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# Gaussian Random Waves in Elastic Medium

**Dmitry N. Maksimov***
**Almas F. Sadreev**
Kirensky Institute of Physics, SB RAS,
Akademgorodok 50, Krasnoyarsk, 660036,
Russia

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*Similar to Berry conjecture for quantum chaos we consider elastic analogue which incorporates longitudinal and transverse random waves. Based on that we derive the intensity correlation function of elastic displacement field. Comparison to numerics in a quarter Bunimovich stadium demonstrates a good agreement. We also consider nodal points (NPs) $u = 0,\ v = 0$ of the in-plane random vectorial displacement field $\mathbf{u} = (u, v)$. We derive the mean density and correlation function of NPs. Consequently, we derive the distribution of the nearest distances between NPs*

*Keywords: Gaussian random waves, wave chaos, billiard, nodal points.*

## Introduction

Attracting interest in the field of wave chaos [1], elastomechanical systems are being studied analytically, numerically, and experimentally [2]. Weaver first measured the few hundred lower eigenfrequencies of an aluminum block and worked out the spectral statistics [3]. Spectral statistics coinciding with random matrix theory were observed in experiments for monocrystalline quartz blocks shaped as three-dimensional Sinai billiards [4], as well as, in experimental and numerical studies of flexural modes [5, 6] and in-plane modes [7, 8] for stadium-shaped plates. Statistical properties of eigenfunctions describing standing waves in elastic billiards were first reported by Schaadt *et al.* [9]. The authors measured the displacement field of several eigenmodes of a thin plate shaped as a Sinai stadium. Due to a good preservation of up-down symmetry in the case of thin plates there are two types of modes. The flexural modes with displacement perpendicular to the plane of the plate are well described by the scalar biharmonic Kirchoff-Love equation. In this case a perfect agreement with theoretical prediction for both intensity statistical distribution and intensity correlation function was found. However in the case of in-plane displacements described by the vectorial Navier-Cauchy equation an agreement between the intensity correlator experimental data and the theory was not achieved [9].

The aim of this work is to present an analogue of Berry conjecture for elastomechanics and investigate statistical properties of random wave fields in vibrating elastic solids. We propose a simple and physically transparent approach based on random superposition of traveling plane waves (Gaussian random wave). We restrict ourselves to the two-dimensional case. However, the method can be easily generalized for the three-dimensional case.

The simplest way to construct Gaussian random field is to use a random superposition of $N$ plane waves. Thus we come to Berry conjecture in the form

$$\phi(\mathbf{x}) = \sqrt{\frac{1}{N}} \sum_{n=1}^{N} \exp[(i(\theta_n + \mathbf{k_n}\mathbf{x})], \tag{1}$$

*mdn@tnp.krasn.ru

where the phases $\theta_n$ are distributed uniformly in range $[0, 2\pi)$ and all the amplitudes are equal (one could assume random independent amplitudes, without any change in the results). The wave vectors $\mathbf{k}_n$ are uniformly distributed on a d-dimensional sphere of radius $k$. It follows now from the central limit theorem that both $\Re\{\phi\}$ and $\Im\{\phi\}$ are independent Gaussian variables. In a finite-size plate (billiard) the conjecture is viewed as a sum of many standing waves, that is simply real or imaginary part of function (1).

In our case one has to construct a Gaussian random wave (GRW) describing acoustic in-plane modes. The mode shapes are given by the two-dimensional Navier-Cauchy equation

$$\mu\nabla^2\mathbf{u} + (\lambda+\mu)\nabla(\nabla\mathbf{u}) + \rho\Omega^2\mathbf{u} = 0 \tag{2}$$

where $\mathbf{u}(x, y)$ is the displacement field in the plate, $\lambda, \mu$ are the material dependent Lamé coefficients, and $\rho$ is the density. Introducing elastic potentials $\psi$ and $\mathbf{A}$ with the help of the Helmholtz decomposition the displacement field $\mathbf{u}$ could be written,

$$\mathbf{u} = \mathbf{u}_l + \mathbf{u}_t, \quad \mathbf{u}_l = \nabla\psi, \ \mathbf{u}_t = \nabla\times\mathbf{A} \tag{3}$$

