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On an Inverse Problem for Quasi-Linear Elliptic Equation

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The identification of an unknown constant coefficient in the main term of the partial differential equation $-kM\psi(u) + g(x)u = f(x)$ with the Dirichlet boundary condition is investigated. Here $\psi(u)$ is a nonlinear increasing function of u , M is a linear self-adjoint elliptic operator of the second order. The coefficient k is recovered on the base of additional integral boundary data. The existence and uniqueness of the solution to the inverse problem involving a function u and a positive real number k is proved.

Keywords: inverse problem, boundary value problem, second-order elliptic equations, existence and uniqueness theorem, filtration.

This paper proceeds the investigation started in [5,6] and is concerned with an inverse problem for the second order differential equation

$$-\operatorname{div}(\mathbf{k}(x, u)\nabla\psi(u)) + \gamma(x, u) = f \quad (0.1)$$

with the Dirichlet boundary data

$$u|_{\partial\Omega} = \beta(x) \quad (0.2)$$

where $\mathbf{k}(x, u)$ is a matrix of functions, $\psi(u)$ and $\gamma(x, u)$ are scalar functions. We assume that Ω is a bounded domain in \mathbb{R}^n with the boundary $\partial\Omega \subset C^2$.

Applications of this problem deal with the recovery of unknown parameters indicating physical properties of a medium (the heat conductivity, the permeability of a porous medium, etc.). Various aspects of such problems are discussed in [1–8] (see also the references given there). Of special interest is the problem of finding the leading coefficients of (0.1) given the additional boundary data on $\partial\Omega$ or a part of $\partial\Omega$. In [1, 2, 7, 8] this problem is investigated in the case where $\psi(u) = u$, $\gamma(x, u) \equiv 0$, $\mathbf{k}(x, u) = k(x)\mathbf{E}$, a function $k(x)$ is unknown, \mathbf{E} is the identical matrix.

In this paper we study the problem of the identification of the constant coefficient k in equation

$$-k\{\operatorname{div}(\mathcal{M}(x)\nabla\psi(u)) + m(x)\psi(u)\} + g(x)u = f(x), \quad x \in \Omega, \quad (0.3)$$

under the boundary data (0.2). Here $\psi(u)$ is a known function, $\mathcal{M}(x)$ is a matrix of functions $m_{ij}(x)$, $i, j = 1, 2, \dots, n$. The functions $m(x)$, $f(x)$, $g(x)$, $\beta(x)$ are given. In physical context the constant coefficient k is interpreted as the average conductivity of a medium.

As an additional data for the recovery of the coefficient k we take the condition of overdetermination

$$k \int_{\partial\Omega} \frac{\partial\psi(u)}{\partial\bar{N}} \omega \, ds = \varphi \quad (0.4)$$

where $\partial u / \partial\bar{N} \equiv (\mathcal{M}(x)\nabla u, \mathbf{n})_R$ is the conormal derivative, \mathbf{n} is a unit outward normal to $\partial\Omega$, ω is a given function, φ is a given real number. Physically, (0.4) describes, for instance, the total flux of the liquid through the surface of the rock [3]. Inverse problems with similar nonlocal boundary

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conditions for elliptic equations were considered in [5,6]. The results of [6] are concerned with the existence and uniqueness of the solution to the problem (0.2)–(0.4) where $\psi(\rho) = (p-1)^{-1}|\rho|^{p-2}\rho$ and $p > 2$ is a real number. The inverse problem for linear elliptic equation (0.3) with $\psi(\rho) = \rho$ is discussed in [5].

The goal of our paper is to establish the existence and uniqueness of a solution to the nonlinear problem (0.2), (0.3), (0.4). We discuss it in Section 2. In Section 3 we present examples and comments to the results of Section 2 and the inverse problem involved.

1. The preliminaries

We start with preliminary results for the direct problem (0.2), (0.3). The results concern the existence, uniqueness and certain properties of the solution to this problem.

From now on we keep the notations: $\|\cdot\|_R, (\cdot, \cdot)_R$ is the norm and the inner product of \mathbb{R}^n ; $\|\cdot\|, (\cdot, \cdot)$ is the norm and the inner product of $L^2(\Omega)$; $\|\cdot\|_j, \langle \cdot, \cdot \rangle_1$ is the norm of $W_2^j(\Omega)$, $j = 1, 2$, and the duality relation between $W_2^1(\Omega)$ and $W_2^{-1}(\Omega)$, respectively; $a(x)$ is the solution of the Dirichlet problem

$$-\operatorname{div}(\mathcal{M}(x)\nabla\psi(a)) + m(x)\psi(a) = 0, \quad x \in \Omega, \quad a|_{\partial\Omega} = \beta(x); \quad (1.1)$$

$b(x)$ is the solution of the Dirichlet problem

$$-\operatorname{div}(\mathcal{M}(x)\nabla b) + m(x)b = 0, \quad x \in \Omega, \quad b|_{\partial\Omega} = \omega(x). \quad (1.2)$$

We introduce the linear operator $M : W_2^1(\Omega) \rightarrow (W_2^1(\Omega))^*$ of the form $M = -\operatorname{div}(\mathcal{M}(x)\nabla) + m(x)I$ where I is the identity operator and reckon that the following assumptions hold throughout the paper.

