EDN: NUMFOQ
УДК 517.55+517.51

$m-cv$ measure $\omega^*(x,E,D)$ and condenser capacity $C(E,D)$ in the class m-convex functions

Azimbay Sadullaev*
National University of Uzbekistan
V. I. Romanovsky Institute of Mathematics
of the Academy of Sciences of Uzbekistan
Tashkent, Uzbekistan

Rasulbek Sharipov†
Urgench State University
Urgench, Uzbekistan

Mukhiddin Ismoilov‡
National University of Uzbekistan
Tashkent, Uzbekistan

Received 10.08.2024, received in revised form 15.09.2024, accepted 24.10.2024

Abstract. In this work we study very basic concepts of potential theory: polar sets and $m-cv$ measures in the class of m-convex functions in real space $\mathbb{R}^n$. We also study capacity of condenser $C(E,D)$ in the class m-convex functions and will prove a number of its potential properties.

Keywords: m-subharmonic function, convex function, m-convex function, $m-cv$ polar set, $m-cv$ measure, Borel measures, Hessians.

Citation: A. Sadullaev, R. Sharipov, M. Ismoilov, $m-cv$ measure $\omega^*(x,E,D)$ and condenser capacity $C(E,D)$ in the class m-convex functions, J. Sib. Fed. Univ. Math. Phys., 2025, 18(3), 385–399. EDN: NUMFOQ.

1. Introduction and preliminaries

Let $u(x)\in C^2(D)$ be a twice smooth function in the domain $D\subset\mathbb{R}^n$. Then the matrix $\left(\dfrac{\partial^2 u}{\partial x_j\partial x_k}\right)$ is symmetric, $\dfrac{\partial^2 u}{\partial x_j\partial x_k}=\dfrac{\partial^2 u}{\partial x_k\partial x_j}$. Therefore, after a suitable orthonormal transformation, it can be transformed into a diagonal form

$$\left(\frac{\partial^2 u}{\partial x_j\partial x_k}\right)\to\begin{pmatrix}\lambda_1 & 0 & \dots & 0\\ 0 & \lambda_2 & \dots & 0\\ \dots & \dots & \dots & \dots\\ 0 & 0 & \dots & \lambda_n\end{pmatrix},$$

where $\lambda_j=\lambda_j(x)\in\mathbb{R}$ are the eigenvalues of the matrix $\left(\dfrac{\partial^2 u}{\partial x_j\partial x_k}\right)$. Let

$$H^k(u)=H^k(\lambda)=\sum_{1\leqslant j_1<\dots<j_k\leqslant n}\lambda_{j_1}\dots\lambda_{j_k}$$

*sadullaev@mail.ru
†sharipovr80@mail.ru https://orcid.org/0000-0002-3033-3047
‡mukhiddin4449@gmail.com https://orcid,o19/0009-0005-9339-0582

be the Hessian of dimension k of the vector $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$.

Definition 1.1. *A twice smooth function $u(x) \in C^2(D)$ is called m-convex in $D \subset \mathbb{R}^n$, $u \in m-cv(D)$, if its eigenvalue vector $\lambda = \lambda(x) = (\lambda_1(x), \lambda_2(x), \dots, \lambda_n(x))$ satisfies the conditions*

$$m - cv \cap C^2(D) = \left\{ H^k(u) = H^k\left(\lambda(x)\right) \geqslant 0,\ \forall x \in D,\ \ k = 1, \dots, n-m+1 \right\}.$$

When $m = n$ the class $n - cv$ coincides with the class of subharmonic functions $sh = \{\lambda_1 + \lambda_2 + \dots + \lambda_n \geqslant 0\}$, when $m = 1$ it coincides with the class of convex functions $cv = \{\lambda_1 \geqslant 0, \lambda_2 \geqslant 0, \dots, \lambda_n \geqslant 0\}$, moreover $cv = 1 - cv \subset 2 - cv \subset \dots \subset n - cv = sh$. The theory of subharmonic functions is a developed and important part of theory functions and mathematical physics. The theory of convex functions is well studied and reflected in the works of A. Aleksandrov, I. Bakelman, A. Pozdnyak and others (see [2–5]). When $m > 1$ this class was studied in the series of works by N. Ivochkina, N. Trudinger, X. Wang et al. [11, 19–21] (see also [8]).

If we want to construct a good theory of $m - cv$ functions, then the class of functions $C^2(D)$ is not enough. For example, if we want to solve the equation

$$\begin{aligned} &H^{n-m+1}\left(u\right) = f\left(u, x\right), \\ &u|_{\partial D} = \varphi \end{aligned}$$

or want to work with extreme $m - cv$ functions, such as maximal $m - cv$ functions, we need to extend the definition of $m - cv$ functions to a wider class of upper semi-continuous functions. In the work of N.Trudinger, X.Wang [21] $m - cv$ functions are introduced in the class of upper semi-continuous functions $u(x)$ in the domain $D \subset \mathbb{R}^n$, using the so-called "viscous" definition, that is $H^k(q) \geqslant 0,\ k = 1, 2, \dots, n-m+1$, for any quadratic polynomial $q(x)$, such that the difference $u(x) - q(x)$ has only a finite number of local maximum in the domain D. In addition, in this work $H^{n-m+1}(u)$ (maximum degree operator) is defined as a Borel measure and with the help of this operator the capacity of condenser $C(E, D)$ was introduced, a number of potential properties of this capacity was proved.

To expand the domain of definition of $m - cv$ functions from $C^2(D)$ to a wider class of semi-continuous functions, we have proposed a completely new approach, the connection of $m-cv$ functions with m-subharmonic (sh_m) functions in complex space $\mathbb{C}^n$. The theory of sh_m-functions is well developed and is currently subject of study by many mathematicians (Z. Błocki [6], S. Dinew and S. Kolodziej [9,10], S. Y. Li [13], H. C .Lu [14,15] and etc). Quite a complete overview of this theory is available in the survey article by A. Sadullaev and B. Abdullaev [1] in proceedings of Mathematical Institute of the RAS.

Let us recall that the theory of the sh_m-functions is based on differential forms and currents $(dd^c u)^k \wedge \beta^{n-k} \geqslant 0,\ \ k = 1, 2, \dots, n-m+1$, where $\beta = dd^c \|z\|^2$ is a standard volume form in $\mathbb{C}^n$. A twice smooth function $u(z) \in C^2(D)$, $D \subset \mathbb{C}^n$ is called strongly m-subharmonic $u \in sh_m(D)$, if at each point of the domain D

$$\begin{aligned} sh_m(D) &= \left\{ u \in C^2 :\ (dd^c u)^k \wedge \beta^{n-k} \geqslant 0,\ \ k = 1, 2, \dots, n-m+1 \right\} = \\ &= \left\{ u \in C^2 :\ dd^c u \wedge \beta^{n-1} \geqslant 0, (dd^c u)^2 \wedge \beta^{n-2} \geqslant 0, \dots, (dd^c u)^{n-m+1} \wedge \beta^{m-1} \geqslant 0 \right\}, \end{aligned} \tag{1}$$

where $\beta = dd^c \|z\|^2$ is a standard volume form in $\mathbb{C}^n$.