Eq. (2) reduces to two Helmholtz equations for the elastic potentials

$$-\nabla^2\psi = k_l^2\psi, \ -\nabla^2\mathbf{A} = k_t^2\mathbf{A}. \tag{4}$$

Here $k_l = \omega/c_l$, $k_t = \omega/c_t$ are the wave numbers for the longitudinal and transverse waves, respectively and $\omega^2 = \rho\Omega^2/E$, where $E$ is Young's modulus. In the two-dimensional case potential $\mathbf{A}$ has only one none-zero component $A_z$ and the dimensionless longitudinal and transverse sound velocities $c_{l,t}$ are given by

$$c_l^2 = \frac{1}{1-\sigma^2}, \ c_t^2 = \frac{1}{2(1+\sigma)}, \tag{5}$$

where $\sigma$ is Poisson's ratio [10]. $E$ and $\sigma$ are functions of the Lamé coefficients [10]. Our conjecture is that both elastic potential are statistically independent Gaussian random waves (1). We write the potentials in the following form

$$\psi(\mathbf{x}) = \frac{1}{ik_l}\sqrt{\frac{\gamma}{N}}\sum_{n=1}^{N}\exp[i(\mathbf{k}_{ln}\mathbf{x} + \theta_{ln})], \tag{6}$$

$$A_z(\mathbf{x}) = \frac{1}{ik_t}\sqrt{\frac{1-\gamma}{N}}\sum_{n=1}^{N}\exp[i(\mathbf{k}_{tn}\mathbf{x} + \theta_{tn})], \tag{7}$$

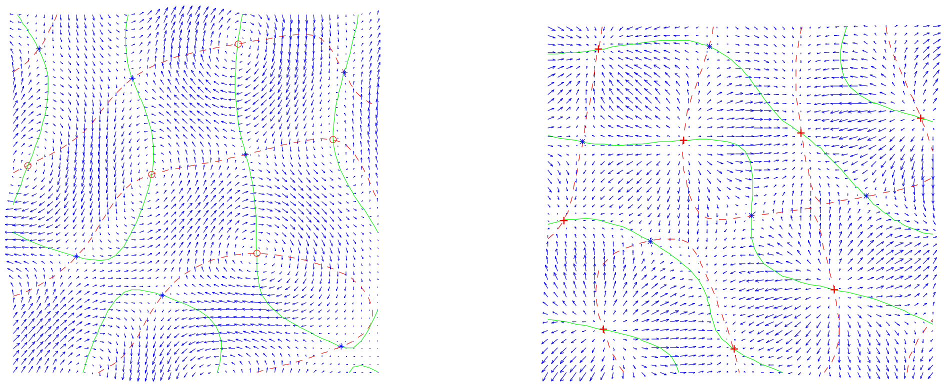
where $\theta_{ln}$, $\theta_{tn}$ are statistically independent random phases. The wave vectors $\mathbf{k}_{ln}$ and $\mathbf{k}_{tn}$ are uniformly distributed on circles of radii $k_l$ and $k_t$ respectively. Prefactors $\sqrt{\gamma}$ and $\sqrt{1-\gamma}$ are chosen from the normalization condition $\langle\mathbf{u}^\dagger\mathbf{u}\rangle = 1$, and $\langle\ldots\rangle$ means average over the random phase ensembles.

One can assume any value of $\gamma$ between 0 and 1. The first case corresponds to a superposition of transverse polarized waves. The second case gives a sum of waves of longitudinal polarization. These exapmles are viewed in Fig. 1.

The article is divided into two parts. In the first part we discuss the apmlitude and intensity correlation functions of the mode shapes describing eigenstates of elastic billiards. The second part deals with nodal points in Gaussian random elastic fields.