- I. The operator M is elliptic. That is, $m(x)$, $m_{ij}(x)$, $\partial m_{ij}/\partial x_l$, $i, j, l = 1, 2, \dots, n$ are bounded, $m(x) \geq 0$ and there exist positive constants m_1, m_2 such that for any $\xi \in \mathbb{R}^n$

$$m_1 \sum_{i=1}^n \xi_i^2 \leq \sum_{i,j=1}^n m_{ij}(x) \xi_i \xi_j \leq m_2 \sum_{i=1}^n \xi_i^2. \quad (1.3)$$

- II. The operator M is self-adjoint. That is, $m_{ij}(x) = m_{ji}(x)$ for $i, j = 1, \dots, n$.

- III. The function $\psi(\rho)$ is a continuous mapping of $(-\infty, +\infty)$ onto itself. For any $\rho_1, \rho_2 \in (-\infty, +\infty)$, $\rho_1 \neq \rho_2$,

$$(\psi(\rho_1) - \psi(\rho_2))(\rho_1 - \rho_2) > 0, \quad (1.4)$$

that is, the function $\psi(\rho)$ is strictly monotone.

From the assumption III it follows that there exists an inverse $\psi^{-1}(\rho)$ of $\psi(\rho)$. The inverse $\psi^{-1}(\rho)$ is strictly monotone and continuous on $(-\infty, +\infty)$.

By the assumptions I–III the problem (1.1) has a unique solution a and $\psi(a) \in W_2^2(\Omega)$ when $\psi(\beta) \in W_2^{3/2}(\partial\Omega)$. If in addition $\beta \in L^\infty(\partial\Omega)$, then, by the continuity of ψ and Theorem 13.1 of [9, Chapter 3], $\psi(a) \in L^\infty(\Omega)$. This means that $a \in L^\infty(\Omega)$.

Lemma 1.1. *Let the assumptions I–III be fulfilled. Let also k be a given positive number, $f \in L^2(\Omega)$, $\psi(\beta) \in W_2^{1/2}(\partial\Omega)$, $g \in C(\bar{\Omega})$ and $g \geq 0$ in $\bar{\Omega}$. Then the following assertions are valid.*

1. *There exists a unique solution u of the direct problem (0.2), (0.3) such that $\psi(u) \in W_2^1(\Omega)$.*

2. If $\psi(\beta) \in W_2^{3/2}(\partial\Omega)$ and

$$|\psi(\rho)| \geq c|\rho|^p \quad (1.5)$$

for any $\rho \in (-\infty, +\infty)$ where $p \geq 1$, $c > 0$ are constants, then $u \in L^{2p}(\Omega) \cap W = \{v \mid \psi(v) \in W_2^2(\Omega)\}$.

3. If $g \equiv 0$ and $\psi(\beta) \in W_2^{3/2}(\partial\Omega)$, then $\psi(u) \in W_2^2(\Omega)$. If in addition $\beta \in L^\infty(\partial\Omega)$, then $u \in L^\infty(\Omega) \cap W$.

Proof. 1. Let $\tilde{a} = \psi(a)$. From (1.1) it follows that \tilde{a} solutions the equation $M\tilde{a} = 0$ and satisfies the boundary condition $\tilde{a}|_{\partial\Omega} = \psi(\beta)$. Multiplying (0.3) by $\tilde{\psi}(u) = \psi(u) - \tilde{a}$ in terms of the inner product of $L^2(\Omega)$ and integrating by parts in the first term we obtain

$$k \int_{\Omega} \left\{ (\mathcal{M}\nabla\tilde{\psi}(u), \nabla\tilde{\psi}(u))_R + m(x)\tilde{\psi}^2(u) \right\} dx + \int_{\Omega} g(u-a)\tilde{\psi}(u) dx = (-ga + f, \tilde{\psi}(u)). \quad (1.6)$$

The estimation of the right side of (1.6) with the help of the Friedrichs inequality

$$\|v\| \leq c_0 \text{mes}^{1/n}\Omega \left(\int_{\Omega} \|\nabla v\|_R^2 dx \right)^{1/2} \quad \text{for } v \in \overset{\circ}{W}_2^1(\Omega) \quad (1.7)$$

leads to the relation

$$\begin{aligned} k \int_{\Omega} \left\{ (\mathcal{M}\nabla\tilde{\psi}(u), \nabla\tilde{\psi}(u))_R + m(x)\tilde{\psi}^2(u) \right\} dx + \int_{\Omega} g(u-a)\tilde{\psi}(u) dx &\leq \\ &\leq \frac{km_1}{2} \int_{\Omega} \|\nabla\tilde{\psi}(u)\|_R^2 dx + \frac{c_0^2 \text{mes}^{2/n}\Omega}{2km_1} \|ga - f\|^2 \end{aligned}$$

whence, by (1.3), (1.4),

$$\begin{aligned} \int_{\Omega} \|\nabla\tilde{\psi}(u)\|_R^2 dx &\leq \frac{1}{m_1} \int_{\Omega} \left\{ (\mathcal{M}\nabla\tilde{\psi}(u), \nabla\tilde{\psi}(u))_R + m\tilde{\psi}^2(u) \right\} dx \leq \\ &\leq \frac{c_0^2 \text{mes}^{2/n}\Omega}{k^2 m_1^2} \|ga - f\|^2 \equiv C. \end{aligned} \quad (1.8)$$

Here the positive constant c_0 depends only on n . (1.7) and (1.8) implies

$$\|\tilde{\psi}(u)\| \leq \frac{c_0^2 \text{mes}^{2/n}\Omega}{km_1} \|ga - f\| \quad (1.9)$$

and

$$\|\psi(u)\|_1 \leq \|\tilde{a}\|_1 + C^{1/2} \max\{1, c_0 \text{mes}^{1/n}\Omega\} \equiv C_1. \quad (1.10)$$

