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Measures on Smashed Products of Quasigroups and their Algebras

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Abstract. We study quasiinvariant measures on smashed and twisted wreath products of quasigroups. The quasiinvariance of measures is investigated relative to isotopies. Specific features are found for quasigroups in comparison with groups. Spaces of measures are scrutinized. Convolution algebras appear to be in general nonassociative because of the nonassociativity of the quasigroup. Ideals of topological convolution algebras are studied.

Keywords: quasigroup, measure, algebra, convolution, topology, invariance.

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Introduction

Harmonic analysis on topological groups plays an important role in mathematics and its applications (see, for example, [1–8] and references therein). New directions of research are related with nonassociative algebra, noncommutative geometry, nonassociative mathematical physics, where quasigroups and loops appear frequently. They are nonassociative group analogs (see [9–11] and references therein). Moreover, quasigroups are frequently and actively used in informatics databases, since they open new possibilities in comparison with groups [18].

Harmonic analysis on nonassociative quasigroups and loops remains a little elaborated. There is very little known about relations between topologies and algebraic structures of quasigroups in comparison with groups. An existence of left- or right-invariant measure was studied earlier on topological groups in [19]. There was obtained a result, that from an existence of a left- or right-invariant nontrivial measure on the topological loop, it follows that it is everywhere dense in a locally compact loop. In particular, on locally compact core quasigroups left-invariant measures were constructed in [20]. Core quasigroups are particular cases of quasigroups. General topological quasigroups are studied in this article, as well as left or right quasigroups and loops. It is worth to emphasize, that the class of left (or right) quasigroups is wider than the class of quasigroups. Therefore this article contains new aspects in this area.

There are specific features of topological quasigroups in comparison with groups. This is caused by a reason, that in the associative case for the topological group G, there exists either left- or right-invariant uniformity on G compatible with its topology [3,5,21]. For the topological quasigroup generally the uniformity need not be neither symmetric, nor left-, nor right-invariant because of its nonassociativity.

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In abstract harmonic analysis on groups a large role is played by invariant and quasiinvariant measures. There is very little known about them in the nonassociative case. Therefore their investigations on quasigroups is important for the development of abstract harmonic analysis in the nonassociative case. The first section is devoted to this for smashed and twisted wreath products of quasigroups. On the other hand, it permits to construct quasiinvariant or invariant measures on a wide class of quasigroups. Moreover, the quasiinvariance of measures relative to isotopies is investigated. Specific features are found for quasigroups in comparison with groups. The second section deals with studies of ideals in the convolution algebra on a space of measures with the help of invariant and quasiinvariant measures on the quasigroup. The convolution algebra appears to be in general nonassociative because of the quasigroup nonassociativity. Minimal ideals of the topological convolution algebra are investigated.

All main results of this article are obtained for the first time. Their applications are discussed in the conclusion section.

We recall a definition in order to avoid any misunderstanding.

Definition 1. Let G be a set with multiplication (that is a single-valued binary operation) $G^2 \ni (a,b) \mapsto ab \in G$ defined on G such that

(i) for each a and b in G a unique $x \in G$ exists with $ax = b$.

The set G with multiplication satisfying condition (i) is called a left quasigroup. Symmetrically is considered the case:

(ii) a unique $y \in G$ exists satisfying $ya = b$.

The set G with multiplication satisfying condition (ii) is called a right quasigroup. Mappings in (i) and (ii) are denoted by $x = a \setminus b = Div_l(a,b)$ and $y = b/a = Div_r(a,b)$ correspondingly. If G is the left and right quasigroup, then it is called a quasigroup. If in addition

(iii) a neutral (that is unit) element exists $e_G = e \in G$: $eg = ge = g$ for each $g \in G$, then

the left (or right) quasigroup G with the unit element is called a left (or right correspondingly) loop. If G is the left and right loop, then it called a loop (or a unital quasigroup). Assume that G is the loop, $\mathcal{C}(G)$ is a center of the loop, $\mathcal{C}_m(G) \subseteq \mathcal{C}(G)$, $\mathcal{C}_m(G)$ is a commutative group such that $(ab)c = t_3(a,b,c)a(bc)$ for each a, b, c belonging to G, where $t_3(a,b,c) \in \mathcal{C}_m(G)$. Then G is called a metagroup.

Let $\mathcal{T}$ be a topology on a left (or right) quasigroup (or loop) G such that multiplication $G \times G \ni (a,b) \mapsto ab \in G$ and the mapping $Div_l(a,b)$ (or $Div_r(a,b)$ correspondingly) are jointly continuous relative to $\mathcal{T}$, then $(G,\mathcal{T})$ is called a left (or right correspondingly) topological quasigroup (or loop correspondingly). If G is the left and right topological quasigroup (loop), then it is called a topological quasigroup (loop correspondingly).