Operators $(dd^c u)^k \wedge \beta^{n-k}$ are closely related to the Hessians. For a twice smooth function $u \in C^2(D)$, the second-order differential $dd^c u = \dfrac{i}{2} \sum\limits_{j,k} \dfrac{\partial^2 u}{\partial z_j \partial \bar{z}_k} dz_j \wedge d\,\bar{z}_k$ (at a fixed point $o \in D$) is a Hermitian quadratic form. After a suitable unitary coordinate transformation, it is reduced to a diagonal form $dd^c u = \frac{i}{2}\left[\lambda_1 dz_1 \wedge d\,\bar z_1 + \cdots + \lambda_n dz_n \wedge d\,\bar z_n\right]$, where $\lambda_1, \ldots, \lambda_n$ are eigenvalues of the Hermitian matrix $\left(\frac{\partial^2 u}{\partial z_j \partial\,\bar z_k}\right)$, which are real: $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$. Note that the unitary transformation does not change the differential form $\beta = dd^c \|z\|^2$. It is easy to see that

$$(dd^c u)^k \wedge \beta^{n-k} = k!(n-k)!H^k(u)\beta^n, \tag{2}$$

where $H^k(u) = \sum_{1\leqslant j_1<\cdots<j_k\leqslant n} \lambda_{j_1}\ldots\lambda_{j_k}$ is the Hessian of dimension k of the vector $\lambda = \lambda(u) \in \mathbb{R}^n$.

Hence, the twice smooth function $u(z) \in C^2(D)$, $D \subset \mathbb{C}^n$ is strongly m-subharmonic if at each point $o \in D$ it satisfies the following inequalities

$$H^k(u) = H^k_o(u) \geqslant 0, \;\; k = 1, 2, \ldots, n-m+1. \tag{3}$$

Note that, the concept of the strongly m-subharmonic functions in a generalized sense is also defined for upper-semicontinuous functions.

Definition 1.2. *The function $u(z)$ defined in a domain $D \subset \mathbb{C}^n$ is called sh_m, if it is upper semi-continuous and for any twice smooth sh_m functions $v_1, \ldots, v_{n-m} \in C^2(D) \cap sh_m(D)$ the current $dd^c u \wedge dd^c v_1 \wedge \cdots \wedge dd^c v_{n-m} \wedge \beta^{m-1}$ defined as*

$$\begin{aligned}&\left[dd^c u \wedge dd^c v_1 \wedge \cdots \wedge dd^c v_{n-m} \wedge \beta^{m-1}\right](\omega) = \\ &= \int u\, dd^c v_1 \wedge \cdots \wedge dd^c v_{n-m} \wedge \beta^{m-1} \wedge dd^c\omega, \;\; \omega \in F^{0,0}.\end{aligned} \tag{4}$$

is positive, $\int u dd^c v_1 \wedge \cdots \wedge dd^c v_{n-m} \wedge \beta^{m-1} \wedge dd^c\omega \geqslant 0 \quad \forall\omega \in F^{0,0}, \; \omega \geqslant 0$. Here $F^{0,0}(D)$ is a family of infinitely smooth finite in D functions.

In the Błocki's work [6] it was proved that, this definition is correct, that for $u \in C^2(D)$ functions this definition coincides with the initial definition of sh_m-functions.

2. Relation between $m-cv$ and sh_m functions

To establish a connection between $m-cv$ functions and sh_m functions, we embed a real space $\mathbb{R}^n_x$ into a complex space $\mathbb{C}^n_z$, $\mathbb{R}^n_x \subset \mathbb{C}^n_z = \mathbb{R}^n_x + i\mathbb{R}^n_y\,(z = x+iy)$, as a real n-dimensional subspace. Then, we extend the function $u(x)$, given in the domain $D \subset \mathbb{R}^n_x$ into domain $\Omega = D \times i\mathbb{R}^n_y \subset \mathbb{C}^n_z$ as $u^c(z) = u^c(x+iy) = u(x)$, by assuming it is a constant on parallel planes $\Pi_{x^0} = \left\{z \in \mathbb{C}^n : \; x = x^0, \; y \in \mathbb{R}^n\right\}$.

Theorem 2.1 (see [16,18])**.** *A twice smooth function $u(x) \in C^2(D)$, $D \subset \mathbb{R}^n_x$, is $m-cv$ in D, if and only if a function $u^c(z) = u^c(x+iy) = u(x)$, that does not depend on variables $y \in \mathbb{R}^n_y$, is sh_m in the domain Ω.*

Theorem 2.1 allows us to define a m-convex function in the class of semi-continuous functions.

Definition 2.1. *An upper semi-continuous function $u(x)$ in a domain $D \subset \mathbb{R}^n_x$ is called m-convex in D, if the function $u^c(z)$ is strongly m-subharmonic, i.e. $u^c(z) \in sh_m(\Omega)$.*

This definition is convenient in the study of m-convex functions, by transferring well-known properties of sh_m-functions to the class $m-cv$. We present some non-trivial ones:

– *(Approximation).* We take a standard kernel $K_\delta(x) = \frac{1}{\delta^n} K\left(\frac{x}{\delta}\right)$, $\delta > 0$, where
– $K(x) = K(|x|)$;
– $K(x) \in C^\infty(\mathbb{R}^n)$;
– $\text{supp}K = B(0,1)$;
– $\int\limits_{\mathbb{R}^n} K(x)dx = \int\limits_{B(0,1)} K(x)dx = 1$.
Then the convolution

$$u_\delta(y) = \int_D u(x)K_\delta(x-y)dx = \int_{\mathbb{R}^n} u(x+y)K_\delta(x)dx \tag{5}$$

has the property, that $u_\delta(x) \in m-cv(D_\delta)$, where $D_\delta = \{x \in D : \, dist(x, \partial D) > \delta\}$, $u_\delta(x)$ decreases as $\delta \downarrow 0$ and converges point wise to the function $u(x) \in m-cv(D)$.

– the limit of a uniformly convergent or decreasing sequence of $m-cv$ functions is $m-cv$;

– the maximum of a finite number of $m-cv$ functions is an $m-cv$ function;

– for an arbitrary locally uniformly bounded family, $\{u_\theta\} \subset m-cv$ the regularization $u^*(x)$ of the supremum $u(x) = \left\{\sup\limits_\theta u_\theta(x)\right\}$ will also be an $m-cv$ function. Since $m-cv \subset sh$, then the set $\{u(x) < u^*(x)\}$ is polar in $\mathbb{C}^n \approx \mathbb{R}^{2n}$. In particular, it has Lebesgue measure zero.

Similarly, for a locally uniformly bounded sequence, $\{u_j\} \subset m-cv$ the regularization $u^*(x)$ of the limit $u(x) = \overline{\lim\limits_{j\to\infty}} u_j(x)$ will also be an $m-cv$ function, and the set $\{u(x) < u^*(x)\}$ is polar;

– if $u(x) \in m-cv(D)$, then for any hyperplane $\Pi \subset \mathbb{R}^n$ the restriction $u|_\Pi \in m-cv(D \cap \Pi)$.

From this property it easily follows that if $u(x) \in m-cv(D)$, then for any plane $\Pi \subset \mathbb{R}^n$, $\dim \Pi = m$, the restriction $u|_\Pi \in sh(D \cap \Pi)$.

For $m = 1$ it is not difficult to prove that a convex function $u(x) \in 1-cv(D)$ belongs to Lipschitz class, i.e. $u(x) \in Lip(D)$. In the work [20] N.Trudinger and X.Wang proved a generalization of this remarkable result, that any m-convex function $u(x) \in m-cv$ at $m < \frac{n}{2}+1$ is Hölder with exponent $\alpha = 2 - \frac{n}{n-m+1}$, $u(x) \in Lip_\alpha(D)$.