# 1. Correlation Functions

First, we calculate the amplitude correlation functions in chaotic elastic plate for in-plane GRW $\mathbf{u} = (u, v)$. Straightforward procedure of averaging over ensembles of random phases



a)    b)

Fig. 1. a) Transverse polarized Gaussian random wave field. b) Longitudinal polarized Gaussian random wave field. Blue arrow show displacement field. Solid green lines and dashed red lines are the nodal lines of vectorial field components. At their intersections lie the nodal points; $*$ — saddles; $+$ — knots; $\circ$ — focuses

$\theta_{ln}$, $\theta_{tn}$ and next, over angles of k-vectors gives

$$\begin{aligned}
\langle u(\mathbf{x}+\mathbf{s})u(\mathbf{x})\rangle &= \tfrac{\gamma}{2}(\cos^2\alpha\ f(k_l s) + \sin^2\alpha\ g(k_l s)) + \\
&+ \tfrac{1-\gamma}{2}(\sin^2\alpha\ f(k_t s) + \cos^2\alpha\ g(k_t s)), \\
\langle v(\mathbf{x}+\mathbf{s})v(\mathbf{x})\rangle &= \tfrac{\gamma}{2}(\sin^2\alpha\ f(k_l s) + \cos^2\alpha\ g(k_l s)) + \\
&+ \tfrac{1-\gamma}{2}(\cos^2\alpha\ f(k_t s) + \sin^2\alpha\ g(k_t s)), \\
\langle u(\mathbf{x}+\mathbf{s})v(\mathbf{x})\rangle &= \sin 2\alpha(\tfrac{1-\gamma}{2}J_2(k_t s) - \tfrac{\gamma}{2}J_2(k_l s)),
\end{aligned} \tag{8}$$

where

$$f(s) = J_0(s) - J_2(s), g(s) = J_0(s) + J_2(s). \tag{9}$$

The correlation functions (8) were obtained for given direction of the vector $\mathbf{s}$ where $\alpha$ is the angle included between vector $\mathbf{s}$ and the x-axis.

We also calculate the intensity correlation functions $P(s) = \langle I\mathbf{x}+\mathbf{s})I(\mathbf{x})\rangle$ where the intensity $I = |\mathbf{u}|^2$ is proportional to the elastic energy of the in-plane oscillations. For the in-plane chaotic GRW of the form $a_l\psi_l + a_t\psi_t$ Schaadt *et al.* [9] derived the intensity correlation function as

$$P(s) = 1 + 2[a_l^2 J_0(k_l s) + a_t^2 J_0(k_t s)]^2. \tag{10}$$

Our calculations give a different result

$$P(s) = 1 + [(\gamma J_0(k_l s) + (1-\gamma)J_0(k_t s)]^2 + [(\gamma J_2(k_l s) - (1-\gamma)J_2(k_t s)]^2 \tag{11}$$

Although the first term in (11) corresponds to (10) there is a different term consisted of the Bessel functions $J_2(x)$. The mathematical origin of deviation is that formula (11) contains the contributions of the components of the wave vectors $\mathbf{k}_l$ and $\mathbf{k}_t$ via space derivatives.

Waves propagate freely inside the billiard, that is, the longitudinal and transverse components are decoupled. The estimate of energy partition between weakly coupled longitudinal and transverse modes was first given by Weaver in [11]. We use a different way to approach this problem. In a billiard wave conversion occurs at the boundary according to Snell's law

$$c_l \sin(\theta_t) = c_t \sin(\theta_l), \tag{12}$$

The reflection amplitudes for each event of the reflection can be easily found from wave equation (2). At first we consider more easy case of the Dirichlet BC (the boundary is clamped). Approximating the boundary as the straight lines for the wavelengths much less than the radius of curvature we have for the reflection amplitudes

$$\begin{aligned} t_{ll} &= \frac{\cos(\theta_t)\cos(\theta_l) - \sin(\theta_t)\sin(\theta_l)}{\cos(\theta_t)\cos(\theta_l) + \sin(\theta_t)\sin(\theta_l)}, \ t_{lt} = \frac{2\sin(\theta_l)\cos(\theta_l)}{\cos(\theta_t)\cos(\theta_l) + \sin(\theta_t)\sin(\theta_l)}, \\ t_{tl} &= \frac{2\sin(\theta_t)\cos(\theta_t)}{\cos(\theta_t)\cos(\theta_l) + \sin(\theta_t)\sin(\theta_l)}, \ t_{tt} = \frac{\cos(\theta_t)\cos(\theta_l) - \sin(\theta_t)\sin(\theta_l)}{\cos(\theta_t)\cos(\theta_l) + \sin(\theta_t)\sin(\theta_l)}. \end{aligned} \quad (13)$$

