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Localization of Solutions of the Equations of Filtration in Poroelastic Medium

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A system of equations of 1D non-stationary fluid motion in poroelastic medium is considered. Localization of solutions of the equations has been established by the integral energy estimates method.

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1. Problem Statement. The Main Results

A quasi-linear system of equations of composite type is considered [1–3]:

$$\frac{\partial(1-\phi)\rho_s}{\partial t}+\frac{\partial}{\partial x}((1-\phi)\rho_s v_s)=0,\quad \frac{\partial(\rho_f\phi)}{\partial t}+\frac{\partial}{\partial x}(\rho_f\phi v_f)=0,$$

$$\phi(v_f-v_s)=-k(\phi)\left(\frac{\partial p_f}{\partial x}-\rho_f g\right),$$

$$\frac{\partial v_s}{\partial x}=-\beta_t(\phi)\left(\frac{\partial p_e}{\partial t}+v_s\frac{\partial p_e}{\partial x}\right),$$

$$\frac{\partial p_{tot}}{\partial x}=-\rho_{tot}g,$$

$$p_{tot}=\phi p_f+(1-\phi)p_s;\quad p_e=p_{tot}-p_f;\quad \rho_{tot}=(1-\phi)\rho_s+\phi\rho_f.$$

This quasi-linear system of equations describes 1D non-stationary isothermal motion of fluid in poroelastic medium. The laws of conservation of mass for each phase, Darcy's law for fluid phase, the rheological Maxwell law and the equation of conservation of momentum for the system describe this process. Here ρ_s, ρ_f, v_s, v_f are, respectively, real density and velocity of solid and fluid phases, ϕ is the porosity, p_f, p_s are, respectively, pressures of the fluid and solid phases; p_e is the effective pressure, p_{tot} is the total pressure, ρ_{tot} is the density of the two-phase medium, g is the density of the mass forces, $k(\phi)$ is the coefficient of filtration, $\beta_t(\phi)$ is the coefficient of bulk compressibility (specified function). The problem is written in the Eulerian coordinates x, t. The real density of the fluid and solid particles ρ_f, ρ_s are assumed constant. The unknown quantities are ϕ, v_s, v_f, p_f, p_s.

Local (with respect to time) solvability of the initial-boundary value problem for the system of equations under consideration has been established in [4], a self-similar solution has been

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found in [5]. Numerical results for this system of equations are given in [1, 2]. In these studies we use Euler variables, additional assumptions about smallness of the speed of solid phase, and the following relations between the functional parameters of the problem: $k(\phi) = \dfrac{k}{\mu}\phi^n$, $\beta_t(\phi) = \beta_\phi \phi^b$, where n, b are positive environment parameters. In this paper a complete system of equations of filtration in a deformable medium is considered. This system of equations can be reduced to a degenerate parabolic equation using transition to Lagrange variables with respect to the speed of the solid phase. To this equation we apply the well-known technique for proving finiteness of the propagation speed of disturbances.

Rewrite the original system in Lagrange variables, following [6]. Suppose that $\bar{x} = \bar{x}(\tau, x, t)$ is a solution of the Cauchy problem

$$\frac{\partial \bar{x}}{\partial \tau} = v_s(\bar{x}, \tau), \quad \bar{x}\,|_{\tau=t} = x.$$

We set $\hat{x} = \bar{x}(0; x, t)$ and take $\hat{x}$ and t for the new variables. Then [6] $1 - \phi(\hat{x}, t) = \left(1 - \phi^0(\hat{x})\right) \hat{J}(\hat{x}, t)$, where $\hat{J}(\hat{x}, t) = \dfrac{\partial \hat{x}}{\partial x}(\hat{x}, t)$ is the Jacobian of the transformation, $\phi^0(x) = \phi|_{t=0}$. The system of equations in the new variables has the form

$$\frac{\partial\left(1-\hat{\phi}\right)}{\partial t} + \frac{\left(1-\hat{\phi}\right)^2}{1-\phi^0}\frac{\partial \hat{v}_s}{\partial \hat{x}} = 0,$$

$$\frac{\partial \hat{\phi}}{\partial t} + \frac{\left(1-\hat{\phi}\right)}{1-\phi^0}\frac{\partial}{\partial \hat{x}}(\hat{\phi}\hat{v}_f) = v_s \frac{\left(1-\hat{\phi}\right)}{1-\phi^0}\frac{\partial \hat{\phi}}{\partial \hat{x}},$$

$$\hat{\phi}(\hat{v}_s - \hat{v}_f) = -k(\phi)\left(\frac{\left(1-\hat{\phi}\right)}{1-\phi^0}\frac{\partial \hat{p}_f}{\partial \hat{x}} - \hat{\rho}_f \hat{g}\right),$$

$$\frac{\left(1-\hat{\phi}\right)}{1-\phi^0}\frac{\partial \hat{v}_s}{\partial \hat{x}} = -\beta_t(\hat{\phi})\frac{\partial \hat{p}_e}{\partial t},$$

$$\frac{\left(1-\hat{\phi}\right)}{1-\phi^0}\frac{\partial \hat{p}_{tot}}{\partial \hat{x}} = -\hat{\rho}\hat{g}.$$