Example 2.1. (*fundamental $m-cv$ function*).

$$\chi_m(x,0) = \begin{cases} |x|^{2-\frac{n}{n-m+1}} & \text{if } m < \frac{n}{2}+1 \\ \ln|x| & \text{if } m = \frac{n}{2}+1 \\ -|x|^{2-\frac{n}{n-m+1}} & \text{if } m > \frac{n}{2}+1 \end{cases} \tag{6}$$

Thus, for $m < \frac{n}{2}+1$ the fundamental function is bounded and Lipschitz, and for $m \geqslant \frac{n}{2}+1$ it is equal to $-\infty$ at the point $x = 0$. Note that for $m = n$, i.e. for the subharmonic case it coincides with the fundamental solution $-\frac{1}{|x|^{n-2}}$ of the Laplace operator Δ.

3. $m-cv$ polar sets and $m-cv$ measure

Definition 3.1. *By analogy with polar sets in classical potential theory, a set $E \subset D \subset \mathbb{R}^n$ is called $m-cv$ polar in D, if there exists a function $u(x) \in m-cv(D)$, $u(x) \not\equiv -\infty$, such that $u|_E = -\infty$.*

From the embedding $m-cv(D)\subset sh(D)$ it follows that every $m-cv$ polar set is polar in the sense of classical potential theory. In particular, for a $m-cv$ polar set E it is true $\mathcal{H}_{2n-2+\varepsilon}(E)=0,\ \forall\varepsilon>0$: and, therefore, the Lebesgue measure of a $m-cv$ polar set E is equal to zero.

$m-cv$ polar sets have another unexpected phenomenon, that when $m<\frac{n}{2}+1$ they are empty, i.e. if the set $E\subset D$ is $m-cv$ polar, $m<\frac{n}{2}+1$, then $E=\varnothing$. This follows from the fact that for $m<\frac{n}{2}+1$ any $m-cv$ function is Hölder continuous (see section 2). However, for $m\geqslant\frac{n}{2}+1$ there are non-empty $m-cv$ polar sets. Therefore, the properties of $m-cv$ polar sets proved below are meaningful only for the cases $m\geqslant\frac{n}{2}+1$.

Theorem 3.1. *The countable union of $m-cv$ polar sets is $m-cv$ polar, i.e. if $E_j\subset D$ is $m-cv$ polar, then $E=\bigcup\limits_{j=1}^{\infty}E_j$ is also $m-cv$ polar.*

The proof is identical to a similar proof for polar sets and we omit it.

Potential theory is usually constructed in regular domains with respect to one or another class of functions.

Definition 3.2. *A domain $D\subset\mathbb{R}^n$ is called $m-cv$ regular if there exists $\rho(x)\in m-cv(D)$ such that $\rho(x)<0,\ \varliminf\limits_{x\to\partial D}\rho(x)=0$. It is called strictly $m-cv$ regular if there exists a twice smooth strictly $m-cv$ function in some neighborhood of the closure $D^+\supset\bar D$ such that $D=\{\rho(x)<0\}$. Strictly m-convexity of the function $\rho(x)$ in D^+ means that for some $\delta>0$ the difference $\rho(x)-\delta\|x\|^2$ is an $m-cv$ function in D^+.*

In the theory of m-convex functions, $m-cv$ measure plays the same role as the harmonic measure in classical potential theory. To exclude trivial cases, $m-cv$ regular or even strictly $m-cv$ regular domains are usually taken as a fixed domain $D\subset\mathbb{R}^n$.

Let $E\subset D$ be some subset of a strictly $m-cv$ regular domain $D\subset\mathbb{R}^n$.

Definition 3.3. *Consider the class of functions*

$$\mathcal{U}(E,D)=\{u(x)\in m-cv(D):\ u|_D\leqslant 0,\ \ u|_E\leqslant -1\}\tag{7}$$

and put $\omega(x,E,D)=\sup\{u(x):\ u\in\mathcal{U}(E,D)\}$. Then the regularization $\omega^*(x,E,D)$ is called $m-cv$ measure of the set E with respect to the domain D.

From the property of the upper envelope of $m-cv$ functions it follows that $\omega^*(x,E,D)\in m-cv(D)$. By Choquet's lemma (see [12, 17]) there is a countable subfamily $\mathcal{U}'\subset\mathcal{U}(E,D)$ such that $\{\sup\{u(x)\}:\ u(x)\in\mathcal{U}'(E,D)\}^*\equiv\omega^*(x,E,D)$. It follows that an $m-cv$ measure $\omega^*(x,E,D)$ can be represented as a limit of a monotonically increasing sequence $\{u_j(x)\}\subset\mathcal{U}(E,D):\ \left[\lim\limits_{j\to\infty}u_j(x)\right]^*\equiv\omega^*(x,E,D)$.

In the particular case when $E\subset\subset D$ is compact, the functions $u_j(x)\in\mathcal{U}(E,D)$ can be chosen to be continuous in D, which can be easily verified by continuing $u_j(x)\in\mathcal{U}(E,D)$ into some fixed neighborhood $D^+\supset\bar D$ and then approximating them with smooth functions $u_{jk}=u_j\circ K_k(x-y)\in m-cv\left(D^+\right)\cap C^\infty\left(D^+\right),\ j,k=1,2,\ldots$, we can fined a sequense $u_{jk_j}\in m-cv\left(D^+\right)\cap C^\infty\left(D^+\right)$ monotonically increasing and $\left\{u_{jk_j}(x)\right\}\subset\mathcal{U}(E,D):\ \left[\lim\limits_{j\to\infty}u_{jk_j}(x)\right]^*\equiv\omega^*(x,E,D)$.

Properties of $m-cv$ measures:

1) (*monotonicity*) if $E_1 \subset E_2$, then $\omega^*(x,E_1,D) \geqslant \omega^*(x,E_2,D)$; if $E \subset D_1 \subset D_2$, then $\omega^*(x,E,D_1) \geqslant \omega^*(x,E,D_2)$.

2) $\omega^*(x,U,D) \in \mathcal{U}(U,D)$ for open sets $U \subset D$ and, therefore $\omega^*(x,U,D) \equiv \omega(x,U,D)$;

This property follows from the fact that for concentric balls $B(x^0,r) \subset B(x^0,R) \subset\subset U$, $0<r<R$, an $m-cv$ measure

$$\omega^*(x,B(x^0,r),\,B(x^0,R)) = \max\left\{-1,\ \frac{\chi_m\left(x,x^0\right)-\chi_m\left(R,x^0\right)}{\chi_m\left(R,x^0\right)-\chi_m\left(r,x^0\right)}\right\}$$

and therefore in both cases $m<\frac{n}{2}+1$ or $m\geqslant\frac{n}{2}+1$ we have $\omega^*(x^0,U,D)=-1$. Here $\chi_m\left(x,x^0\right)$ is a fundamental $m-cv$ function (see (6)).

3) If $U \subset D$ is an open set, $U=\bigcup\limits_{j=1}^{\infty} K_j$, where $K_j \subset \overset{\circ}{K}_{j+1}$, then $\omega^*(x,K_j,D) \downarrow \omega(x,U,D)$ (easily follows from property 2).

4) If $E \subset D$ an arbitrary set, then there is a decreasing sequence of open sets $U_j \supset E$, $U_j \supset U_{j+1}$ $(j=1,2,\dots)$, such that $\omega^*(x,E,D)=\left[\lim\limits_{j\to\infty}\omega(x,U_j,D)\right]^*$.