To prove the existence and uniqueness of the solution of the problem (0.2), (0.3) we denote $\tilde{\psi}(u)$ by h . In view of the definition of $\tilde{\psi}(u)$ the function h satisfies the equation

$$\tilde{M}h \equiv Mh + g\psi^{-1}(h + \tilde{a}) - ga = f - ga \equiv \tilde{f} \quad (1.11)$$

and $h|_{\partial\Omega} = 0$. One can show that the operator \tilde{M} as a map from $\overset{\circ}{W}_2^1(\Omega)$ into $\overset{\circ}{W}_2^{-1}(\Omega)$ is demi-continuous, coercive and strongly monotone. Hence, there exists a solution $h \in \overset{\circ}{W}_2^1(\Omega)$ of equation (1.11) (see Theorem 2.1 [10, Chapter III]). Moreover, the solution h is unique. Indeed, let h_1 and h_2 be two solutions of (1.11) in $\overset{\circ}{W}_2^1(\Omega)$. Then by (1.3) and the strong monotonicity of \tilde{M} we have

$$0 = \langle \tilde{M}h_1 - \tilde{M}h_2, h_1 - h_2 \rangle_1 \geq m_1 \|\nabla(h_1 - h_2)\|^2,$$

which implies $h_1 = h_2$.

Coming back to the original problem we conclude that the problem (0.2), (0.3) has a solution $u = \psi^{-1}(h) \in W$ and this solution is unique.

2. In the hypotheses of the lemma from (1.9) it follows that $u \in L^{2p}(\Omega)$ and

$$\|u\|_{L^{2p}(\Omega)} \leq \|\psi(u)\| \leq \frac{c_0^2 \text{mes}^{2/n} \Omega}{km_1} \|ga - f\| + \|\psi(a)\| \equiv C_2.$$

Multiplying (0.3) by $M\psi(u)$ in terms of the inner product of $L^2(\Omega)$ we obtain the equality

$$k\|M\psi(u)\|^2 = -(gu, M\psi(u)) + (f, M\psi(u)),$$

from which it follows by (1.9) and the Cauchy inequality that

$$\begin{aligned} k\|M\psi(u)\|^2 &\leq \frac{1}{k} \left[\|g\|_{C(\bar{\Omega})}^2 \|u\|_{L^{2p}(\Omega)}^2 \text{mes}^{(p-1)/p} \Omega + \|f\|^2 \right] + \frac{k}{2} \|M\psi(u)\|^2 \leq \\ &\leq \frac{1}{k} \left[\|g\|_{C(\bar{\Omega})}^2 C_2^2 \text{mes}^{(p-1)/p} \Omega + \|f\|^2 \right] + \frac{k}{2} \|M\psi(u)\|^2 \end{aligned}$$

whence

$$\|M\psi(u)\| \leq \frac{\sqrt{2}}{k} \left[\|g\|_{C(\bar{\Omega})}^2 C_2^2 \text{mes}^{(p-1)/p} \Omega + \|f\|^2 \right]^{1/2} \equiv C'_2. \quad (1.12)$$

In view of (1.8), (1.10) and Theorem 5.1 of [11, Chapter 2] the last inequality implies

$$\|\psi(u)\|_2 \leq c(\|M\psi(u)\| + \|\psi(u) - \psi(a)\|_1) + \|\psi(a)\|_2 \leq c(C'_2 + C_1) + \|\psi(a)\|_2(c+1) \quad (1.13)$$

where the constant c depends on $n, m_1, m_2, \text{vrai max}_{\Omega} |\partial m_{ij}/\partial x_l|, i, j, l = 1, 2, \dots, n$ and $\text{mes} \Omega$. Thus, the solution $u \in L^{2p}(\Omega) \cap W$.

3. In the hypotheses of this part of the lemma $M\psi(u) \in W_2^2(\Omega)$ and the estimate (1.13) takes the form

$$\|\psi(u)\|_2 \leq c(k^{-1}\|f\| + C_1) + \|\psi(a)\|_2(c+1).$$

If in addition $\beta \in L^\infty(\Omega)$, then the boundedness of u follows from the assumption IV and Theorem 13.1 [9, Chapter 3]. \square

Lemma 1.2. *Let the assumptions of Lemma 1.1 be fulfilled and $ga - f \geq 0$. Then the solution u of the problem (0.2), (0.3) satisfies the inequality $u \leq a$ for almost all $x \in \Omega$.*

Proof. Let us consider the function $w = a - u$. By (0.3) and (1.1), we have

$$-k \text{div}(\mathcal{M}(x)(\nabla \psi(a) - \nabla \psi(u))) + g(x)w = ga - f.$$

We multiply this equation by $\psi \bar{a} - \psi u$ where

$$\bar{a} = \begin{cases} u & \text{if } u \leq a, \\ a & \text{if } u > a, \end{cases}$$

in terms of the inner product of $L^2(\Omega)$ and integrate by parts in the first term of the left side. This yields

$$\begin{aligned} k \int_{\Omega} \left[\left(\mathcal{M} \nabla (\psi(\bar{a}) - \psi(u)), \nabla (\psi(\bar{a}) - \psi(u)) \right)_R + m(\psi(\bar{a}) - \psi(u))^2 \right] dx + \\ + \int_{\Omega} (gw - ga + f)(\psi(\bar{a}) - \psi(u)) dx = 0. \end{aligned} \quad (1.14)$$