Since

$$v_s \frac{\partial \hat{\phi}}{\partial \hat{x}} = \frac{\partial}{\partial \hat{x}}(\hat{\phi} v_s) - \hat{\phi}\frac{\partial v_s}{\partial \hat{x}},$$

it follows that the continuity equation for liquid phase can be reduced to the form

$$\frac{1}{\left(1-\hat{\phi}\right)}\frac{\partial \hat{\phi}}{\partial t} + \frac{1}{1-\phi^0}\frac{\partial}{\partial \hat{x}}\left(\hat{\phi}(\hat{v}_f - v_s)\right) + \frac{1}{1-\phi^0}\hat{\phi}\frac{\partial v_s}{\partial \hat{x}} = 0.$$

Using the continuity equation for the solid phase, we find that

$$\frac{\partial}{\partial t}\left(\frac{\hat{\phi}}{1-\hat{\phi}}\right) + \frac{1}{(1-\phi^0)}\frac{\partial}{\partial \hat{x}}\left((\hat{\phi}(\hat{v}_f - \hat{v}_s)\right) = 0.$$

Finally, passing from $(\hat{x}, t)$ to the mass Lagrangian variables (y, t) by the rule

$$(1 - \phi^0(\hat{x}))d\hat{x} = dy, \quad y(\hat{x}) = \int_0^{\hat{x}} \left(1 - \phi^0(\eta)\right) d\eta \in [0, 1],$$

and formally replacing y by x, we obtain

$$\frac{\partial}{\partial t}\left(\frac{\phi}{1-\phi}\right)+\frac{\partial}{\partial x}(\phi(v_f-v_s))=0, \tag{1}$$

$$\frac{\partial(1-\phi)}{\partial t}+(1-\phi)^2\frac{\partial v_s}{\partial x}=0, \tag{2}$$

$$\phi(v_s-v_f)=k(\phi)\left((1-\phi)\frac{\partial p_f}{\partial x}-\rho_f g\right), \tag{3}$$

$$(1-\phi)\frac{\partial v_s}{\partial x}=-\beta_t(\phi)\frac{\partial p_e}{\partial t}, \tag{4}$$

$$(1-\phi)\frac{\partial p_{tot}}{\partial x}=-\rho_{tot}g. \tag{5}$$

Introduce a function $G(\phi)$ defined by the equation $\dfrac{\partial G(\phi)}{\partial \phi}=\dfrac{1}{\beta_t(\phi)(1-\phi)}$. Therefore, from (2) and (4), we obtain

$$\frac{\partial p_e}{\partial t}=-\frac{\partial G(\phi)}{\partial t},$$

and hence

$$p_e=-G(\phi)+G_0+p_e^0,\quad G_0=G(\phi^0),\quad \phi|_{t=0}=\phi^0,\quad p_e|_{t=0}=p_e^0. \tag{6}$$

Therefore, from (1) and (3), we have

$$\frac{\partial}{\partial t}\left(\frac{\phi}{1-\phi}\right)=\frac{\partial}{\partial x}\left(k(\phi)((1-\phi)\frac{\partial p_f}{\partial x}-\rho_f g)\right).$$

Taking into account the equality $p_f=p_{tot}-p_e$ and equations (5), (6) we have

$$\frac{\partial}{\partial t}\left(\frac{\phi}{1-\phi}\right)=\frac{\partial}{\partial x}\left(\frac{k(\phi)}{\beta_t(\phi)}\frac{\partial\phi}{\partial x}\right)-\frac{\partial}{\partial x}\left(k(\phi g((1-\phi)\rho_s-(1+\phi)\rho_f)\right).$$

Further on, we introduce a new function $s=\phi/(1-\phi)$ instead of $\phi\in[0,1)$, and assume [2] that

$$k(\phi)=\frac{k}{\mu}\phi^n,\quad \beta_t(\phi)=\beta_\phi\phi^b,$$

where k is the permeability, μ is the dynamic viscosity of the fluid, β_ϕ is the coefficient of bulk compressibility of solid phase, b,n are positive environment parameters (in what follows it is assumed that $0\leqslant n+b-2,\ 0<n-b\leqslant 2$). Then the equation for s can be expressed as

$$\frac{\partial s}{\partial t}=\frac{\partial}{\partial x}\left(d(s)\frac{\partial s}{\partial x}\right)+\frac{\partial f(s)}{\partial x}, \tag{7}$$

it is assumed that there is a constant $M>0$ such that we have the following estimates

$$0\leqslant s\leqslant M<\infty,\quad \frac{k}{\mu\beta_\phi}s^{n-b}(1+M)^{b-n-2}\leqslant d(s)\leqslant\frac{k}{\mu\beta_\phi}s^{n-b},\quad g\geqslant 0,$$