In fact, if $\{u_j(x)\} \subset \mathcal{U}(E,D)$ is monotonically increasing such that $\left[\lim\limits_{j\to\infty}u_j(x)\right]^* \equiv \omega^*(x,E,D)$, then an open set $U_j=\left\{u_j<-1+\frac{1}{j}\right\}$ has the property as $U_j \supset E$, $U_j \supset U_{j+1}$ $(j=1,2,\dots)$ and

$$\omega^*(x,E,D) \leqslant \omega(x,U_j,D) \leqslant u_j(x)+\frac{1}{j}.$$

Hence $\omega^*(x,E,D)=\left[\lim\limits_{j\to\infty}\omega(x,U_j,D)\right]^*$.

5) a $m-cv$ measure $\omega^*(x,E,D)$ is either nowhere equal to zero or identically equal to zero. $\omega^*(x,E,D)\equiv 0$ if and only if E is $m-cv$ polar in D.

Remark 3.1. Property 5 is meaningful only if $m\geqslant\frac{n}{2}+1$. At $m<\frac{n}{2}+1$ non-empty $m-cv$ polar set does not exist, so the trivial $m-cv$ measure $\omega^*(x,E,D)\equiv 0$ does not exist.

Example 3.1. Consider $m=1$, a ball $B=B(0,1)$ and a set in it $E=\{0\}$, consisting of one point. Consider a $1-cv$ measure $\nu=\omega^*(x,E,B)$, $x\in\mathbb{R}^n$, $\nu\in\mathbb{R}$ as a function in $\mathbb{R}^{n+1}_{(x,\nu)}$. Then it is easy to see that the convex function $\nu=\omega^*(x,E,B)$, $x\in\mathbb{R}^n$, $\nu\in\mathbb{R}$ will be a cone, with a vertex at point $(0,-1)$ and a base at $\{x\in\partial B, \nu=0\}$. Thus, $1-cv$ measure $\omega^*(x,E,B)\not\equiv 0$.

Definition 3.4. *A point $x^0\in K$ is called $m-cv$ regular of a compact set K (relatively to D), if $\omega^*(x^0,K,D)=-1$. A compact set $K\subset D$ is called $m-cv$ regular compact if each of its points $x^0\in K$ is $m-cv$ regular.*

Since $m-cv(D)\subset sh(D)$, then $m-cv$ measure of a pair (K,D) is always no greater than the harmonic measure of this pair. Consequently, regular compacts in the sense of classical potential theory are always $m-cv$ regular. Therefore, the closure of the domain $G\subset\subset D$, with a twice smooth boundary ∂G is a $m-cv$ regular compact. It follows that for any compact $K\subset U\subset D$, where U is an open set, there is always a $m-cv$ regular compact $F: K\subset F\subset\subset U\subset D$. All this shows that the family $m-cv$ regular compact is quite rich.

6) If the set E lies compactly in a strictly $m - cv$ regular domain $D = \{\rho(x) < 0\}$, $E \subset\subset D$, then $m - cv$ measure $\omega^*(x, E, D)$ continues as $m - cv$ function to a neighborhood $\rho(x) < \delta$, $\delta > 0$, of the closure $\bar{D}$.

Actually, since $E \subset\subset D$ is a compact set, then there is a constant $C > 0$ such that $C\rho(x) < -1$, $\forall x \in E$. It follows that $C\rho(x) \in \mathcal{U}(E, D)$ and $C\rho(x) \leqslant \omega^*(x, E, D)$. Therefore, the function

$$w(x) = \begin{cases} \max\{C\rho(x), \ \omega^*(x, E, D)\} & \text{if } x \in D \\ C\rho(x) & \text{if } x \notin D \end{cases}$$

is $m - cv$ in some neighborhood $D^+ \supset \bar{D}$, $w(x) = \omega^*(x, E, D)$, $\forall x \in D$.

The following theorem plays an important role in the introducing condenser capacity and further studying the potential properties of m-convex functions.

Theorem 3.2. *If a compact set $E \subset D$ is $m - cv$ regular, then a $m - cv$ measure $\omega^*(x, E, D) \equiv \omega(x, E, D)$ and is a continuous function in D, $\omega^*(x, E, D) \in C(D)$.*

Proof. According to property 6) a $m - cv$ measure $\omega^*(x, E, D)$ continues to the neighborhood $\rho(x) < \delta$, $\delta > 0$, of the closure $\bar{D}$ and approximating $\omega^*(x, E, D)$ in some neighborhood $D^+ \supset \bar{D}$ we fined $u_j(x) \in C^\infty(D^+) \cap m - cv(D^+)$: $u_j(x) \downarrow \omega^*(x, E, D)$.

We fix a number $\varepsilon > 0$ and two neighborhoods $U = \{\omega^*(x, E, D) < -1 + \varepsilon\} \supset E$, $\breve{D} = \{\omega^*(x, E, D) < \varepsilon\} \supset \bar{D}$. Applying Hartogs' lemma twice to the sequence $u_j(x) \downarrow \omega^*(x, E, D)$ and $U \supset E$, $\breve{D} \supset \bar{D}$ find the number $j_0 \in \mathbb{N}$: $u_j(x) < -1 + 2\varepsilon$, $\forall x \in K$, $u_j(x) < 2\varepsilon$, $\forall x \in \bar{D}$, $j \geqslant j_0$. Then $u_j(x) - 2\varepsilon < -1$, $\forall x \in E$, $u_j(x) - 2\varepsilon < 0$, $\forall x \in D$, $j \geqslant j_0$, i.e. $u_j(x) - 2\varepsilon \in \mathcal{U}(E, D)$. From here, $\omega^*(x, E, D) - 2\varepsilon \leqslant u_j(x) - 2\varepsilon \leqslant \omega^*(x, E, D)$. This means that the sequence of smooth functions $u_j(x) \downarrow \omega^*(x, E, D)$ converges uniformly and $\omega^*(x, E, D) \in C(D)$. □

4. Capacity value of a pair (E, D)

We fix a set $E \subset D$, considering, as above, the domain $D \subset \mathbb{R}^n$ to be strongly m-convex. Let $\omega^*(x, E, D)$ be a $m - cv$ measure of $E \subset D$. Then the integral

$$\mathcal{P}_{mcv}(E, D) = -\int_D \omega^*(x, E, D) dV$$

is called $m - cv$ capacity of the set E with relation to D.

$m - cv$ capacity expresses the capacity value of a pair (E, D). It has the following obvious properties: $\mathcal{P}_{mcv}(E, D) \geqslant 0$ and $\mathcal{P}_{mcv}(E, D) = 0$ if and only if E is a polar set in D.

Theorem 4.1. *The value $\mathcal{P}_{mcv}(E, D)$ is an increasing and countably subadditive function of the set: $\mathcal{P}_{mcv}(E_1, D) \leqslant \mathcal{P}_{mcv}(E_2, D)$ for $E_1 \subset E_2$ and*

$$\mathcal{P}_{mcv}\left(\bigcup_{j=1}^{\infty} E_j, D\right) \leqslant \sum_{j=1}^{\infty} \mathcal{P}_{mcv}(E_j, D). \tag{8}$$

Moreover, $\mathcal{P}_{mcv}(E, D)$ is continuous on the right, i.e. for any set $E \subset D$ and for any $\varepsilon > 0$ there is an open set $U \supset E$ such that $\mathcal{P}_{mcv}(U, D) - \mathcal{P}_{mcv}(E, D) < \varepsilon$.

