 + \mu_2$$

are fundamental weights of $\mathfrak{sp}_4(\mathbb{C})$. Note that, we also have the following simple relations;

$$\begin{aligned} \mu_1 &= \lambda_1 \\ \mu_2 &= -\lambda_1 + \lambda_2 \\ R_1 &= 2\lambda_1 - \lambda_2 \\ R_2 &= -2\lambda_1 + 2\lambda_2 \\ \lambda_1 &= R_1 + \frac{1}{2}R_2 \\ \lambda_2 &= R_1 + R_2. \end{aligned}$$

We denote the Weyl group by $\mathcal{W}$. It is generated by the reflection σ_α, $\alpha \in \Pi$, where

$$\sigma_\alpha(\beta) = \beta - \frac{2(\beta, \alpha)}{(\alpha, \alpha)}\alpha,$$

for $\beta \in \Phi$. Let $\sigma_1 = \sigma_{R_1}$ and $\sigma_2 = \sigma_{R_2}$. In the following table, we give the elements od $\mathcal{W}$ by their actions on the elements μ_1 and μ_2, their expressions as products of σ_1 and σ_2, and their sign, $\epsilon(\omega)$.

The Weyl group of $\mathfrak{sp}_4(\mathbb{C})$

ω	$\omega(\mu_1)$	$\omega(\mu_2)$	presentation	$\epsilon(\omega)$
ω_1	μ_1	μ_2	1	1
ω_2	μ_2	μ_1	σ_1	-1
ω_3	μ_1	$-\mu_2$	σ_2	-1
ω_4	μ_2	$-\mu_1$	$\sigma_1\sigma_2$	1
ω_5	$-\mu_2$	μ_1	$\sigma_2\sigma_1$	1
ω_6	$-\mu_1$	μ_2	$\sigma_1\sigma_2\sigma_1$	-1
ω_7	$-\mu_2$	$-\mu_1$	$\sigma_2\sigma_1\sigma_2$	-1
ω_8	$-\mu_1$	$-\mu_2$	$(\sigma_1\sigma_2)^2$	1

Let $\lambda = a\lambda_1 + b\lambda_2$ be an integral dominant weight of $\mathfrak{sp}_4(\mathbb{C})$. We denote the corresponding $\mathfrak{sp}_4(\mathbb{C})$-module by $V(\lambda)$. Using Weyl's dimension formula, (see [1], page 267), we have

$$\dim V(\lambda) = \frac{1}{6}(a+1)(b+1)(a+b+2)(a+2b+3).$$

Equivalently, if $\lambda = p\mu_1 + q\mu_2$, we have

$$\dim V(\lambda) = \frac{1}{6}(p+2)(q+1)(p+q+3)(p-q+1).$$

Now, we are going to compute the weight multiplicities for $\mathfrak{sp}_4(\mathbb{C})$. These are very important for decomposition of a $\mathfrak{sp}_4(\mathbb{C})$-module in to the direct sum of irreducible constituents. For this purpose, we use the Kostant multiplicity formula;

$$\dim V(\lambda)_\mu = \sum_{\omega\in\mathcal{W}} \epsilon(\omega)\mathfrak{P}(\omega(\lambda+\rho) - (\mu+\rho)),$$

where $\rho = \lambda_1 + \lambda_2$ and $\mathfrak{P}$ is the partition function, i.e. $\mathfrak{P}(\mu)$ is the number of ways to write μ as a linear combination of positive roots with non-negative integer coefficients.

Suppose $\lambda = p\mu_1 + q\mu_2$ and $\mu = r\mu_1 + s\mu_2$. Since we have $\rho = 2\mu_1 + \mu_2$, so using the above table,

$$\begin{aligned}
\dim V(\lambda)_\mu &= \sum_{i=1}^{8} \epsilon(\omega_i)\mathfrak{P}(\omega_i(\lambda+\rho) - (\mu-\rho)) \\
&= \sum_{i=1}^{8} \epsilon(\omega_i)\mathfrak{P}(\omega_i((p+2)\mu_1 + (q+1)\mu_2) \\
&\quad -((r+2)\mu_1 + (s+1)\mu_2)) \\
&= \ \mathfrak{P}((p-r)\mu_1 + (q-s)\mu_2) \\
&\quad -\mathfrak{P}((q-r-1)\mu_1 + (p-s+1)\mu_2) \\
&\quad -\mathfrak{P}((p-r)\mu_1 - (q+s+2)\mu_2) \\
&\quad +\mathfrak{P}((q-r-1)\mu_1 + (p+s+3)\mu_2).
\end{aligned}$$

It is enough to know the values of $\mathfrak{P}(\gamma)$ for the following cases,

Case 1: $\gamma = i\mu_1 + j\mu_2$ such that $i, j \geqslant 0$ and $i + j$ is even.