$$|f(s)|\leqslant\frac{k}{\mu}s^n g\left(\rho_s+(1+2M)\rho_f\right).$$

The main result of this paper can be formulated as follows: let $s(x,t)$ be a weak solution of (7) in $K_{\rho_0}(x_0)\times(0,\infty)$, $K_{\rho_0}(x_0)=\{(x,x_0):|x-x_0|<\rho_0\}$ such that $s_0(x)\equiv s(x,0)=0$

in $K_{\rho_0}(x_0)$. Then there exist $T > 0$ and $\rho(t) \in (0, \rho_0)$ such that $s(x,t) = 0$ for all $t \leqslant T$ and $x \in K_\rho(x_0)$. Under additional assumptions on the character of vanishing of $s_0(x)$ it is proved that $s(x,t) = 0$ in $K_{\rho_0}(x_0)$. Questions of the existence of the corresponding solution are not considered here. The local energy method developed in the papers [7, 8] is used for the proof .

On Ω and Q_T we consider several function spaces following the notation from [9]. Suppose that $||\cdot||_{q,\Omega}$ is the norm on the Lebesgue space $L_q(\Omega)$, $q \in [1,\infty]$. For brevity, let $||\cdot||_q = ||\cdot||_{q,\Omega}$, $||\cdot|| = ||\cdot||_{2,\Omega}$. We also use the space $\overset{o}{C}{}^\infty$ of infinitely differentiable functions with compact support in Ω, and the Sobolev spaces $W_p^l(\Omega)$, where l is a natural number and $p \in [1,\infty]$, with norms $||f||_{W_p^l(\Omega)} = \sum\limits_{m=0}^{l} ||D_x^m f||_{p,\Omega}$.

Definition 1. *By a weak solution of the equation* (7) *with initial condition* $s_0(x)$ *we mean a non-negative bounded measurable function* $s(x,t)$ $(0 \leqslant s(x,t) \leqslant M)$ *on* $\Omega \times (0,\infty)$, *if* $\forall T > 0$ *and any open subset* $\Omega_1 \subset R^1$ *the following conditions are fulfilled*

$$s \in L_\infty(0,T,W_2^1(\Omega)), \quad \frac{\partial}{\partial x}\left(s^{n-b+1}\right) \in L_2[(0,T)\times\Omega_1], \tag{8}$$

$$\lim_{t\to 0}\int_Q s dx = \int_Q s_0 dx, \tag{9}$$

and $\forall\varphi(x,t) \in \overset{o}{C}{}^\infty((0,T)\times\Omega_1)$

$$\int_0^\infty\int_\Omega\left[d(s)\frac{\partial s}{\partial x}\frac{\partial\varphi}{\partial x} - \frac{\partial f(s)}{\partial x}\varphi\right]dx\,dt = \int_0^\infty\int_\Omega s\frac{\partial\varphi}{\partial t}\,dx\,dt + \int_\Omega s(x,0)\,\varphi(x,0)\,dx. \tag{10}$$

We introduce the notation

$$A(\rho,t) \equiv \int_{K_\rho(x_0)} s^2(x,t)dx, \quad B(\rho,t) \equiv \int_{K_\rho(x_0)} s^{n-b}\left(\frac{\partial s}{\partial x}\right)^2 dx,$$

and without loss of generality we assume that $x_0 = 0$.

Lemma. *Suppose that* (8), (9) *are fulfilled. Then for* $s(\rho,t)$ *we have the estimates*

$$s^\sigma(\rho,t) \leqslant C_i A^{\frac{1-\theta}{r}}(\rho,t)[B^{\frac{1}{2}}(\rho,t) + \rho^{-\delta}A^{\frac{1}{r}}(\rho,t)]^\theta, \quad i = 1,2, \tag{11}$$

where

$$\sigma = \frac{n}{2} - \frac{b}{2} + 1 > 0, \quad \theta = \frac{2}{2+r}, \quad \delta = \frac{1}{\theta r}.$$

If $i = 1$ *then* $r \in (1,2)$, $0 < n - b < 2$,

$$C_1 = CM^{\frac{(r\sigma-2)(1-\theta)}{2}}\max(\sigma, M^{\frac{r\sigma-2}{r}}),$$

and if $i = 2$, $n - b = 2$,

$$r = \frac{4}{n-b+2} = 1, \quad C_2 = C\max(\sigma,1),$$

C *is a positive constant independent of the radius* ρ.

Proof follows [10]. For all $u \in W_q^1(K_\rho(0))$ we have the estimate [11]

$$|u(\rho)| \leqslant C \cdot \left(\|u_x\|_{q,K_\rho(0)} + \rho^{-\delta}\|u\|_{r,K_\rho(0)}\right)^\theta \|u\|_{r,K_\rho(0)}^{1-\theta}, \tag{12}$$

$$\theta = \frac{q}{q-r+qr}, \quad \delta = \frac{1}{\theta r}, \quad q \geqslant 1, \quad 1 \leqslant r \leqslant \infty.$$