By (1.3), (1.4) and the nonnegativity of $ga - f$ it follows from (1.14) that

$$km_1 \int_{\Omega} (\nabla \psi(\bar{a}) - \nabla \psi(u))^2 dx \leq 0,$$

which implies $\nabla(\psi(\bar{a}) - \psi(u)) = 0$. In view of (0.2) and (1.1) $\bar{a}|_{\partial\Omega} = u|_{\partial\Omega}$. Hence $\bar{a} - u = 0$ that is $u \leq a$ for almost all $x \in \Omega$. \square

2. The main result

In this section we prove the existence and uniqueness theorem for the problem (0.2), (0.3), (0.4). By a solution of this problem is meant the pair involving a function $u \in W$ and a positive real number k which satisfy (0.2)–(0.4).

Hereafter we keep the following notation: for $v_1, v_2 \in W_2^1(\Omega)$

$$\langle Mv_1, v_2 \rangle_M \equiv \int_{\Omega} [(\mathcal{M}\nabla v_1, \nabla v_2)_R + mv_1 v_2] dx.$$

Theorem 2.1. *Let the assumptions I–III and the condition (1.5) be fulfilled. Let also*

$$(i) \ f \in L^2(\Omega), \quad \psi(\beta) \in W_2^{3/2}(\partial\Omega), \quad \omega(x) \in W_2^{3/2}(\partial\Omega), \quad g(x) \in C(\overline{\Omega});$$

$$(ii) \ \omega(x) \text{ be nonnegative in } \partial\Omega;$$

$$0 \leq g(x) \leq g_1 = \text{const} < +\infty \quad x \in \Omega; \quad (2.1)$$

$$\Psi \equiv \langle M\nabla\psi(a), \nabla b \rangle_M > 0; \quad (2.2)$$

$$F(x) \equiv ga - f \geq 0; \quad (2.3)$$

$$\Phi \equiv \varphi - (ga - f, b) > 0. \quad (2.4)$$

Then the problem (0.2)–(0.4) has a solution $(u(x), k)$. Moreover, $u(x) \in L^{2p}(\Omega)$ and

$$u(x) \leq a(x) \quad (2.5)$$

for almost all $x \in \Omega$. If in addition $g \equiv 0$ or the inequalities

$$B \equiv \Psi^{-1} \langle M\psi(a), \psi(a) \rangle_M^{1/2} \langle Mb, b \rangle_M^{1/2} < 2 \quad (2.6)$$

and

$$\Phi > 2c_0 m_1^{-1/2} \text{mes}^{1/n} \Omega \|F\| (2 - B)^{-1} \langle Mb, b \rangle_M^{1/2} \quad (2.7)$$

are valid, then the solution of the problem (0.2)–(0.4) is unique.

Proof. We multiply (0.3) by b in terms of the inner product in $L^2(\Omega)$ and integrate by parts in the first term of the resulting equality twice. In view of (0.3) we obtain

$$-\varphi + k\Psi + \int_{\Omega} g(x)ub \, dx = \int_{\Omega} fb \, dx. \quad (2.8)$$

By (2.2) and the definition of Φ in (2.4) we can rewrite (2.8) as

$$k = (\Phi + (g(a - u), b))\Psi^{-1}. \quad (2.9)$$

If $g \equiv 0$, then $k = \Phi/\Psi > 0$ is known function. By Lemma 2.1 the problem (0.2), (0.3) with such k has a unique solution $u \in L^{2p} \cap W$.

Let now $g \neq 0$ and (2.1), (2.4) holds. We introduce the operator A mapping the set \mathbf{R}^+ of the positive real numbers into \mathbf{R} according to the following role: for any $y \in \mathbf{R}^+$

$$A(y) = (\Phi + (g(a - u_y), b))\Psi^{-1} \quad (2.10)$$

where u_y is a solution of the direct problem (0.2), (0.3) with $k = y$. According to Lemma 1.1, the problem (0.2), (0.3) has a unique solution $u_y \in W$ and hence the value of $A(y)$ defined by (2.10) is meaningful for any $y \in \mathbf{R}^+$. Therefore (2.9) can be consider as the operator equation

$$k = A(k). \quad (2.11)$$

Following the idea of [12, Chapter 1] one can show that the problem (0.2)–(0.4) has a solution if and only if the operator equation (2.11) is solvable.

According to Lemma 1.2 in the hypotheses of the theorem 3.1 $u_y \leq a$ that is (2.5) is valid. By (2.2) and (2.5),

$$0 < k_0 \equiv \frac{\Phi}{\Psi} \leq A(y) \leq \frac{|\varphi| + |(f, b)| + g_1 \|u_y\| \|b\|}{\Psi} \quad (2.12)$$

for $y \in \mathbf{R}^+$. The left inequality in (2.12) enables to estimate $\|u_y\|$. As shown in Lemma 1.1, the estimate (1.12) is valid for $\psi(u_y)$. Therefore for every $y \geq k_0$

$$\|u_y\| \leq c^{-1} (\text{mes} \Omega)^{\frac{p-1}{2p}} \|\psi(u_y)\| \leq c^{-1} (\text{mes} \Omega)^{\frac{p-1}{2p}} \left[\frac{c_0^2 \text{mes}^{2/n} \Omega}{k_0^2 m_1^2} \|ga - f\| + \|\psi(a)\| \right] \equiv C_3. \quad (2.13)$$

Then for $y \geq k_0$

$$k_0 \leq A(y) \leq \frac{|\varphi| + |(f, b)| + g_1 C_3 \|b\|}{\Psi} \equiv K_0. \quad (2.14)$$

This inequality implies that the operator A maps $[k_0, K_0]$ into itself.