Case 2: $\gamma = i\mu_1 - j\mu_2$ such that $i \geqslant j > 0$ and $i + j$ is even.

2. Values of the Partition Function

In what follows, we obtain the exact values of $\mathfrak{P}(\gamma)$. We know that the positive roots of $\mathfrak{sp}_4(\mathbb{C})$ are

$$\beta_1 = \mu_1 - \mu_2,\ \beta_2 = \mu_1 + \mu_2,\ \beta_3 = 2\mu_1,\ \beta_4 = 2\mu_2.$$

Now, we have

$$\mathfrak{P}(\gamma) = |\{(r_1, r_2, r_3, r_4) : r_i \in \mathbb{Z}, r_i \geqslant 0, \gamma = \sum_{i=1}^{4} r_i\beta_i\}|.$$

Let $\gamma = i\mu_1 + j\mu_2$. If it is possible to write γ as non-negative integer linear combination of β_is, then we have

$$\begin{aligned}
i\mu_1 + j\mu_2 &= r_1\beta_1 + r_2\beta_2 + r_3\beta_3 + r_4\beta_4 \\
&= (r_1 + r_2 + 2r_3)\mu_1 + (-r_1 + r_2 + 2r_4)\mu_2,
\end{aligned}$$

so we must have

$$i = r_1 + r_2 + 2r_3, \quad j = -r_1 + r_2 + 2r_4.$$

Hence we obtain the the following results;

1. If $i + j$ is odd, then $\mathfrak{P}(\gamma) = 0$.
2. If $i < 0$, then $\mathfrak{P}(\gamma) = 0$.
3. If $-j > i \geqslant 0$, then $\mathfrak{P}(\gamma) = 0$.

It is enough to know the values of $\mathfrak{P}(\mu)$ for the following cases,

Case 1: $\gamma = i\mu_1 + j\mu_2$ such that $i, j \geqslant 0$ and $i + j$ is even.

Case 2: $\gamma = i\mu_1 - j\mu_2$ such that $i \geqslant j > 0$ and $i + j$ is even.

Let $\gamma = i\mu_1 + j\mu_2$ such that $i, j \geqslant 0$ and $i+j$ is even. Then $\mathfrak{P}(\gamma)$ is the number of non-negative integer solutions of

$$\begin{array}{rcl} r_1 + r_2 + 2r_3 & = & i \\ -r_1 + r_2 + 2r_4 & = & j \end{array}$$

Adding up two equations, we obtain

$$r_2 + r_3 + r_4 = \frac{i + j}{2}.$$

Also, subtraction gives

$$r_1 = \frac{i - j}{2} + r_4 - r_3.$$

So, $\mathfrak{P}(\gamma)$ is the number of integer solutions of the following system;

$$\begin{array}{rcl} x + y + z & = & \frac{i+j}{2} \\ x + 2y & \leqslant & i \\ x + 2z & \geqslant & j \\ x, y, z & \geqslant & 0 \end{array}$$

We consider two cases: $j < i$ and $i \leqslant j$. In the first case, we see that $\mathfrak{P}(\gamma)$ is the number of solutions of the system;

$$\begin{array}{rcl} x + y & \leqslant & \frac{i+j}{2} \\ x + 2y & \leqslant & i \\ x, y & \geqslant & 0 \end{array}$$

Lemma 2.1. *The number of integer solutions of the system*

$$\begin{array}{rcl} x + y & \leqslant & k \\ x, y & \geqslant & 0 \end{array}$$

is equal to

$$n = \frac{(k+1)(k+2)}{2}.$$

Proof. For $0 \leqslant l \leqslant k$, let

$$A_l = \{(x,l) : x \geqslant 0,\ x+l \leqslant k\}.$$

So, we have $|A_l| = k-l+1$. Thus

$$n = \sum_{l=1}^{k} |A_l| = \frac{(k+1)(k+2)}{2}.$$

□

Corollary 2.1. *The number of the integer solutions of the system*

$$\begin{array}{rcl} x+y & \leqslant & k \\ 0 \leqslant y & \leqslant & l \\ x & \geqslant & 0 \end{array}$$

is equal to

$$\frac{(l+1)(2k+2-l)}{2}.$$

Lemma 2.2. *The number of the integer solutions of the system*

$$\begin{array}{rcl} x+2y & \leqslant & k \\ x,y & \geqslant & 0 \end{array}$$

is equal to

$$n = \frac{(k+2)^2}{4},$$

if k is even, otherwise

$$n = \frac{(k+1)(k+3)}{4}.$$

Proof. For $0 \leqslant l \leqslant k$, suppose

$$A_l = \{(x,l) : x \geqslant 0, x+2l \leqslant k\}.$$

Then, we have

$$|A_l| = k-2l+1,$$

and hence, for even k

$$n = 1+3+\cdots+(k+1) = \frac{(k+2)^2}{4},$$

and for odd k, we have

$$n = 2+4+\cdots+(k+1) = \frac{(k+1)(k+3)}{4}.$$

□

Corollary 2.2. *Suppose n is the number of integer solutions of the system*

$$\begin{array}{rcl} x+2y & \leqslant & k \\ 0 \leqslant l & \leqslant & y \\ x & \geqslant & 0. \end{array}$$