Take in (12) $u = s^\sigma$ and $q = 2$, then

$$s^\sigma(\rho,t) \leqslant C \cdot \left(\sigma\left(\int_{K_\rho(0)} s^{2\sigma-2}\left(\frac{\partial s}{\partial x}\right)^2 dx\right)^{\frac{1}{2}} + \rho^{-\delta}\left(\int_{K_\rho(0)} s^{r\sigma}\,dx\right)^{\frac{1}{r}}\right)^\theta \left(\int_{K_\rho(0)} s^{r\sigma}\,dx\right)^{\frac{1-\theta}{r}}. \tag{13}$$

Let us strengthen the right-hand side of (13). If $0 < n-b < 2$, then

$$s^{r\sigma} = s^2\, s^{r\sigma-2} \leqslant M^{r\sigma-2}\, s^2, \quad r \in \left(1, \frac{4}{n-b+2}\right).$$

If $n-b=2$, then take $r = 4/(n-b+2) = 1$ in (13), and, given that $s^{r\sigma} = s^2\, s^{r\sigma-2} \leqslant s^2$, we deduce

$$s^\sigma(\rho,t) \leqslant C\, M^{\frac{(r\sigma-2)(1-\theta)}{r}} \cdot \left(\sigma\left(\int_{K_\rho(0)} s^{n-b}\left(\frac{\partial s}{\partial x}\right)^2 dx\right)^{\frac{1}{2}} + \right.$$

$$\left. + M^{\frac{r\sigma-2}{r}} \rho^{-\delta}\left(\int_{K_\rho(0)} s^2\,dx\right)^{\frac{1}{r}}\right)^\theta \left(\int_{K_\rho(0)} s^2\,dx\right)^{\frac{1-\theta}{r}},$$

if $0 < n-b < 2$, $1 < r < 2$, and if $n-b=2$, $r=1$, then

$$s^\sigma(\rho,t) \leqslant C \cdot \left(\sigma\left(\int_{K_\rho(0)} s^{n-b}\left(\frac{\partial s}{\partial x}\right)^2 dx\right)^{\frac{1}{2}} + \rho^{-\delta}\left(\int_{K_\rho(0)} s^2\,dx\right)^{\frac{1}{r}}\right)^\theta \left(\int_{K_\rho(0)} s^2\,dx\right)^{\frac{(1-\theta)}{r}},$$

that is, we come to (11). □

Theorem 1. *Assume that the conditions* (8)–(10) *are fulfilled and additionally* $t \in [0,T]$, $T \leqslant T^*$, *where*

$$T^* \leqslant \min\left(4M^{2-b-n}F_1^{-2}\left(\min\left(1, \frac{k}{\mu\beta_\phi}(1+M)^{b-n-2} - \frac{1}{2}\right)\right)^2\right),$$

$$\left(\left(\rho_0^{1+2\delta} - \rho^{1+2\delta}\right)\frac{(2\theta-1)\mu\beta_\phi}{(2\delta+1)4kK_i^2} w^{1-2\theta}(\rho_0,t)\right)^{\frac{1}{1-\theta}}, \quad i = 1,2.$$

If $s(x,t)$ *is a weak solution of (7) and* $s_0(x) = 0$ *in* $K_{\rho_0}(x_0)$, $0 < \rho_0 < dist(x_0, \partial G)$, *then* $s(x,t) = 0$ *almost everywhere in* $K_{\rho_1(t)}(x_0)$, $0 \leqslant t \leqslant T \leqslant T^*$. *Moreover*

$$\rho_1(t) = \left({\rho_0}^{1+2\delta} - L\, t^{1-\theta}(w(\rho_0,t))^{2\theta-1}\right)^{\frac{1}{1+2\delta}},$$

where if $0 < n-b < 2$, *then*

$$L = 4C_1^2 \cdot Q(r), \quad r \in (1,2),$$

and if $n-b=2$, *then*

$$L = 4C_2^2 \cdot Q(r), \quad r = \frac{4}{n-b+2} = 1.$$

In both cases

$$w(\rho_0,t)=\sup_{0\leqslant\tau\leqslant t}\int_0^\tau B(\rho_0,s)ds,\quad Q(r)=\frac{2\delta+1}{2\theta-1}\left(\frac{1}{2}\rho_0^\delta+T^{\frac{1}{2}}M^{2(\delta-1)}\rho_0^{\delta-1}\right)^{2\theta}\left(\frac{k}{\mu\beta_\phi}\right)^2,$$

$$K_i=C_i\left[\frac{1}{2}{\rho_0}^\delta+T^{\frac{1}{2}}\,{\rho_0}^{\delta-1}\,M^{2(\delta-1)}\right]^\theta,\quad i=1,2,\quad F_1=\frac{kng}{\mu}\left(\rho_s+(1+2M)\rho_f\right),$$

and constants C_1 and C_2 are determined in (11).

Theorem 2. *Assume that in addition to the conditions of Theorem 1 we have*

$$\int_0^t B(\rho,\tau)d\tau\leqslant C_0,\quad \int_{K_\rho(x_0)}s_0^2(x)dx\leqslant K_3\Big(\rho-\rho_0\Big)^{\frac{2+r}{2-r}},\quad \forall\rho\in(\rho_0,R). \tag{14}$$

Then there exists T_0 depending on the data of the problem such that $s(x,t)=0$ for almost all $x\in K_{\rho_0}(x_0)$ and $t\in[0,T_0]$.