Next, we assume that all directions of waves are statistically equivalent. Then we have for the energy density of reflected wave

$$\rho_{out} = \gamma(\overline{T}_{ll} + \overline{T}_{lt}) + (1-\gamma)(\overline{T}_{tt} + \overline{T}_{tl}), \quad (14)$$

where

$$\overline{T}_{ij} = \frac{1}{\pi}\int_0^\pi t_{ij}^2 d\theta_i, \ i = l, t.$$

Substituting Eq. (13) into Eq. (14) one can obtain after elementary calculations

$$\begin{gathered} \overline{T}_{ll} = 1 - \tfrac{c_t}{c_l} I_1, \ \overline{T}_{lt} = I_2, \\ \overline{T}_{tt} = 1 - \tfrac{2}{\pi}\arcsin\tfrac{c_t}{c_l} + \left(\tfrac{c_t}{c_l}\right)^3 I_1, \ \overline{T}_{tl} = \tfrac{2}{\pi}\arcsin\tfrac{c_t}{c_l} - \left(\tfrac{c_t}{c_l}\right)^2 I_2. \end{gathered} \quad (15)$$

We do not present here integrals $I_1, I_2$, since after substitution of (15) into (14) they cancel each other. The equality $\rho_{in} = 1 = \rho_{out}$ gives a very simple evaluation

$$\gamma = \frac{c_t^2}{c_t^2 + c_l^2}. \quad (16)$$

which is the same as given in [10]. The next remarkable result is that although the reflection amplitudes for the free BC have the form different from (13), the evaluation of $\gamma$ has the same form as for the fixed BC i.e. the result does not depend on either the free BC or the clamped BC is applied. An additional remark should be made to the amplitude correlation function. As it was shown in [12] the amplitude correlation function is equal to the imaginary part of Green's function of the corresponding wave equation. Therefore, it is not surprising that after substitution of (16) into (8) one gets the imaginary part of 2D elastomechanical Green's function.

The results obtained were verified numerically. To do this we computed 200 in-plane eigenmodes of an elastic plate shaped as a qurter of Bunimovich stadium (see Fig. 2) for several values of $\gamma$. Fig. 3 shows $\gamma$ versus Poisson ratio $\sigma$ in comparison with our numerical results obtained in averaging over 200 eigenstates. In Fig. 3 we also show the intensity correlation function (11) compared to the intensity correlation function of the eigenstate shown in Fig 2.