Furthermore, A is a continuous operator on $[k_0, K_0]$. Indeed, let $y_1, y_2 \in [k_0, K_0]$ and u_{y_1}, u_{y_2} be solutions to problem (0.2), (0.3) with y_1 and y_2 instead of k , respectively. In accordance with the definition of the operator A , we have

$$A(y_1) - A(y_2) = \Psi^{-1} (g(u_{y_1} - u_{y_2}), b). \quad (2.15)$$

On the other hand, the difference $\bar{w} = u_{y_1} - u_{y_2}$ obeys equation

$$y_1 M(\psi(u_{y_1}) - \psi(u_{y_2})) + g\bar{w} = (y_1 - y_2) M\psi(u_{y_2}) \quad (2.16)$$

and the boundary condition $(u_{y_1} - u_{y_2})|_{\partial\Omega} = 0$. Multiplying (2.16) by $\psi(u_{y_1}) - \psi(u_{y_2})$ in terms of the inner product of $L^2(\Omega)$ and integrating by parts in the first term of the resulting equation yields

$$\begin{aligned} y_1 \langle M(\psi(u_{y_1}) - \psi(u_{y_2})), \psi(u_{y_1}) - \psi(u_{y_2}) \rangle_1 + (g\bar{w}, \psi(u_{y_1}) - \psi(u_{y_2})) = \\ = -(y_1 - y_2) \langle M\psi(u_{y_2}), \psi(u_{y_1}) - \psi(u_{y_2}) \rangle_1. \end{aligned} \quad (2.17)$$

The second term of the left side of (2.17) is nonnegative by the assumption (1.4). Multiplying (0.3) for u_{y_2} by $\psi_a = \psi(u_{y_2}) - \psi(a)$ in terms of the inner product of $L^2(\Omega)$ and integrating by parts in the first term one can obtain

$$y_2 \langle M\psi(u_{y_2}), \psi(u_{y_2}) \rangle_M + (g(u_{y_2} - a), \psi_a) = y_2 \langle M\psi(u_{y_2}), \psi(a) \rangle_M + (f - ga, \psi_a)$$

from where in view of (1.4), (1.9) and the Cauchy inequality it follows that

$$\langle M\psi(u_{y_2}), \psi(u_{y_2}) \rangle_M \leq \langle M\psi(a), \psi(a) \rangle_M + \frac{c_0^2 \text{mes}^{2/n} \Omega}{y_2^2 m_1} \|ga - f\|^2. \quad (2.18)$$

The right-hand side of (2.17) can be estimated with the help of (2.5) and (2.18).

$$\begin{aligned} |y_1 - y_2| \left| \langle M\psi(u_{y_2}), \psi(u_{y_1}) - \psi(u_{y_2}) \rangle_1 \right| \leq \frac{y_1}{2} \langle M(\psi(u_{y_1}) - \psi(u_{y_2})), \psi(u_{y_1}) - \psi(u_{y_2}) \rangle_1 + \\ + \left[\langle M\psi(a), \psi(a) \rangle_M + \frac{c_0^2 \text{mes}^{2/n} \Omega}{y_2^2 m_1} \|ga - f\|^2 \right] \frac{|y_1 - y_2|^2}{2y_1}. \end{aligned} \quad (2.19)$$

Then, by (2.5), (2.18) and (2.19), the equality (2.17) yields

$$\begin{aligned} y_1 \langle M(\psi(u_{y_1}) - \psi(u_{y_2})), \psi(u_{y_1}) - \psi(u_{y_2}) \rangle_1 &\leq \left[\langle M\psi(a), \psi(a) \rangle_M \right. \\ &\quad \left. + \frac{c_0^2 \text{mes}^{2/n} \Omega}{k_0^2 m_1} \|ga - f\|^2 \right] \frac{|y_1 - y_2|^2}{y_1} \equiv C_4 |y_2 - y_1|^2. \end{aligned} \quad (2.20)$$

Next, multiplying (2.16) by b in terms of the inner product of $L^2(\Omega)$ and integrating by parts with the use of the fact that

$$\int_{\partial\Omega} \frac{\partial(y_1\psi(u_{y_1}) - y_2\psi(u_{y_2}))}{\partial N} \omega ds = 0, \quad (2.21)$$

we obtain

$$(g(u_{y_1} - u_{y_2}), b) = -y_1 \langle M(\psi(u_{y_1}) - \psi(u_{y_2})), b \rangle_M - (y_1 - y_2) \langle M\psi(u_{y_2}), b \rangle_M,$$

whence, by (2.18) and (2.20),

$$\begin{aligned} |(g(u_{y_1} - u_{y_2}), b)| &\leq \left[y_1 \langle M(\psi(u_{y_1}) - \psi(u_{y_2})), \psi(u_{y_1}) - \psi(u_{y_2}) \rangle_M^{1/2} + \right. \\ &\quad \left. + |y_1 - y_2| \langle M\psi(u_{y_2}), \psi(u_{y_2}) \rangle_M^{1/2} \right] \langle Mb, b \rangle_M^{1/2} \leq C_5 |y_1 - y_2|. \end{aligned} \quad (2.22)$$

Joining (3.15) and (2.22) we are led to the inequality

$$|A(y_1) - A(y_2)| \leq \frac{C_5}{\Psi} |y_1 - y_2|$$

proving the continuity of A on $[k_0, K_0]$. By Brouwer's theorem, equation (2.11) has a solution $k \in [k_0, K_0]$. This in turn implies the existence of the solution $\{u(x), k\}$ of problem (0.2)–(0.4). The solution satisfies (2.5), (2.13), (2.14). Moreover, by Lemma 1.1 the estimates (1.8)–(1.12) holds with k_0 instead of k in the constants C_1 and C_2 .