2. Finite Propagation Speed of Disturbances

Proof of Theorem 1. Suppose in (10) $\varphi(x,t)=\varphi_n(|x-x_0|)\xi_k(t)\dfrac{1}{h}\displaystyle\int_t^{t+h}T_l(s(x,\tau))d\tau$, where $h\in(0,T-t)$,

$$T_l(s)\ =\min(|s|,l)sign\,s,$$

$$\varphi_n(r)=\begin{cases}1, & r\in[0,\rho-1/n],\\ n(\rho-r), & r\in[\rho-1/n,\rho],\\ 0, & r\in[\rho,\rho_0],\end{cases}$$

$$\xi_k(r)=\begin{cases}1, & r\in[0,t-1/k],\\ k(t-r), & r\in[t-1/k,t],\\ 0, & r\in[t,T^*].\end{cases}$$

We have

$$\int_0^\infty\int_{K_{\rho_0}(x_0)}\left[d(s)\frac{\partial s}{\partial x}\,\xi_k(\tau)\frac{\partial}{\partial x}\left(\varphi_n\frac{1}{h}\int_t^{t+h}T_l(s(x,\psi))d\psi\right)-\frac{\partial f(s)}{\partial x}\,\varphi(x,\tau)\right]dx\,d\tau=$$

$$=\int_0^\infty\int_{K_{\rho_0}(x_0)}s\,\varphi_n\frac{\partial}{\partial\tau}\left(\xi_k\frac{1}{h}\int_t^{t+h}T_l(s(x,\psi))d\psi\right)dx\,d\tau+\int_{K_{\rho_0}(x_0)}s(x,0)\,\varphi(x,0)\,dx. \tag{15}$$

Taking into account the Lebesgue theorem with $k\to\infty$ we get

$$\lim_{k\to\infty}\int_0^\infty\int_{K_{\rho_0}(x_0)}s\,\varphi_n\frac{\partial}{\partial\tau}\left(\xi_k\frac{1}{h}\int_t^{t+h}T_l(s(x,\psi))d\psi\right)dx\,d\tau=$$

$$=\lim_{k\to\infty}\int_0^\infty\int_{K_{\rho_0}(x_0)}s\,\varphi_n\frac{\partial\xi_k}{\partial\tau}\frac{1}{h}\int_t^{t+h}T_l(s(x,\psi))d\psi\,dx\,d\tau+$$

$$+\lim_{k\to\infty}\int_0^\infty\int_{K_{\rho_0}(x_0)}s\,\varphi_n\xi_k\frac{1}{h}\Big(T_l(s(x,\tau+h))-T_l(s(x,\tau))\Big)\,dx\,d\tau=$$

$$= -\lim_{k\to\infty} k \int_{t-1/k}^{t} \int_{K_{\rho_0}(x_0)} s\,\varphi_n \frac{1}{h} \int_t^{t+h} T_l(s(x,\psi))d\psi\,dx\,d\tau +$$

$$+ \int_0^{\infty} \int_{K_{\rho_0}(x_0)} s\,\varphi_n \frac{1}{h} \Big(T_l(s(x,\tau+h)) - T_l(s(x,\tau)) \Big)\,dx\,d\tau =$$

$$= -\int_{K_{\rho_0}(x_0)} s\,\varphi_n \frac{1}{h} \int_t^{t+h} T_l(s(x,\psi))d\psi\,dx + \int_0^{\infty} \int_{K_{\rho_0}(x_0)} s\,\varphi_n \frac{1}{h} \Big(T_l(s(x,\tau+h)) - T_l(s(x,\tau)) \Big)\,dx\,d\tau$$

and, therefore, at $h \to 0$ the identity (15) can be written as

$$\int_0^t \int_{K_{\rho_0}(x_0)} \left[d(s) \frac{\partial s}{\partial x} \frac{\partial \varphi_n}{\partial x} T_l + \left(d(s) \frac{\partial s}{\partial x} \varphi_n \frac{\partial T_l}{\partial x} - \frac{\partial f(s)}{\partial x} \right) \varphi_n T_l \right] dx\,d\tau =$$

$$= -\int_{K_{\rho_0}(x_0)} s\varphi_n T_l\,dx + \int_{K_{\rho_0}(x_0)} s_0(x)\varphi_n T_l(s(x,0))\,dx.$$

Therefore, after passing to the limit as $l \to \infty$, we obtain

$$\int_0^t \int_{K_{\rho_0}(x_0)} \left[s d(s) \frac{\partial s}{\partial x} \frac{\partial \varphi_n}{\partial x} + \left(d(s) \left(\frac{\partial s}{\partial x} \right)^2 \varphi_n - \frac{\partial f(s)}{\partial x} \right) \varphi_n s \right] dx\,d\tau =$$

$$= -\int_{K_{\rho_0}(x_0)} s^2 \varphi_n\,dx + \int_{K_{\rho_0}(x_0)} s_0^2(x)\varphi_n\,dx.$$