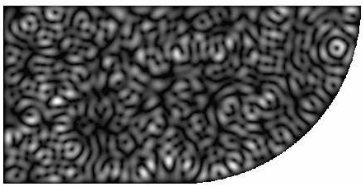


Fig. 2. The energy density of a chaotic eigenstate

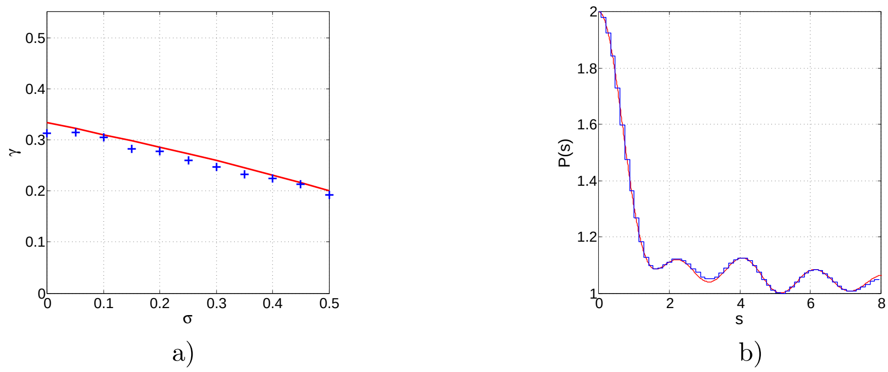



Fig. 3. a) $\gamma$ against $\sigma$. b) The intensity correlation function

## 2. Nodal Points Statistics

Now consider nodal points (NPs) in the random elastic field. Our aim is to find the nodal density of different types of nodal points and derive nodal density corrlation function. The statistcs of nodal points in the context of quantum and wave chaos was discused in [13, 14]. In our study we follow the way outlined in [14]. Our final goal is the distribution of the nearest distances between the nodal points.

The in-plane eigenmodes of a billiard discribed by displacement vector

$$\mathbf{u}(x, y) = (u(x, y),\ v(x, y))$$

have NPs at the points whrere nodal lines $u(x, y) = 0$, $v(x, y) = 0$ of both components intersect each other (see Fig. 1). NPs of 2D vectorial field are specified by the Poincaré index (topological charge) [15]

$$q = sign(\det M_{\mathbf{x}_0}) = sign(\lambda_1\lambda_2),\ M = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}, \tag{17}$$

where $\lambda_{1,2}$ are eigenvalues of matrix $M$ at NP $\mathbf{x}_0$. Depending on these eigenvalues NPs split on the four types: 1) centers for imaginary $\lambda_{1,2}$ with the index $q = 1$; 2) knots for real $\lambda_{1,2}$ with the same sign and $q = 1$; 3) focuses for complex $\lambda_1 = \lambda_2^*$ with $q = 1$ ; and 4) saddles for real $\lambda_{1,2}$ with opposite sign and $q = -1$. Eigenvalues of matrix $M$ are

$$\lambda_{1,2} = \frac{u_x + v_y}{2} \pm \sqrt{\left(\frac{u_x + v_y}{2}\right)^2 - \det M} = \frac{u_x + v_y}{2} \pm \sqrt{D}/4 \tag{18}$$

where we introduced $D = (u_x + v_y)^2 - 4 \det M$. Therefore, for $D > 0$ NP can be classified as a knot, while for $D < 0$ we have a focus (center for particular case $u_x + v_y = 0$). At last for $M < 0$ NP is a saddle.

Averaging over Gaussian random fields makes possible to derive the density of NPs in elastic vectorial field. For the nodal density we have [13, 14]

$$\rho = \langle \delta(u)\delta(v)|M|\rangle, \tag{19}$$

where the Jacobian $|M|$ is found from Eq. (17). We derived the densities of all nodal point types. The results are shown in Fig. 4. One can see that if $\gamma = 1$ as nodal points of positive charge we have knots only. In case $\gamma = 0$ we have only fucuses. As it have been shown in the previous section in any realistic system $\gamma$ is given by Eq. (16). I what follows we will use this evaluation.

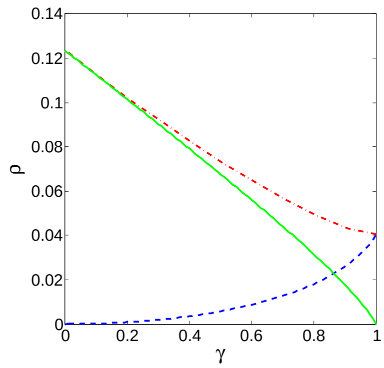



Fig. 4. The density of saddles, red dash-dot line; focuses, green solid line; and knotes, blue dotted line. The material parameter $\sigma = 0.345$

The corresponding correlation function is [13, 14]

$$G(s) = \frac{1}{\rho^2}\langle \delta(u)\delta(v)|M|\delta(u_s)\delta(v_s)|M_s|\rangle, \tag{20}$$

where for brevity we omitted coordinate arguments of values except index $s$ which implies the distance between the points $\mathbf{x}$ and $\mathbf{x}+\mathbf{s}$. Similarly, the charge correlation function [13, 14] gives the correlation of density, but weighted with their charge $q$,

$$G_q(s) = \frac{1}{\rho^2}\langle \delta(u)\delta(v)M\delta(u_s)\delta(v_s)M_s\rangle. \tag{21}$$