The only point remaining concerns the uniqueness of the solution. Let (u_1, k_1) and (u_2, k_2) be two solutions of problem (0.2)–(0.4). By (0.2), $(u_1 - u_2)|_{\partial\Omega} = 0$. Subtracting (0.3) for (u_2, k_2) from (0.3) for (u_1, k_1) , we obtain

$$k_1 M(\psi(u_1) - \psi(u_2)) + g\tilde{u} = -(k_1 - k_2) M\psi(u_2). \quad (2.23)$$

Multiplying this difference by $\psi(u_1) - \psi(u_2)$ in terms of the inner product of $L^2(\Omega)$, integrating by parts and applying the arguments proving (2.20) to the resulting identity one can derive a similar inequality. Namely,

$$\begin{aligned} \langle M(\psi(u_1) - \psi(u_2)), \psi(u_1) - \psi(u_2) \rangle_1 &\leq \\ &\leq \left[\langle M\psi(a), \psi(a) \rangle_M + \frac{c_0^2 \text{mes}^{2/n} \Omega}{k_0^2 m_1} \|ga - f\|^2 \right] \frac{|k_1 - k_2|^2}{k_1^2}. \end{aligned} \quad (2.24)$$

Furthermore, we multiply (2.23) by b in terms of the inner product of $L^2(\Omega)$. Integrating by parts and taking into account (2.21) for $y_1 = k_1$ and $y_2 = k_2$ we are lead to the identity

$$(g(u_1 - u_2), b) = -k_1 \langle M(\psi(u_1) - \psi(u_2)), b \rangle_M - (k_1 - k_2) [\langle M(\psi(u_2) - \psi(a), b) \rangle_M + \Psi]. \quad (2.25)$$

Subtracting (2.9) for (u_2, k_2) from (2.9) for (u_1, k_1) , we obtain

$$k_1 - k_2 = Ak_1 - Ak_2 = \frac{(g(u_1 - u_2), b)}{\Psi}. \quad (2.26)$$

Without loss of generality we can suppose that $k_1 \geq k_2$. Then $(g(u_1 - u_2), b) \geq 0$. Let us rewrite (2.25) as

$$\begin{aligned} (g(u_1 - u_2), b) + \Psi(k_1 - k_2) = \\ = -\langle k_1 M(\psi(u_1) - \psi(u_2)) + (k_1 - k_2)M(\psi(u_2) - \psi(a), b) \rangle_M. \end{aligned} \quad (2.27)$$

We can estimate the right term of (2.27) with the help of (1.8), (2.14), (2.24).

$$\begin{aligned} -k_1 \langle M(\psi(u_1) - \psi(u_2)), b \rangle_M - (k_1 - k_2) \langle M(\psi(u_2) - \psi(a), b) \rangle_M \leq \\ \leq k_1 \langle M(\psi(u_1) - \psi(u_2)), \psi(u_1) - \psi(u_2) \rangle_M^{1/2} \langle Mb, b \rangle_M^{1/2} + \\ + (k_1 - k_2) \langle M(\psi(u_2) - \psi(a), \psi(u_2) - \psi(a)) \rangle_M^{1/2} \langle Mb, b \rangle_M^{1/2} \leq \\ \leq \left[\frac{2c_0 \Psi \text{mes}^{1/n} \Omega}{\Phi m_1^{1/2}} \|ga - f\| + \langle M\psi(a), \psi(a) \rangle_M^{1/2} \right] \langle Mb, b \rangle_M^{1/2} (k_1 - k_2). \end{aligned} \quad (2.28)$$

Thus, (2.26)–(2.28) lead to the inequality

$$Ak_1 - Ak_2 \leq \frac{1}{2} \left[\frac{2c_0 \text{mes}^{1/n} \Omega}{\Phi m_1^{1/2}} \|ga - f\| + \Psi^{-1} \langle M\psi(a), \psi(a) \rangle_M^{1/2} \right] \langle Mb, b \rangle_M^{1/2} (k_1 - k_2)$$

proving the contractibility of the operator A by (2.6) and (2.7), from which and (2.26) it follows that $k_1 - k_2 = 0$ and in view of (1.4), (2.24) $u_1 - u_2 = 0$ for almost all $x \in \Omega$. \square

The condition $\psi(u) \geq c|u|^p$ for $p > 0$ provides that $u \in L^{2p}(\Omega)$ and $\psi(u) \in W_2^2(\Omega)$. This condition can be ignored under additional assumptions on f , β and ψ .