Finally, passing to the limit as $n \to \infty$, we have

$$\lim_{n\to\infty} \int_0^t \int_{K_{\rho_0}(x_0)} s d(s) \frac{\partial s}{\partial x} \frac{\partial \varphi_n}{\partial x}\,dx\,d\tau =$$

$$= -\lim_{n\to\infty} n \int_0^t \int_{\rho-1/n<|x-x_0|<\rho} s d(s) \frac{\partial s}{\partial x}\,dx\,d\tau = -\int_0^t s d(s) \frac{\partial s}{\partial x}\,d\tau.$$

Therefore, we arrive at the equality($x_0 = 0$)

$$\int_0^\rho s^2 dx + \int_0^t \int_0^\rho \left[d(s) \left(\frac{\partial s}{\partial x}\right)^2 - s \frac{\partial f}{\partial x} \right] dx\,d\tau = \int_0^\rho s_0^2\,dx + \int_0^t s d(s) \frac{\partial s}{\partial x}(\rho,\tau)\,ds]\,d\tau. \tag{16}$$

Let

$$a(\rho,t) = \sup_{0\leqslant\tau\leqslant t} A(\rho,\tau).$$

It follows from (16) that

$$a(\rho,t) + \frac{k}{\mu\beta_\phi}(1+M)^{b-n-2} w(\rho,t) \leqslant \frac{k}{\mu\beta_\phi} I_1 + I_2, \tag{17}$$

where

$$I_1 = \int_0^t s(\rho,\tau)^{n-b+1} \left| \frac{\partial s}{\partial x}(\rho,\tau) \right| d\tau,$$

$$I_2 = \int_0^t \int_0^\rho s(\rho,\tau) \left| \frac{\partial f(s)}{\partial x} \right| dx d\tau.$$

Applying the Hölder inequality and (11), we obtain

$$I_1 = \int_0^t s(\rho,\tau)^{n-b+1} \frac{\partial s}{\partial x}(\rho,s)ds \leqslant$$

$$\leqslant \left(\int_0^t s^{n-b}(\rho,\tau) \left(\frac{\partial s}{\partial x}(\rho,\tau)\right)^2 d\tau \right)^{\frac{1}{2}} \left(\int_0^t s^{n-b+2}(\rho,\tau)\, d\tau \right)^{\frac{1}{2}} = \left(\frac{\partial w}{\partial \rho}\right)^{\frac{1}{2}} a_1(\rho,t),$$

$$a_1(\rho,t) \equiv \left(\int_0^t s^{n-b+2}(\rho,\tau) \right)^{\frac{1}{2}} \leqslant C_i \left(\int_0^t \left(B^{\frac{1}{2}}(\rho,\tau) + \rho^{-\delta} A^{\frac{1}{r}}(\rho,\tau) \right)^2 d\tau \right)^{\frac{\theta}{2}} \left(\int_0^t A^{\frac{2}{r}}(\rho,\tau) d\tau \right)^{\frac{1-\theta}{2}} \leqslant$$

$$\leqslant C_i \left(\left(\int_0^t B(\rho,\tau) d\tau \right)^{\frac{1}{2}} + \rho^{-\delta} \left(\int_0^t A^{\frac{2}{r}}(\rho,\tau) d\tau \right)^{\frac{1}{2}} \right)^{\theta} t^{\frac{1-\theta}{2}}\, a^{\frac{1-\theta}{r}}(\rho,t) \leqslant$$

$$\leqslant C_1 \rho^{-\delta} t^{\frac{1-\theta}{2}} \left(w^{\frac{1}{2}}(\rho,t)\, a^{\frac{1-\theta}{r\theta}}(\rho,t) \rho^{\delta} + T^{\frac{1}{2}} a^{\frac{1}{r\theta}}(\rho,t) \right)^{\theta}.$$

But $\rho < \rho_0$, and moreover

$$2w^{\frac{1}{2}}\, a^{\frac{1-\theta}{r\theta}} = 2w^{\frac{1}{2}}\, a^{\frac{1}{2}} \leqslant a + w,$$

$$a^{\frac{1}{r\theta}}(\rho,t) \leqslant a(\rho,t)\, a^{\delta-1}(\rho_0,t) \leqslant a(\rho,t)\, {\rho_0}^{\delta-1}\, M^{2(\delta-1)}, \quad \delta > 1.$$

Consequently,

$$a_1(\rho,t) \leqslant C_i \rho^{-\delta} t^{\frac{1-\theta}{2}} \left(\frac{\rho_0^{\delta}}{2}(a+w) + T^{\frac{1}{2}} a \rho_0^{\delta-1} M_0^{2(\delta-1)} \right)^{\theta} \leqslant$$

$$\leqslant C_i \rho^{-\delta} t^{\frac{1-\theta}{2}} (a+w)^{\theta} \left(\frac{1}{2}\rho_0^{\theta} + T^{\frac{1}{2}} \rho_0^{\delta-1} M^{2(\delta-1)} \right)^{\theta}.$$

Correspondingly,

$$I_1 \leqslant K_i t^{\frac{1-\theta}{2}}\, \rho^{-\delta}\, [a(\rho,t) + w(\rho,t)]^{\theta} \left(\frac{\partial w}{\partial \rho}\right)^{\frac{1}{2}},$$

where

$$K_i = C_i \left[\frac{1}{2}{\rho_0}^{\delta} + T^{\frac{1}{2}}\, {\rho_0}^{\delta-1}\, M^{2(\delta-1)} \right]^{\theta}, \quad i = 1, 2.$$

Now we estimate I_2.