The charge-charge correlation function was found analytically with the use of the method de-

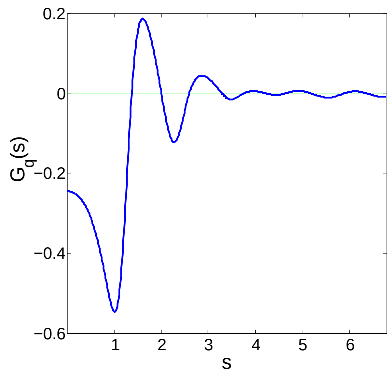



Fig. 5. Topological charge correlation function

scribed in [16]. the correlation function is shown in Fig. 5. Unfortunately, we could not derive density-density correlation function (20) in any analytical form. A numerically computed result was used instead in the following computation.

The distribution of the nearest distances between NPs can be derived from the density correlation function (20). Following [14] we use Poisson approximation for which the distribution is

$$f(s) \approx 2\pi s G(s)\exp(-\langle n(s)\rangle), \tag{22}$$

where the mean number of NPs inside the circle of radius $s$ around given one is

$$\langle n(s)\rangle = 2\pi\rho\int_0^s rG(r)dr, \tag{23}$$

with $\rho$ as mean density of NPs. Fig. 6 shows the histogram of the distribution function of the nearest distances between NPs for the random in-plane elastic waves obtained in numerical experiment with the information about more than 500000 NPs acummulated. For comparison we present the distribution of nearest distances between uncorrelated random points [17]

$$f(x) = \frac{\pi x}{2}\exp(-\pi x^2/4) \tag{24}$$

One can see that the approximate distribution discribes with a sufficient accuracy. The main point is that the NPs newerthwleless being embedded in a random field experience repulsion which is described be the nodal density correlation function (20).

## Conclusion

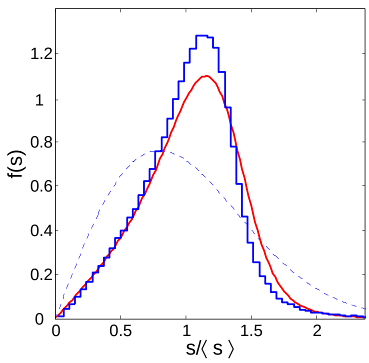



Fig. 6. Histogram of the distribution function of the nearest distances between NPs for the in-plane random elastic waves; $\sigma = 0.345$. Solid line shows the distribution function calculated in the Poisson approximation (22). Dashed line shows the special case of uniformly distributed and completely random points (24)

Based on random waves theory we investigated in-plane eigenmodes of chaotic elastic billiards. With the help of the Helmholtz decomposition we succeeded to present an adequate description of intensity correlation properties of Gaussian random elastic waves. Moreover, in case of billiards we found that double ray splitting at the billiard boundary affects the partition of energy between longitudinal and transverse waves. The results obtained were verified numerically. Next, we considered the statistical properties of nodal points in random elastic displacement field. We investigated their topological properties, derived the charge-charge correlation function, and numerically computed the density-density correlation function. These results were the starting point for finding the distribution of the nearest distances between the nodal points. We shown that our approximate result demonstrates a good accuracy. For further reading see [18, 19].

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## Случайные гауссовы волны в изотропной упругой среде


**Дмитрий Н. Максимов**
**Алмаз Ф. Садреев**



*В данной работе рассматриваются собственные колебательные моды упругих пластин, выполненных в форме хаотических биллиардов. Для описания статистических свойств таких волновых полей была рассмотрена модель упругих колебаний, описывающихся потенциалами Гельмогольца, являющимися случайными гауссовыми функциями. В рамках этого подхода нами получены корреляционная функция интенсивности колебаний, корреляционная функция нодальной плотности и распределение ближайших расстояний между нодальными точками. Нами также выполнена проверка полученных результатов при помощи численного моделирования.*



*Ключевые слова: гауссовы случайные волны, волновой хаос, биллиард, нодальные точки.*