Theorem 2.2. *Let the assumptions I–III and the hypotheses (i) and (ii) of Theorem 3.1 be fulfilled. Let also $\beta \in L^\infty(\Omega)$, $f \geq 0$ and $\beta \geq \rho_0 = \min\{0, \rho^*\}$ for almost all $x \in \partial\Omega$ where a real number ρ^* is such that $\psi(\rho^*) = 0$. Then the problem (0.2), (0.3), (0.4) has a solution $(u(x), k)$. Moreover, $u(x) \in L^\infty(\Omega)$ and*

$$\rho_0 \leq u(x) \leq a(x) \quad (2.29)$$

for almost all $x \in \Omega$. If in addition $g \equiv 0$ or the inequalities (2.6) and (2.7) are valid, then the solution of the problem (0.2), (0.3), (0.4) is unique.

Proof. The proof of the theorem repeats the proof of Theorem 3.1 almost at all. We need only establish (2.29) and show that the solution of the direct problem (0.2), (0.3) belongs to W for every $k > 0$ because in the hypotheses of the theorem this fact is not evidenced by Lemma 2.1.

We first note that by Lemma 2.1 the direct problem (0.2), (0.3) has a unique solution u and $\psi(u) \in W_2^1(\Omega)$ for every $k > 0$. By Lemma 2.2 the solution satisfy the right inequality of (2.29). Moreover, by the maximum principle for linear elliptic equations, $\psi(a) \geq \psi(\rho_0)$ and hence $a \geq \rho_0$ for almost all $x \in \Omega$. In order to prove the left part of (2.29) we define the function

$$\bar{u} = \begin{cases} u & u < \rho_0, \\ \rho_0 & u \geq \rho_0 \end{cases}$$

and multiply (0.3) by $\bar{h} = \psi(\bar{u}) - \psi(\rho_0)$ in terms of the inner product of $L^2(\Omega)$. Integrating by parts in the first term of the resulting identity gives

$$k \langle M\bar{h}, \bar{h} \rangle_M + \int_{\Omega} \{k m\psi(\rho_0) + g\bar{u} - f\} \bar{h} dx = 0.$$

In the hypotheses of the theorem the second term is nonnegative. Therefore, by (1.3), the last equality implies $\nabla \bar{h} = 0$ from which we conclude that $\bar{h} = 0$ for almost all $x \in \Omega$ because $\bar{h}|_{\partial\Omega} = 0$. In view of monotonicity of ψ this means that $u \geq \rho_0$ for almost all $x \in \Omega$.

As mentioned above, if $\beta \in L^\infty(\partial\Omega)$, then $\psi(a) \in L^\infty(\Omega)$ by the continuity of ψ and Theorem 13.1 of [9, Chapter 3]. In the hypotheses of the theorem this means that $a \in L^\infty(\Omega)$. Hence $u \in L^\infty(\Omega)$ and $\|u\|_{L^\infty(\Omega)} \leq \max\{|\rho_0|, \|a\|_{L^\infty(\Omega)}\}$ in view of (2.29). Repeating the arguments led to (1.12) one can obtain the inequality

$$\|M\psi(u)\| \leq \frac{\sqrt{2}}{k} \left[\|g\|_{C(\bar{\Omega})}^2 \max\{|\rho_0|^2, \|a\|_{L^\infty(\Omega)}^2\} \text{mes}^2\Omega + \|f\|^2 \right]^{1/2} \equiv C_2''.$$

By (1.8), (1.9) and Theorem 5.1 of [11, Chapter 2] the last inequality implies

$$\|\psi(u)\|_2 \leq c(C_2'' + C_1) + \|\psi(a)\|_2(c + 1),$$

which proves that the solution u belongs to W . \square

Certain inverse problems can be reduced to the problem (0.2)–(0.4). This allows to establish the existence and uniqueness theorems for such problems with the help of Theorems 2.1 and 2.2. In particular, the problem of the identification of the constant coefficient ν in equation

$$-\nu \text{div}(\mathcal{M}(x) \nabla \psi_1(v)) + g(x) \psi_2(v) = f(x) \quad (2.30)$$

with the boundary condition $v|_{\partial\Omega} = \mu(x)$ and the condition of overdetermination

$$\nu \int_{\partial\Omega} \frac{\partial \psi_1(v)}{\partial \bar{N}} \omega \, ds = \varphi_1$$

are reduced to the problem (0.2)–(0.4) if the function $\psi_2(\rho)$ is an injection in $(-\infty, +\infty)$. Indeed, we can define a function $u = \psi_2(v)$ and rewrite this problem as (0.2)–(0.4) with $k = \nu$, $\psi(\rho) = \psi_1(\psi_2^{-1}(\rho))$, $\beta = \psi_2(\mu)$ and $\varphi = \varphi_1$.

The inverse problem (0.2), (0.3) with the condition of overdetermination given only on a part Γ of the boundary $\partial\Omega$, that is

$$k \int_{\Gamma} \frac{\partial \psi(u)}{\partial \bar{N}} \omega \, ds = \varphi, \quad (2.31)$$

can also be reduced to the problem (0.2), (0.3), (0.4). If the function $\omega \in W_2^{3/2}(\Gamma)$ is finite on Γ and $\text{supp } \omega \subset \Gamma$ then one can extend ω into $\partial\Omega$ setting $\omega = 0$ in $\partial\Omega \setminus \Gamma$ and consider the integral in (2.31) over the whole boundary $\partial\Omega$. In this case Theorem 2.1 is formulated as follows.