We have

$$\left| \frac{\partial f}{\partial x} \right| \leqslant s^{n-1} \left| \frac{\partial s}{\partial x} \right| F_1,$$

where

$$F_1 = \frac{kn}{\mu} g(\rho_s + (1+2M)\rho_f).$$

Therefore,

$$I_2 \leqslant F_1 \left(\int_0^t \int_0^{\rho} \left(\frac{\partial s}{\partial x}\right)^2 s^{n-b} dx d\tau \right)^{\frac{1}{2}} \left(\int_0^t \int_0^{\rho} s^{n+b} dx\, d\tau \right)^{\frac{1}{2}} \leqslant$$

$$\leqslant F_1 w^{\frac{1}{2}} M^{\frac{n+b-2}{2}} \left(\int_0^t \int_0^{\rho} s^2 dx\, d\tau \right)^{\frac{1}{2}} \leqslant F_1 M^{\frac{n+b-2}{2}} w^{\frac{1}{2}} a^{\frac{1}{2}} t^{\frac{1}{2}} \leqslant \frac{1}{2} F_1 M^{\frac{n+b-2}{2}} t^{\frac{1}{2}} (a+w).$$

Consequently, (17) takes the form

$$\min(1, \frac{k}{\mu\beta_\phi}(1+M)^{b-n-2})(a(\rho,t)+w(\rho,t)) \leqslant \frac{k}{\mu\beta_\phi} I_1 + I_2 \leqslant$$

$$\leqslant t^{\frac{1-\theta}{2}} \frac{k}{\mu\beta_\phi} K_i \rho^{-\delta} \left(a(\rho,t)+w(\rho,t)\right)^\theta \left(\frac{\partial w}{\partial \rho}\right)^{\frac{1}{2}} + \frac{1}{2} F_1 M^{\frac{b+n-2}{2}} t^{\frac{1}{2}} (a+w), \quad i=1,2.$$

Now choose t in such a way that

$$\frac{1}{2} F_1 M^{\frac{b+n-2}{2}} t^{\frac{1}{2}} \leqslant \min(1, \frac{k}{\mu\beta_\phi}(1+M)^{b-n-2}) - \frac{1}{2}.$$

Therefore,

$$\frac{1}{2}(a+w) \leqslant \frac{k}{\mu\beta_\phi} K_i t^{\frac{1-\theta}{2}} \rho^{-\delta} (a+w)^\theta \left(\frac{\partial w}{\partial \rho}\right)^{\frac{1}{2}}.$$

Accordingly,

$$\rho^\delta w^{1-\theta} \leqslant (a+w)^{1-\theta} \rho^\delta \leqslant 2 \frac{k}{\mu\beta_\phi} K_i t^{\frac{1-\theta}{2}} \left(\frac{\partial w}{\partial \rho}\right)^{\frac{1}{2}},$$

and hence

$$\rho^{2\delta} w^{2(1-\theta)} \leqslant K_i^* t^{1-\theta} \frac{\partial w}{\partial \rho}, \tag{18}$$

where $K_i^* = 4(\frac{k}{\mu\beta_\phi} K_i)^2,\ i=1,2.$

Integrating (18) by ρ from ρ_1 to ρ_0, we find that $(1 \leqslant r < 2)$

$$\frac{1}{2\delta+1}(\rho_0^{1+2\delta} - \rho_1^{1+2\delta}) \leqslant K_i^* t^{1-\theta} (w^{2\theta-1}(\rho_0,t) - w^{2\theta-1}(\rho_1,t)) \frac{1}{2\theta-1}.$$

Therefore, we have

$$\rho_1^{1+2\delta} - \rho_0^{1+2\delta} + \frac{2\delta+1}{2\theta-1} K_i^* t^{1-\theta} w^{2\theta-1}(\rho_0,t) \geqslant \frac{2\delta+1}{2\theta-1} K_i^* t^{1-\theta} w^{2\theta-1}(\rho_1,t). \tag{19}$$