It is supposed in this article that $\mathcal{T}$ is the $T_1 \cap T_{3.5}$ topology, if something other will not be specified. For subsets A and B in G by means of $A - B$ is denoted their difference $A - B = \{a \in A : a \notin B\}$.

Remark 1. The notation $\mathcal{B}(X)$ is used in this article for the Borel σ-algebra on a topological space X, $\mathcal{F}_\mu(X)$ denotes a completion of $\mathcal{B}(X)$ by means of $|\mu|$, where μ is a measure on $\mathcal{B}(X)$ with values in $\bar{\mathbf{R}} = [-\infty, \infty]$ or $\mathbf{C}$, $|\mu|$ is a variation of the measure μ. Henceforth it assumed that $\mathcal{F}_\mu(X) \supseteq \mathcal{B}(X)$, σ-additive measures are considered on σ-algebras, on locally compact spaces Radon measures are investigated, if something other will not be specified. The measure $\lambda : \mathcal{F}_\lambda(G) \to \bar{\mathbf{R}}$ or $\lambda : \mathcal{F}_\lambda(G) \to \mathbf{C}$ is called left-quasiinvariant (or left-invariant) relative to H, if λ^{L_x} is equivalent to λ (or $\lambda^{L_x} = \lambda$ correspondingly) for each $x \in H$, where $H \subseteq G$, $\lambda^{L_x}(\Omega) = \lambda(x\Omega)$ for each $\Omega \in \mathcal{F}_\lambda(G)$. For $H = G$ "relative to G" is frequently omitted for short.

Definitions of smashed products and twisted wreath products are given in the appendix.

1. Quasiinvariant measures on smashed products of quasigroups

Theorem 1. *Assume that A and B are locally compact left T_1 quasigroups with nontrivial left-quasiinvariant measures λ_A and λ_B both taking values either in $\bar{\mathbf{R}} = [-\infty, +\infty]$ or in $\mathbf{C}$ such that λ_B is also quasiinvariant relative to $\phi_1(a_1)$ for each $a_1 \in A$; $C = A\wp^{\xi_1,\xi_2,\phi_1,\phi_2,\phi_3}B$ is a smashed product of A and B, with (jointly) continuous smashing mappings ξ_i, ϕ_j for each $i \in \{1,2\}$ and $j \in \{1,2,3\}$. Then a nontrivial left-quasiinvariant measure λ_C exists on C induced by λ_A and λ_B.*

Proof. In view of Theorem 5.3 in [17] C as the topological space $A \times B$ is supplied with the Tychonoff product topology $\mathcal{T}_C$, consequently, C is locally compact, since A and B are locally compact. Multiplication on C is given by the formula

$\mu((a_1,b_1),(a_2,b_2)) = ((a_1a_2),[(\xi_1(a_1,b_1,a_2)b_1^{(a_2)})\xi_2(a_1,b_1,a_2)]^{\{a_1\}}b_2^{a_1})$ for each a_1, a_2 in A; b_1, b_2 in B, where

$b_2^{a_1} := \phi_1(a_1)b_2$, $b_1^{(a_2)} := \phi_2(a_2)b_1$, $b_2^{\{a_1\}} := \phi_3(a_1)b_2$, $\phi_j : A \to \mathcal{A}(B)$, where $\mathcal{A}(B)$ denotes a family of all homeomorphisms from B on B, $\xi_i : A\times B\times A \to B$ and $A\times B \ni (a,b) \mapsto \phi_j(a)b \in B$ are (jointly) continuous mappings for each $i \in \{1,2\}$ and $j \in \{1,2,3\}$, $(a_1,b_1) \in C$, a product on C is shortly denoted by $(a_1,b_1)(a_2,b_2)$ instead of $\mu((a_1,b_1),(a_2,b_2))$. From the Radon–Nikodym and the conditions of this theorem it follows that there exist factors of a quasiinvariance (that is Radon–Nikodym derivatives) $d_{\lambda_A}(L_a,w_1) := \dfrac{\lambda_A^{L_a}(dw_1)}{\lambda_A(dw_1)}$, $d_{\lambda_B}(\phi_1(a_1),w_2) := \dfrac{\lambda_B^{\phi_1(a_1)}(dw_2)}{\lambda_B(dw_2)}$, and $d_{\lambda_B}(L_{b_1},w_2) := \dfrac{\lambda_B^{L_{b_1}}(dw_2)}{\lambda_B(dw_2)}$, where $\lambda_B^g(W_2) = \lambda_B(g(W_2))$ for a mapping $g : B \to B$ such that $g : \mathcal{F}_{\lambda_B}(B) \to \mathcal{F}_{\lambda_B}(B)$, $d_{\lambda_B}(g,w_2) := \dfrac{\lambda_B^g(dw_2)}{\lambda_B(dw_2)}$, $L_{b_1}b_2 = b_1b_2$ for each b_1 and b_2 belonging to B, $W_2 \in \mathcal{F}_{\lambda_B}(B)$, where $\mathcal{F}_{\lambda_B}(B)$ denotes a completion o the Borel σ-algebra $\mathcal{B}(B)$ relative to the variation $|\lambda_B|$ of the measure λ_B. Then it is possible to provide the product measure $\lambda_C = \lambda_A \times \lambda_B$ on C by virtue of theorem 1 in Ch. 3, Sect. 5 in [14], since $\mathcal{B}(C) \supset \mathcal{B}(A)\times\mathcal{B}(B)$. For each $f \in \mathcal{K}(C,\mathbf{F})$ according to the Fubini theorem