Theorem 2.3. *Let the assumptions I–III and (1.5) be fulfilled, $f \in L^2(\Omega)$, $\psi(\beta) \in W_2^{3/2}(\partial\Omega)$, $\omega(x) \in W_2^{3/2}(\Gamma)$, $g(x) \in C(\bar{\Omega})$. Let also $\omega(x)$ be nonnegative and finite on Γ , $\text{supp } \omega \subset \Gamma$, $\omega(x) = 0$ on $\partial\Omega \setminus \Gamma$ and (2.1)–(2.4) hold. Then the problem (0.2), (0.3), (2.31) has a solution $(u(x), k)$. Moreover, $u(x)$ satisfies (2.5) for almost all $x \in \Omega$. If in addition $g \equiv 0$ or (2.6), (2.7) are fulfilled, then the solution of the problem (0.2), (0.3), (2.31) is unique.*

Theorem 2.2 can be reformulate for the problem (0.2), (0.3), (2.31) in a similar manner.

Theorem 2.4. *Let the assumptions I–III be fulfilled, $f \in L^2(\Omega)$, $\psi(\beta) \in W_2^{3/2}(\partial\Omega)$, $\omega(x) \in W_2^{3/2}(\Gamma)$, $g(x) \in C(\bar{\Omega})$, $\beta \in L^\infty(\Omega)$. Let also (2.1)–(2.4) hold, $f \geq 0$ for almost all $x \in \Omega$, the function $\omega(x)$ be nonnegative and finite on Γ , $\text{supp } \omega \subset \Gamma$, $\omega(x) = 0$ on $\partial\Omega \setminus \Gamma$ and $\beta \geq \rho_0$ for almost all $x \in \partial\Omega$. Then the problem (0.2), (0.3), (2.31) has a solution $(u(x), k)$. Moreover, $u(x) \in L^\infty(\Omega)$ and (2.29) is fulfilled for almost all $x \in \Omega$. If in addition $g \equiv 0$ or the inequalities (2.6) and (2.7) are valid, then the solution of the problem (0.2), (0.3), (2.31) is unique.*

3. Examples and comments

As mentioned above the interest to the identification of coefficients in the elliptic equations, among them (0.3), is due to its extensive applications. Certain examples of such problems were considered in [6]. In particular, the special case of (0.3)

$$-k \operatorname{div}(\mathcal{M}(x)|u|^{p-2}\nabla u) = 0 \quad (3.1)$$

describes the steady-state isothermal gas filtration [13] where $p = 3$, $\mathcal{M}(x) = (m\mu_1)^{-1}\mathbf{E}$, u is the pressure of a liquid in the pores, μ_1 is the viscosity of liquid, k and m are the average permeability and the porosity of the rock. In this case $\psi(\rho) = (p-1)^{-1}|\rho|^{p-2}\rho$ and $g \equiv 0$. The equation (4.1) satisfies the assumptions I–III. If the other hypotheses of Theorem 2.1 coincide, then the inverse problem (3.1), (0.2), (0.4) has a unique solution.

In [14] the physical model of the non-equilibrium effects in a simultaneous flow of immiscible fluids in porous media is presented. For the steady-state flow with capillary counter-current imbibition the basic equation for the effective water saturation assumes the form

$$k\Delta\psi(u) = 0 \quad (3.2)$$

where u is the actual water saturation, the coefficient k depends on the air permeability of the porous medium, its porosity (the relative volume occupied by the pores), the surface tension and a (conventional) contact angle at the triple water-oil-solid boundary,

$$\psi(\rho) = - \int_0^\rho \frac{f_1(s)f_2(s)}{f_1(s) + f_2(s)} J'(s) ds.$$

The dimensionless nonnegative quantities f_1 and f_2 are called the relative permeabilities and satisfy the inequality $0 \leq f_i \leq 1$. The function Leverett J is the dimensionless capillary pressure. $f_1(s)$ is a monotonic non-decreasing smooth function. The function $f_2(s)$ is a monotonic non-increasing smooth one. Finally, the nonnegative function $J(u)$ monotonically decreases. The function ψ is continuous and strictly increases on $(-\infty, +\infty)$. Therefore under the hypotheses of Theorem 3.2 the inverse problem (0.2), (0.4), (3.2) has a unique solution $(u(x), k)$.

Some kinds of equations (2.30) are involved in modeling the electric fields of semiconductors. In the absence of an external electric field the stationary nonlinear dissipative equation for the potential u of the self-consistent electric field of a semiconductor takes the form [15]

$$-k\Delta(u + \frac{\varepsilon}{2}u^2 + \frac{\varepsilon^2}{6}u^3) + \lambda|u|^q u = 0$$

where the positive parameters k and λ depends on the electric susceptibility of the semiconductor, the parameter $\varepsilon > 0$ depends on the temperature of free electrons, $q \geq 0$. These equations can be reduced to (0.3) with the appropriate function $\psi(\rho)$ satisfying the assumption IV.

The last point of our considerations is the stability of the solution to (0.2), (0.3), (0.4) with respect to φ . One can prove it much as the contractibility of the operator A was established in Theorem 3.1.

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Об одной обратной задаче для квазилинейного эллиптического уравнения

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Исследуется задача идентификации неизвестного постоянного коэффициента в старшем члене уравнения с частными производными $-kM\psi(u) + g(x)u = f(x)$ при граничном условии Дирихле. Здесь $\psi(u)$ – нелинейная возрастающая функция от u , M – линейный самосопряженный эллиптический оператор второго порядка. Коэффициент k восстанавливается по дополнительным интегральным данным на границе. Доказывается существование и единственность решения обратной задачи, включающего функцию u и положительное действительное число k .

Ключевые слова: обратная задача, краевая задача, эллиптическое уравнение второго порядка, теорема существования и единственности, фильтрация.