Proof. Monotonicity of $\mathcal{P}_{mcv}(E, D)$ obviously follows from the monotonicity of the $m - cv$ measure. Proof of (8) follows from a similar inequality $-\omega\left(x, \bigcup_{j=1}^{\infty} E_j, D\right) \leqslant -\sum_{j=1}^{\infty} \omega(x, E_j, D)$

for $m-cv$ measures: for any sets $E_j \subset D$ and $u_j(x) \in \mathcal{U}(E_j, D)$ the sum $\sum\limits_{j=1}^{\infty} u_j(x)$ is $m-cv$ function in the broad sense (i.e., it can also equal $-\infty$). Besides $\sum\limits_{j=1}^{\infty} u_j(x) \in \mathcal{U}\left(\bigcup\limits_{j=1}^{\infty} E_j, D\right)$ and therefore, $\sum\limits_{j=1}^{\infty} u_j(x) \leqslant \omega\left(x, \bigcup\limits_{j=1}^{\infty} E_j, D\right)$. On the other side,

$$\sup\left\{\sum_{j=1}^{\infty} u_j(x):\ u_j(x) \in \mathcal{U}(E_j, D)\right\} =$$

$$= \sum_{j=1}^{\infty} \sup\left\{u_j(x):\ u_j(x) \in \mathcal{U}(E_j, D)\right\} = \sum_{j=1}^{\infty} \omega(x, E_j, D),$$

i.e.

$$\sum_{j=1}^{\infty} \omega(x, E_j, D) \leqslant \omega\left(x, \bigcup_{j=1}^{\infty} E_j, D\right).$$

Integrating this inequality and using Levy's theorem, we get

$$-\int \omega\left(x, \bigcup_{j=1}^{\infty} E_j, D\right) dV \leqslant -\sum_{j=1}^{\infty} \int \omega(x, E_j, D)\, dV,$$

so that (8) is true.

It remains to show the right continuity of the set function $\mathcal{P}_{mcv}(E,D)$. We fix a set $E \subset D$ and according to the $m-cv$ measure property, construct a sequence of open sets $U_j \supset E$, $U_j \supset U_{j+1}$: $\left[\lim\limits_{j\to\infty} \omega(x, U_j, D)\right]^* \equiv \omega^*(x, E, D)$. So, as $\omega(x, U_j, D)$ increasing, then again by Levy's theorem

$$\lim_{j\to\infty} \mathcal{P}_{mcv}(U_j, D) = -\lim_{j\to\infty} \int \omega(x, U_j, D)\, dV = -\int \lim_{j\to\infty} \omega(x, U_j, D) =$$

$$= -\int \left[\lim_{j\to\infty} \omega(x, U_j, D)\right]^* dV = \mathcal{P}_{mcv}(E, D).$$

Hence, for any $\varepsilon > 0$, there is a number j_0 such that for $j \geqslant j_0$ the inequality $\mathcal{P}_{mcv}(U_j, D) - \mathcal{P}_{mcv}(E, D) < \varepsilon$ is true. The theorem is proved. □

Corollary 4.1. For any decreasing sequence of compacts $K_1 \supset K_2 \supset \ldots$ the following right continuity holds

$$\mathcal{P}_{mcv}\left(\bigcap_{j=1}^{\infty} K_j,\ D\right) = \lim_{j\to\infty} \mathcal{P}_{mcv}(K_j,\ D).$$

For arbitrary given sets $G_1 \subset G_2 \subset \ldots,\ G = \bigcup\limits_{j=1}^{\infty} G_j$, the left continuity holds

$$\mathcal{P}_{mcv}\left(\bigcup_{j=1}^{\infty} G_j, D\right) = \lim_{j\to\infty} \mathcal{P}_{mcv}(G_j, D).$$

From Corollary 4.1 it follows that the introduced capacity satisfies the Choquet axioms on the measurability of a capacity quantity $\mathcal{P}_{mcv}(E,D)$ (see [12,17]).

Theorem 4.2 (Choquet). *If a set function $C(E)$ satisfies the following Choquet conditions*

a) $0 \leqslant C(E) < \infty, \forall E \subset\subset D$;

b) if $E_1 \subset E_2$, *then* $C(E_1) \leqslant C(E_2)$;

c) for any set $E \subset D$ *and number* $\varepsilon > 0$ *there exists an open set* $U \supset E$ *such that* $C(U) - C(E) < \varepsilon$;

d) for any increasing sequence $E_j \subset E_{j+1}$ *holds*

$$C\left(\bigcup_{j=1}^{\infty} E_j\right) = \lim_{j\to\infty} C\left(E_j\right),$$

then any Borel set $E \in \mathcal{B}$ *is measurable, i.e. if* $E \in \mathcal{B}$, *then*

$$C(E) = C_*(E) = \sup\left\{C(K):\ K \subset E - compact\right\}.$$

Thus, we have obtained that the capacity value $\mathcal{P}_{mcv}(E,D)$ we introduced above is a measurable function of the sets $E \subset D$, $\mathcal{P}_{mcv}(E,D) = \sup\left\{\mathcal{P}_{mcv}(K,D):\ K \subset E -\ compact\right\}$.

5. Hessians H^k and condenser capacity

Although the $\mathcal{P}_{mcv}(E,D)$-capacity of sets is simpler to define, measurable and has many properties of capacities, the concept of a condenser capacity is more natural, which is defined using the Hessians H^k as total mass of the measure.

Let us first recall the definition of Hessians H^k for a bounded semi-continuous function $u(x) \in m - cv(D) \cap L^\infty(D)$ as positive Borel measures (see [16]). We embed $\mathbb{R}^n_x$ in $\mathbb{C}^n_z$, $\mathbb{R}^n_x \subset \mathbb{C}^n_z = \mathbb{R}^n_x + i\mathbb{R}^n_y\,(z = x+iy)$, as a real $n-$dimensional subspace of the complex space $\mathbb{C}^n_z$. Then an upper semi-continuous function $u(x)$ in the domain $D \subset \mathbb{R}^n_x$ will be m-convex in D, if the function $u^c(z) = u^c(x+iy) = u(x)$ which does not depend on the variables $y \in \mathbb{R}^n_y$, is strongly m-subharmonic, $u^c(z) \in sh_m\left(D \times i\mathbb{R}^n_y\right)$ in the domain $D \times i\mathbb{R}^n_y$ (Theorem 2.1).

If an m-convex function $u(x) \in m - cv(D)$ is locally bounded in the domain $D \subset \mathbb{R}^n_x$, then $u^c(z)$ will also be a locally bounded, strongly m-subharmonic function in the domain $D \times i\mathbb{R}^n_y \subset \mathbb{C}^n_z$. As it is known, the operators

$$\left(dd^c u^c\right)^k \wedge \beta^{n-k},\ \ k = 1,2,\dots,n-m+1$$

are defined for any bounded function $u \in sh_m\left(D \times i\mathbb{R}^n_y\right)$ as Borel measures in the domain $D \times i\mathbb{R}^n_y \subset \mathbb{C}^n_z$, $\mu_k = \left(dd^c u^c\right)^k \wedge \beta^{n-k}$, $k = 1,2,\dots,n-m+1$.