$$\int_C f(L^{-1}_{(a_1,b_1)}x)\lambda_C(dx) = \int_A \left(\int_B f((w_1,w_2))\lambda_B^{L_{\beta(a_1,b_1,w_1)}}(dw_2^{a_1})\right)\lambda_A^{L_{a_1}}(dw_1),$$

where $L_c^{-1}x = c \setminus x$, $x \in C$, $c \in C$, $w_1 \in A$, $w_2 \in B$,

$\beta = \beta(a_1,b_1,w_1) = [(\xi_1(a_1,b_1,w_1)b_1^{(w_1)})\xi_2(a_1,b_1,w_1)]^{\{a_1\}}$, $\mathcal{K}(C,\mathbf{F})$ denotes the space of all continuous functions $h : C \to \mathbf{F}$ with a compact support, either $\mathbf{F} = \mathbf{R}$ or $\mathbf{F} = \mathbf{C}$ correspondingly. Since $\lambda_A^{L_{a_1}}$ is equivalent to λ_A, also $\lambda_B^{L_b}$ and $\lambda_B^{\phi_1(a_1)}$ are equivalent λ_B for each $a_1 \in A$, $b \in B$, then $\lambda_C^{L_{(a_1,b_1)}}$ is equivalent to λ_C. Moreover

$$d_{\lambda_C}(L_{(a_1,b_1)},w) = d_{\lambda_A}(L_{a_1},w_1)d_{\lambda_B}(L_\beta,w_2^{a_1})d_{\lambda_B}(\phi_1(a_1),w_2),$$

where $w = (w_1,w_2) \in C$, $w_1 \in A$, $w_2 \in B$, $\beta = \beta(a_1,b_1,w_1)$. □

Corollary 1. *If the conditions of theorem 1 are satisfied and in addition λ_A and λ_B are left-invariant, $\lambda_B^{\phi_1(a_1)} = \lambda_B$ for each $a_1 \in A$, then λ_C is left-invariant.*

Example 1. In particular, $\phi_1(a_1)$ may be in theorem 1 compositions of left shift operators of the form $L_{c_1}L^n_{\pi(a_1)}L_{c_2}$ with fixed c_1 and c_2, where $\pi : A \to B$ is a continuous mapping,

$L_cL_bx = L_c(L_bx)$, $L_bx = bx$, $L_b^{-1}x = b \setminus x$, $L_b^{n+1} = L_bL_b^n$ for each $a_1 \in A$, for each b, b_1, c_1, c_2, x belonging to B. From the left quasiinvariance of the λ_B it follows in this case, that it is quasiinvariant relative to the transformations $\phi_1(a_1)$.

Theorem 2. *Assume that a left topological loop $C = D\Delta_A^{\phi,\eta,\kappa,\xi}F$ is smashed twisted wreath product of D with F and (jointly) continuous smashed mappings $\xi : A \times F \times A \to \mathcal{C}_1$ and ϕ, η, κ, where $F = B^V$, D and B are locally compact T_1 metagroups, A is a submetagroup in D with a finite discrete transversal set V for A in D with a continuous transversal mapping $\tau : D \to V$, $\mathcal{C}_1 \hookrightarrow \mathcal{C}(D) \cap \mathcal{C}(B)$. Let λ_D and λ_B be nontrivial left-quasiinvariant measures taking values both either in $\bar{\mathbf{R}}$ of in $\mathbf{C}$, such that λ_B is also quasiinvariant relative to $\phi(a_1)$ for each $a_1 \in A$. Then there exists a nontrivial left-quasiinvariant measure λ_C on C induced by λ_D and λ_B.*

Proof. Since V is finite, then F is a locally compact T_1 metagroup by Theorem 4 and Corollary 3 in [15]. From the continuity of $\tau : D \to V$ it follows that the mapping $\psi : D \to A$ is continuous by remark 1 in [16], since $q = q^\psi q^\tau$, $q^\psi = \psi(q)$, $q^\tau = \tau(q)$ for each $q \in D$, the mapping $D^2 \ni (q,r) \mapsto q/r \in D$ is (jointly) continuous. By virtue of theorem 3 in [16] C is the locally compact $T_1 \cap T_3$ loop, since D are F locally compact, since V is finite and discrete. Moreover, the local compactness and regularity imply that C Tychonoff $T_1 \cap T_{3\frac{1}{2}}$ by Theorems 3.3.11 and 3.2.6 in [21].