Choosing t such that the equality

$$\rho_1^{1+2\delta} = \rho_0^{1+2\delta} - \frac{2\delta+1}{2\theta-1} K_i^* t^{1-\theta} w^{2\theta-1}(\rho_0,t),$$

holds, we obtain that $w(\rho,t)=0$ for all $\rho \leqslant \rho_1$, i.e. $s(x,t)=0$ almost everywhere in $K_\rho(0)$ for $\rho \leqslant \rho_1$ and

$$0 \leqslant t \leqslant \min\left(4M^{2-b-n} F_1^{-2} \left(\min\left(1, \frac{k}{\mu\beta_\phi}(1+M)^{b-n-2} - \frac{1}{2}\right)\right)^2, \right.$$
$$\left. \left(\left(\rho_0^{1+2\delta} - \rho^{1+2\delta}\right) \frac{2\theta-1}{(2\delta+1)K_i^*} w^{1-2\theta}(\rho_0,t)\right)^{\frac{1}{1-\theta}}\right).$$

3. Metastable Localization of Solutions

Proof of Theorem 2. Following the initial reasoning in Theorem 1 and formally replacing in (15) ρ by R, for all $\rho \in (0, R)$ we deduce the initial equality (16). According to the conditions of the theorem, $s_0(x) = 0$ in the ball $K_{\rho_0}(x_0)$. Therefore, the first integral on the right hand side of the equality (16) (of s_0^2) is in fact (for $\rho \in (\rho_0, R)$) taken over the interval (ρ_0, ρ), and hence the estimate (14) is valid. Other terms of the right hand side of (16) are estimated in the same way as in Theorem 1. So, instead of (19) we have for all $t < T$

$$\frac{1}{2}(a+w) \leqslant \frac{k}{\mu\beta_\phi} K_i (a+w)^\theta \rho^{-\delta}\, t^{\frac{1-\theta}{2}} \left(\frac{\partial w}{\partial \rho}\right)^{\frac{1}{2}} + K_3 \Big(\rho - \rho_0\Big)^{\frac{1}{2\theta-1}}.$$

The first term of the right hand side is estimated by using Young's inequality

$$\frac{1}{2}(a+w) \leqslant \varepsilon^{\frac{1}{\theta}} \theta (a+w) + \frac{1-\theta}{\varepsilon^{\frac{1}{1-\theta}}}\, t^{\frac{1}{2}}\, \rho^{-\frac{\delta}{1-\theta}} \left(\frac{\partial w}{\partial \rho}\right)^{\frac{1}{2(1-\theta)}} \left(\frac{k}{\mu\beta_\phi} K_i\right)^{\frac{1}{1-\theta}} + K_3 \Big(\rho - \rho_0\Big)^{\frac{1}{2\theta-1}}.$$

Choosing $\varepsilon^{1/\theta} = \dfrac{1}{4\theta} > 0$, we obtain

$$\frac{1}{4}(a+w) \leqslant \frac{1-\theta}{\varepsilon^{\frac{1}{1-\theta}}}\, t^{\frac{1}{2}}\, \rho^{-\frac{\delta}{1-\theta}} \left(\frac{\partial w}{\partial \rho}\right)^{\frac{1}{2(1-\theta)}} \left(\frac{k}{\mu\beta_\phi} K_i\right)^{\frac{1}{1-\theta}} + K_3 \Big(\rho - \rho_0\Big)^{\frac{1}{2\theta-1}}.$$

Using the inequality $a^p + w^p \geqslant (a+w)^p,\ 0 < p < 1,\ a, w \geqslant 0$, we have

$$w^{2(1-\theta)} \leqslant (4(1-\theta))^{2(1-\theta)} \varepsilon^{-2}\, \rho^{-2\delta} \left(\frac{k}{\mu\beta_\phi} K_i\right)^2 t^{1-\theta}\, \frac{\partial w}{\partial \rho} + (4K_3)^{2(1-\theta)} \Big(\rho - \rho_0\Big)^{\frac{2(1-\theta)}{2\theta-1}}.$$

The result is a special case of the inequality

$$w^\sigma \leqslant C\, t^\kappa\, w'_\rho + C \Big(\rho - \rho_0\Big)^{\frac{\sigma}{1-\sigma}}, \quad 0 < \sigma < 1, \quad \rho \in [\rho_0, R], \tag{20}$$

studied in [12]. As shown in the cited paper, (20) implies the equality $w(\rho_0, t) = 0$ for all $t \in [0, t_0]$, where t_0 is calculated from the relation

$$t_0 = ((1-\sigma)\, 2^{1-\sigma},\ \ R/C_i^* C_0^{1-\sigma})^{2/\sigma},\ \ C_i^* = (4(1-\theta))^{2(1-\theta)} \varepsilon^{-2}\, \rho^{-2\delta} \left(\frac{k}{\mu\beta_\phi} K_i\right)^2,\ j = 1, 2.$$

Therefore $s(x, t) = 0$ for almost all $x \in K_{\rho_0}(x_0)$ and $t \in [0, T_0]$, $T_0 = \min(t_0, T^*)$.

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Локализация решений уравнений фильтрации в пороупругой среде

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В работе рассматривается система уравнений одномерного нестационарного движения жидкости в пороупругой среде. Методом интегральных энергетических оценок устанавливается локализация решений уравнений.

Ключевые слова: фильтрация, закон Дарси, пороупругость, локализация, метастабильная локализация.