Since for a twice smooth function $\left(dd^c u^c\right)^k \wedge \beta^{n-k} = k!\,(n-k)!H^k\left(u^c\right)\beta^n$, then for a bounded, strongly m-subharmonic function $u^c(z)$ in the domain $D \times i\mathbb{R}^n_y \subset \mathbb{C}^n_z$, it is natural to determine its Hessians, equating to the measure

$$H^k\left(u^c\right) = \frac{\mu_k}{k!\,(n-k)!} = \frac{1}{k!\,(n-k)!}\left(dd^c u^c\right)^k \wedge \beta^{n-k}. \tag{9}$$

Since $u^c(z) \in sh_m\left(D \times i\mathbb{R}^n_y\right)$ does not depend on $y \in \mathbb{R}^n_y$, then for any Borel sets $E_x \subset D$, $E_y \subset \mathbb{R}^n_y$ the measures $\dfrac{1}{mes\,E_y}\mu_k\left(E_x \times E_y\right)$ do not depend on the set $E_y \subset \mathbb{R}^n_y$, i.e. $\dfrac{1}{mes\,E_y}\mu_k\left(E_x \times E_y\right) = \nu_k\left(E_x\right)$. Borel measures $\nu_k:\ \nu_k\left(E_x\right) = \dfrac{1}{mes\,E_y}\mu_k\left(E_x \times E_y\right)$, $k = 1,2,\dots,n-m+1$, we call hessians H^k, $k = 1,2,\dots,n-m+1$ for bounded, m-convex function

$u(x)\in m-cv(D)$ in the domain $D\subset\mathbb{R}^n_x$. For a twice smooth function, $u(x)\in m-cv(D)\cap C^2(D)$ the Hessians are ordinary functions; however, for a non-twice smooth but bounded semi-continuous function, $u(x)\in m-cv(D)\cap L^\infty(D)$ the Hessians $H^k,\ k=1,2,\ldots,n-m+1$, are positive Borel measures.

Now we can define the concept of condenser capacity

Definition 5.1. *Let K be a compact in the domain $D\subset\mathbb{R}^n$. Then the value*

$$\begin{gathered}C_m(K)=C_m(K,D)=\\=\inf\left\{H_u^{n-m+1}(D):\ u\in m-cv(D)\bigcap C(D),\ u|_K\leqslant -1,\ \varliminf_{x\to\partial D}u(x)\geqslant 0\right\}\end{gathered}\tag{10}$$

is called the condenser capacity (m-capacity of condenser) of (K,D). For easiness of writing below, we omit the index "m" in the notation $C_m(K)$.

Let us prove the following properties of capacity $C(K)=C_m(K)=C_m(K,D)$

1) The capacity is monotonic, i.e. $C(E)\geqslant C(K)\ \forall E\supset K$ (obviously).

2) For any $m-cv$ regular compact $K\subset D$ holds $C(K)=H^{n-m+1}_{\omega*}(K)$.

Actually, since compact $K\subset D$ is $m-cv$ regular, then $\omega^*(x,K,D)\equiv\omega(x,K,D)\in C(D)$ and $\omega^*(x,K,D)=-1\ \ \forall x\in K$. Consequently,

$$C(K)=\inf\left\{H_u^{n-m+1}(D):\ u\in m-cv(D)\cap C(D),\ u|_K\leqslant -1,\ \varliminf_{x\to\partial D}u(x)\geqslant 0\right\}\leqslant H^{n-m+1}_{\omega*}(K).$$

Conversely, for any fixed $\varepsilon,\ 0<\varepsilon<1$ and for any $u\in m-cv(D)\cap C(D),\ u|_K\leqslant -1$, $\varliminf_{x\to\partial D}u(x)\geqslant 0$, an open set $F=\left\{x\in D:\ u(x)+\frac{\varepsilon}{2}<(1-\varepsilon)\,\omega^*(x,K,D)\right\}\subset\subset D$. Therefore, according to the comparison principle,

$$H_u^{n-m+1}(F)\geqslant(1-\varepsilon)^{n-m+1}H^{n-m+1}_{\omega*}(F).$$

In addition, $K\subset F$ and $H^{n-m+1}_{\omega*}(D\backslash K)=0$ in $D\backslash K$. So that

$$\begin{gathered}(1-\varepsilon)^{n-m+1}H^{n-m+1}_{\omega*}(D)=(1-\varepsilon)^{n-m+1}H^{n-m+1}_{\omega*}(K)=(1-\varepsilon)^{n-m+1}H^{n-m+1}_{\omega*}(F)\leqslant\\ \leqslant H_u^{n-m+1}(F)\leqslant H_u^{n-m+1}(D).\end{gathered}$$

Due to the arbitrariness ε, from here we get

$$H^{n-m+1}_{\omega*}(D)\leqslant H_u^{n-m+1}(D),$$

i.e. inf on the right side of (10) reaches at $m-cv$ measure $\omega^*(x,K,D)$.

3) For any compact $K\subset D$

$$C(K)=\inf\left\{C(E):\ E\supset K,\ \ E\ \ m-cv\ \textit{regular}\right\}.\tag{11}$$

In fact, from the monotonicity of capacity (property 1), the left side of (11) does not exceed the right side, i.e.

$$C(K)\leqslant\inf\left\{C(E):\ E\supset K,\ \ E\ \ m-cv\ \textit{regular}\right\}.\tag{12}$$

Now, for any $\varepsilon,0<\varepsilon<1$ there exists $u\in m-cv(D)\cap C(D)$ such that $u|_K\leqslant -1$, $\varliminf_{x\to\partial D}u(x)\geqslant 0$ and

$$H_u^{n-m+1}(D)-C(K)<\varepsilon.\tag{13}$$

Let $U=\{x\in D:\, u(x)<-1+\varepsilon\}$ a neighborhood of a compact K and E is a $m-cv$ regular compact set, such that $K\subset E\subset\subset U$. Consider the open set

$$F=\left\{x\in D:\, u(x)+\frac{\varepsilon}{2}<(1-\varepsilon)\,\omega^*(x,E,D)\right\}\subset\subset D.$$

Since E is $m-cv$ regular compact, then $E\subset F\subset\subset D$. Therefore, according to the comparison principle and (13), we obtain

$$C(E)=H_{\omega^*}^{n-m+1}(E)=H_{\omega^*}^{n-m+1}(F)\leqslant\frac{1}{(1-\varepsilon)^{n-m+1}}H_u^{n-m+1}(F)\leqslant$$

$$\leqslant\frac{1}{(1-\varepsilon)^{n-m+1}}H_u^{n-m+1}(D)\leqslant\frac{1}{(1-\varepsilon)^{n-m+1}}\left(C\left(K\right)+\varepsilon\right).$$

Hence, the right side of (11) does not exceed $\frac{1}{(1-\varepsilon)^{n-m+1}}\left(C(K)+\varepsilon\right)$. Since ε it is arbitrary, it does not exceed $C(K)$, i.e.

$$C(K)\geqslant\inf\left\{C(E):\, E\supset K,\;\; E\quad m-cv\ \textit{regular}\right\}.$$

This inequality, together with (12), gives us the required statement.

4) If a compact $K\subset D$ is $m-cv$ regular, then

$$C(K)=\sup\left\{H_u^{n-m+1}(K):\, u\in m-cv(D)\cap C(D),\;-1\leqslant u<0\right\}. \tag{14}$$

Proof. Since $C(K)=H_\omega^{n-m+1}(K)$, then

$$C(K)\leqslant\sup\left\{H_u^{n-m+1}(K):\, u\in m-cv(D)\cap C(D),\;-1\leqslant u<0\right\}. \tag{15}$$

On the other hand, for any function $u\in m-cv(D)\cap C(D)$, we set $v(x)=$ $=\max\left\{(1+\varepsilon)\omega(x,K,D),\, u(x)\right\},\ 0<\varepsilon<1$. Then $v\in m-cv(D)\cap C(D),\ -1\leqslant v<0$ and $\lim\limits_{x\to\partial D}v(x)=0$. Therefore, according to the comparison principle

$$(1+\varepsilon)^{n-m+1}H_\omega^{n-m+1}(D)\geqslant H_v^{n-m+1}(D)\geqslant H_v^{n-m+1}(K).$$

Since $H_\omega^{n-m+1}(D\backslash K)=0$, then

$$H_v^{n-m+1}(K)=H_u^{n-m+1}(K).$$

From here,

$$(1+\varepsilon)^{n-m+1}H_\omega^{n-m+1}(D)\geqslant H_v^{n-m+1}(K)\geqslant H_u^{n-m+1}(K)$$

and tending $\varepsilon\to0$ we will receive

$$C(K)=H_\omega^{n-m+1}(K)\geqslant H_u^{n-m+1}(K).$$

Due to the arbitrariness of the function u

$$C(K)\geqslant\sup\left\{H_u^{n-m+1}(K):\, u\in m-cv(D)\cap C(D),\;-1\leqslant u<0\right\},$$

which together with (15) gives us (14).