If k is even, then we have

$$n = \frac{(k-2l+2)^2}{4},$$

and otherwise

$$n = \frac{(k-2l+1)(k-2l+3)}{4}.$$

We return to the system

$$\begin{aligned} x+y &\leqslant \frac{i+j}{2} \\ x+2y &\leqslant i \\ x,y &\geqslant 0 \end{aligned}$$

where $j \leqslant i$ and $i+j$ is even. One can split this system into two complementary systems as follows,

$$\begin{aligned} x+y &\leqslant \frac{i+j}{2} \\ 0 \leqslant y &\leqslant \frac{i-j}{2} \\ x &\geqslant 0 \end{aligned}$$

and

$$\begin{aligned} x+2y &\leqslant i \\ y &\geqslant \frac{i-j}{2}+1 \\ x &\geqslant 0. \end{aligned}$$

So, using Lemmas 3-2 and 3-4, we obtain

Corollary 2.3. *Let $\gamma = i\mu_1 + j\mu_2$ such that $0 \leqslant j < i$ and $i+j$ is even. If j is even, then*

$$\mathfrak{P}(\gamma) = \frac{(i-j+2)(i+3j+4)+2j^2}{8},$$

and if j is odd, then we have

$$\mathfrak{P}(\gamma) = \frac{(i-j+2)(i+3j+4)+2(j^2-1)}{8}.$$

Now, suppose we have $i \leqslant j$. It is easy to see that in this case, $\mathfrak{P}(\gamma)$ is the number of solutions of the system

$$\begin{aligned} x+2y &\leqslant i \\ x,y &\geqslant 0. \end{aligned}$$

Hence, we obtain;

Corollary 2.4. *Let $\gamma = i\mu_1 + j\mu_2$ such that $0 \leqslant i \leqslant j$ and $i+j$ is even. If i is even, then*

$$\mathfrak{P}(\gamma) = \frac{(i+2)^2}{4},$$

while in the other case

$$\mathfrak{P}(\gamma) = \frac{(i+1)(i+3)}{4}.$$

Finally, we consider the case $\gamma = i\mu_1 - j\mu_2$, with $i + j$ even and $0 < j \leqslant i$. Again, $\mathfrak{P}(\gamma)$ is the number of non-negative integer solutions of the system

$$\begin{aligned} r_1 + r_2 + 2r_3 &= i \\ -r_1 + r_2 + 2r_4 &= -j \end{aligned}$$

Adding up two equations, we obtain

$$r_2 + r_3 + r_4 = \frac{i-j}{2}.$$

Also, subtraction gives

$$r_1 = \frac{i+j}{2} + r_4 - r_3.$$

We see that if (r_2, r_3, r_4) satisfies

$$\begin{aligned} r_2 + r_3 + r_4 &= \frac{i-j}{2} \\ r_2 &\geqslant 0 \\ r_3 &\geqslant 0 \\ r_4 &\geqslant 0 \end{aligned}$$

then (r_1, r_2, r_3, r_4) is a non-negative integral solution to the above system, and moreover every solution can be obtain by this method. Hence the required number is just the number of solutions of the simple sysytem

$$\begin{aligned} x + y &\leqslant \frac{i-j}{2} \\ x, y &\geqslant 0 \end{aligned}$$

and so we have

Corollary 2.5. *Suppose $\gamma = i\mu_1 - j\mu_2$, with $i + j$ even and $0 < j \leqslant i$.Then we have*

$$\mathfrak{P}(\gamma) = \frac{(i-j+2)(i-j+4)}{8}.$$

References

[1] R.W.Carter, Lie algebras of finite and affine type, Cambridge University Press, 2005.

Функция разбиения Костанта для $\mathfrak{sp}_4(\mathbb{C})$

Хасан Рефагат
Мохаммад Шахрири

В этой статье получены точные значения функции разбиения Костанта для простой алгебры Ли $\mathfrak{sp}_4(\mathbb{C})$. Используя значения функции разбиения простыми вычислениями найдены весовые кратности неприводимых представлений $\mathfrak{sp}_4(\mathbb{C})$.

Ключевые слова: симплектическая алгебра Ли, функция разбиения Костанта, группа Вейля, весовая кратность.