We define the external capacity in a standard way by assuming

$$C^*(E)=\inf\left\{C(U):U\supset E-\textit{open}\right\},$$

where the capacity of an open set is

$$C(U) = \sup\{C(K): K \subset U\} = \sup\{C(K): K \subset U,\ K \quad m-cv \ \ regular\}.$$

Let us note the following properties of the external capacity

5) For any compact, $K \subset D$ its external capacity $C^*(K) = C(K)$.

This follows from property 3).

The following property of capacity is very important in practice.

Theorem 5.1. *If a set $U \subset D$ is open, then*

$$C(U) = \sup\left\{H_u^{n-m+1}(U): u \in m-cv(D) \cap C(D),\ -1 \leqslant u < 0\right\} =$$
$$= \sup\left\{H_u^{n-m+1}(U): u \in m-cv(D) \cap C^\infty(D),\ -1 \leqslant u < 0\right\}. \tag{16}$$

Proof. For any $m-cv$ regular compact set $K \subset U$ we have

$$C(K) = \sup\left\{H_u^{n-m+1}(K): u \in m-cv(D) \cap C(D),\ -1 \leqslant u < 0\right\}.$$

Therefore, $C(U) \geqslant C(K) \geqslant H_u^{n-m+1}(K)$ for any fixed $u \in m-cv(D) \cap C(D),\ -1 \leqslant u < 0$. Since $K \subset U$ is an arbitrary $m-cv$ regular compact, then $C(U) \geqslant H_u^{n-m+1}(U)$. From here,

$$C(U) \geqslant \sup\left\{H_u^{n-m+1}(U): u \in m-cv(D) \cap C(D),\ -1 \leqslant u < 0\right\} \geqslant$$
$$\geqslant \sup\left\{H_u^{n-m+1}(U): u \in m-cv(D) \cap C^\infty(D),\ -1 \leqslant u < 0\right\}. \tag{17}$$

On the other hand, we fix an arbitrary $m-cv$ regular compact set $K \subset U$. According to property 7) of the $m-cv$ measure, the $\mathcal{P}_{mcv}(E,D)$-measure $\omega(x,K,D)$ $m-cv$ continues into a certain neighborhood $G \supset \overline{D}$. It follows, that $\omega(x,K,D)$ can be approximated in some neighborhood of $\bar{D}$ by infinitely smooth $m-cv$ convex functions $u_j(x) \downarrow \omega(x,K,D)$. Since the compact $K \subset U$ is $m-cv$ regular, then $\omega(x,K,D)$ is continuous in D. From this the convergence $u_j(x) \downarrow \omega(x,K,D)$ will be uniform and the sequence of Borel measures $H_{u_j}^{n-m+1}$ weakly converges to the measure H_ω^{n-m+1}, $H_{u_j}^{n-m+1} \mapsto H_\omega^{n-m+1}$.

From the properties of convergent Borel measures we have

$$C(K) = H_\omega^{n-m+1}(K) = H_\omega^{n-m+1}(U) \leqslant \underline{\lim}_{j\to\infty} H_{u_j}^{n-m+1}(U). \tag{18}$$

Let's us now fix a $\varepsilon > 0$ and put it down $v_j = \dfrac{u_j - \varepsilon}{1+\varepsilon}$. Then $-1 \leqslant v_j < 0$, for large $j \geqslant j_0$ and therefore,

$$H_{u_j}^{n-m+1}(U) = (1+\varepsilon)^{n-m+1} H_{v_j}^{n-m+1}(U) \leqslant$$
$$\leqslant (1+\varepsilon)^{n-m+1} \sup\left\{H_w^{n-m+1}(U): w \in m-cv(D) \cap C^\infty(D),\ -1 \leqslant w < 0\right\}.$$

From here and according to (18) we have

$$C(K) \leqslant \underline{\lim}_{j\to\infty} H_{u_j}^{n-m+1}(U) \leqslant$$
$$\leqslant (1+\varepsilon) \sup\left\{H_w^{n-m+1}(U): w \in m-cv(D) \cap C^\infty(D),\ -1 \leqslant w < 0\right\}.$$

Due to the arbitrariness of the number $\varepsilon > 0$

$$C(K) \leqslant \sup\left\{H_w^{n-m+1}(U): w \in m-cv(D) \cap C^\infty(D),\ -1 \leqslant w < 0\right\}$$

and taking here the supremum over all $m - cv$ regular compacts $K \subset U$ we get

$$C(U) \leqslant \sup\left\{ \int_U (dd^c w)^n : w \in m - cv(D) \cap C^\infty(D), \ -1 \leqslant w < 0 \right\},$$

which together with (17) proves the theorem completely. □

Remark 5.1. If $U \subset\subset D$ and $K \subset U$ is an arbitrary fixed compact, then $m - cv$ measure $\omega^*(x, K, D)$ $m - cv$ continues into a fixed neighborhood $D^+ \supset \overline{D}$ such that the extended function does not exceed 1 in D^+. According to properties 2) and 4) we have

$$C(K) \leqslant \sup\left\{ H_u^{n-m+1}(K) : u \in m - cv(G) \cap C(G), -1 \leqslant u < 0 \ in\, D \ and \ \ |u| < 1 \ in \ \ D^+ \right\} \leqslant$$
$$\leqslant \sup\left\{ H_u^{n-m+1}(U) : \ u \in m - cv(D) \cap C(D), \ -1 \leqslant u < 0 \right\} = C(K).$$

So that

$$C(K) = \sup\left\{ H_u^{n-m+1}(K) : \ u \in m - cv(G) \cap C(G), \ -1 \leqslant u < 0 \ in \ D \ and \ |u| < 1 \ in \ D^+ \right\}.$$

Using $C(U) = \sup\{C(K) : \ K \subset U, \ K \quad m - cv \ \ regular\}$, for an open set $U \subset\subset D$ we get

$$C(U) = \sup\left\{ H_u^{n-m+1}(U) : \ u \in m - cv(D) \cap C(D), \ -1 \leqslant u < 0 \ \text{ in } \ D \ and \ \ |u| < 1 \ \ in \ G \right\}.$$

Moreover, approximating $\omega^*(x, K, D)$ in the neighborhood $\bar{D}$ by infinitely smooth functions, just as in the proof of Theorem 2.1, we obtain

Corollary 5.1. If $U \subset\subset D-$ an open set lying compactly in D, then

$$C(U) = \sup\left\{ H_u^{n-m+1}(U) : \ u \in m - cv(D) \cap C^\infty(D), \ -1 \leqslant u < 0 \ in \ D \ and \ |u| < 1 \ in \ G \supset\supset D \right\}.$$

This relation is useful in practice because the Hessian H_u^{n-m+1} here is an ordinary function, defined in the neighborhood of $\bar{D}$.

6) The external capacity of condenser $C^*(E)$ is monotonic, i.e. if $E_1 \subset E_2$, then $C^*(E_1) \leqslant C^*(E_2)$; it is countably subadditive, i.e. $C^*\left(\bigcup_j E_j \right) \leqslant \sum_j C^*(E_j)$.

In fact, monotonicity C^* follows from monotonicity $C(K)$ in the class of pluriregular compacts. Let us show countably subadditivity: firstly let $E_j \subset D$ are open sets and $E = \bigcup_j E_j$.

According to Theorem 5.1

$$C(E) = \sup\left\{ H_u^{n-m+1}(E) : \ u \in m - cv(D) \cap C^\infty(D), \ -1 \leqslant u < 0 \right\} \leqslant$$
$$\leqslant \sup\left\{ \sum_j H_u^{n-m+1}(E_j) : \ u \in m - cv(D) \cap C^\infty(D), \ -1 \leqslant u < 0 \right\} \leqslant$$
$$\leqslant \sum_j \sup\left\{ H_u^{n-m+1}(E_j) : \ u \in m - cv(D) \cap C^\infty(D), \ -1 \leqslant u < 0 \right\} \leqslant \sum_j C(E_j).$$

For arbitrary sets $E_j \subset D$, for a fixed $\varepsilon > 0$ we will construct open sets $U_j \supset E_j$ such that $C(U_j) - C^*(E_j) < \frac{\varepsilon}{2^j}$. Then

$$\sum_j C^*(E_j) \geqslant \sum_j C(U_j) - \varepsilon \geqslant C\left(\bigcup_j U_j \right) - \varepsilon \geqslant C^*(E) - \varepsilon$$

and from here, at $\varepsilon \to 0$ we obtain the required statement.

7) For any increasing sequence of open sets $U_j \subset U_{j+1}$, $C\left(\bigcup_j U_j\right) = \lim_{j\to\infty} C(U_j)$.

It obviously follows from the fact that any compact space $K \subset \bigcup_j U_j$ belongs to U_j, starting from some $j \geqslant j_0$.

We prove that the introduced outer condenser capacity $C^*(E)$ satisfies the Choquet axioms on the measurability (see Theorem 4.2).

Theorem 5.2. *Any Borel set $E \in \mathcal{B}$ is measurable, i.e. if $E \in \mathcal{B}$, then*

$$C^*(E) = C_*(E) = \sup \{C(K) : \quad K \subset E - \textit{compact}\}.$$

References

[1] B.Abdullaev, A.Sadullaev, Potential theory in the class of m-subharmonic functions. Analytic and geometric issues of complex analysis, *Trudy Mat. Inst. Steklova, MAIK Nauka/Interperiodica, Moscow*, **279**(2012), 155–180. DOI: 10.1134/S0081543812080111

[2] A.D.Aleksandrov, Intrinsic geometry of convex surfaces. OGIZ, Moscow, 1948 (in Russian); German transl., Akademie Verlag, Berlin, 1955.

[3] A.D.Aleksandrov, Dirichlet's problem for the equation $\det(z_{ij}) = \varphi$, *Vestnic Leningrad University*, **13**(1958), 5–24 (in Russian).

[4] I.J.Bakelman, Variational problems and elliptic Monge-Ampere equations, *J. Diff. Geo*, **18**(1983), 669–999.

[5] I.J.Bakeman, Convex Analysis and Nonlinear Geometric Elliptic Equations, *Springer-Verlag, Berlin*, 1994. DOI: 10.1007/978-3-642-69881-1

[6] Z.Błocki, Weak solutions to the complex Hessian equation, *Ann.Inst. Fourier, Grenoble*, **5**(2005), 1735–1756. DOI: 10.5802/aif.2137

[7] M.Brelot, Éléments de la théorie classique du potentiel, Les Cours de Sorbonne. 3e cycle, Centre de Documentation Universitaire, Paris, 1959.

[8] K.S.Chou, X.J.Wang, Variational theoryfor Hessian equations, *Comm. Pure Appl. Math.*, **54**(2001), 1029–1064. DOI: 10.1002/CPA.1016

[9] S.Dinew, S.Kolodziej, A priori estimates for the complex Hessian equation, *Anal. PDE*, **7**(2014), 227–244. DOI: 10.2140/apde.2014.7.227

[10] S. Dinew, S.Kolodziej, Non standard properties of m-subharmonic functions, *Dolom. Res. Not. Approx.*, **11**(2018), 35–50.

[11] N.Ivochkina, N.S.Trudinger, X.J.Wang, The Dirichlet problem for degenerate Hessian equations, *Comm. Partial Diff. Equations*, **29**(2004), 219–235. DOI: 10.1081/PDE-120028851

[12] N.S.Landkof, Foundations of Modern Potential Theory, Springer Berlin Heidelberg, 1972.

[13] S.Y.Li, On the Dirichlet problems for symmetric function equations of the eigenvalues of the complex Hessian, *Asian J.Math.*, **8**(2004), 87–106. DOI: 10.4310/AJM.2004.v8.n1.a8

[14] C.H.Lu, A variational approach to complex Hessian equations in $\mathbb{C}^n$, *Journal of Mathematical Analysis and Applications,* **431**(2015), no. 1, 228–259. DOI: 10.1016/j.jmaa.2015.05.067

[15] H.Ch.Lu, Solutions to degenerate Hessian equations, *Journal de Mathematique Pures et Appliques,* **100**(2013), no. 6, 785–805. DOI: 10.1016/j.matpur.2013.03.002

[16] A.Sadullaev, Definition of Hessians for m-convex functions as Borel measures, *Analysis and Applied Mathematics. AAM 2022. Trends in Mathematics, Birkhauser, Cham*, **6**(2024), 13–19. DOI: 10.1007/978-3-031-62668-5_2

[17] A.Sadullaev, Potential theory, Universitet, Tashkent, 2022 (in Russian).

[18] R.A.Sharipov, M.B.Ismoilov, m-convex ($m - cv$) functions, *Azerbaijan Journal of Mathematics*, **13**(2023), no. 2, 237-247. DOI: 10.59849/2218-6816.2023.2.237

[19] N.S.Trudinger, X.J.Wang, Hessian measures I, *Topological Methods in Nonlinear Analysis, Topol. Methods Nonlinear Anal.*, **10**(1997), no. 2, 225–239.

[20] N.S.Trudinger, X.J.Wang, Hessian measures II, *Anal. Math.*, **150**(1999), 579–604.

[21] N.S.Trudinger, X.J.Wang, Hessian measures III, *J. Funct. Anal.*, **193**(2002), 1–23.

$m - cv$ мера $\omega^*(x, E, D)$ и емкость конденсатора $C(E, D)$ в классе m-выпуклых функций

Азимбай Садуллаев
Национальный университет Узбекистана
Институт математики им. В. И. Романовского Академии наук Узбекистана
Ташкент, Узбекистан

Расулбек Шарипов
Ургенчский государственный университет
Ургенч, Узбекистан

Мухиддин Исмоилов
Национальный университет Узбекистана
Ташкент, Узбекистан

Аннотация. В данной работе изучаются самые начальные понятия теории потенциала: полярные множества и $m - cv$ меры в классе m-выпуклых функций в вещественном пространстве $\mathbb{R}^n$. Мы также изучаем емкость конденсатора $C(E, D)$ в классе m-выпуклых функций и будем доказывать некоторые ее потенциальные свойства.

Ключевые слова: m-субгармонические функции, выпуклые функции, m-выпуклые функции, $m-$ cv полярное множество, $m - cv$ мера, борелевские меры, гессианы.