On the metagroup F the measure $\lambda_F = \prod_{v\in V}(\lambda_B)_v$ exists, where $(\lambda_B)_v = \lambda_B$ for each $v \in V$ by theorem 1 in Chapter 3, Section 5 in [14], since the set V is finite. Evidently the measure λ_F is F left-quasiinvariant and also quasiinvariant relative to $\phi(a_1)$ for each $a_1 \in A$. Then $d_{\lambda_F}(L_x, y) = \prod_{v\in V} d_{\lambda_B}(L_{x_v}, y_v)$, where $y = \{y_v \in B : v \in V\} \in F$.

Multiplication on C has the form:

$(d_1, f_1)(d_2, f_2) = (d_1d_2, \xi((d_1^\psi, f_1), d_2^\psi)f_1f_2^{\{d_1\}})$,

where $\xi((d_1^\psi, f_1), d_2^\psi)(v) = \xi((d_1^\psi, f_1(v)), d_2^\psi) \in \mathcal{C}_1$ for each d_1 and d_2 in D, f_1 and f_2 in F, $v \in V$, $\mathcal{C}_1 \hookrightarrow \mathcal{C}(D) \cap \mathcal{C}(B)$, $\mathcal{C}(B) = Com(B) \cap N(B)$ is the center of the metagroup B, $(d_1, f_1) \in C$ (see remark 3 and theorem 3 in [16]), since by the conditions of this theorem $\xi : A \times F \times A \to \mathcal{C}_1$, in the considered case ξ is independent of f_2. Moreover, $f_2^{\{d_1\}}(v) = f_2^{s(d_1,v)}(v^{[d_1\setminus e]})$, $s(d_1, v) = e/(v/d_1)^\psi$, $(a^\tau)^{[c]} = (a^\tau c)^\tau$ for each a and c in D, $v \in V$, $b^{a_1} = \phi(a_1)b$ for each $b \in B$ and $a_1 \in A$, $\mathcal{A}(B)$ denotes a family of all homeomorphisms from B onto B, $\phi : A \to \mathcal{A}(B)$; η, κ, ξ and $A \times B \ni (a_1, b) \mapsto \phi(a_1)b \in B$ are the (jointly) continuous mappings.

As a measure on C it is possible to take the product measure $\lambda_C = \lambda_D \times \lambda_F$, consequently, $\lambda_C(W) = \lambda_D(W_1)\lambda_F(W_2)$ for each $W = W_1 \times W_2$, $W_1 \in \mathcal{B}(D)$, $W_2 \in \mathcal{B}(F)$. Therefore $\lambda_C^{L_{(d_1,f_1)}}(dw) = \lambda_D^{L_{d_1}}(dw_1)\lambda_F^{L_\beta}(dw_2^{\{d_1\}})$ for each $w = (w_1, w_2) \in C$, $w_1 \in D$, $w_2 \in F$, $(d_1, f_1) \in C$, where $\beta = \beta(d_1, f_1, w_1) = \xi((d_1^\psi, f_1), w_1^\psi)f_1$. Hence for each $f \in \mathcal{K}(C, \mathbf{F})$ by virtue of the Fubini theorem

$$\int_C f(L^{-1}_{(d_1,f_1)}x)\lambda_C(dx) = \int_D \left(\int_F f((w_1, w_2))\lambda_D^{L_{\beta(d_1,f_1,w_1)}}(dw_2^{\{d_1\}})\right)\lambda_D^{L_{d_1}}(dw_1)$$

On the other hand, $w_2^{\{d_1\}}(v) = \phi(e/(v/d_1)^\psi)(w_2((v(d_1 \setminus e))^\tau))$ for each $v \in V$.

Therefore $\lambda_F(U_1 \times \cdots \times U_m) = \lambda_F(U_{g(1)} \times \cdots \times U_{g(m)})$ for each $U_1, \ldots, U_m$ belonging to $\mathcal{B}(B)$ and each bijection g of the set $\{1, \ldots, m\}$, where $m = card(V)$. Then $d_{\lambda_F}(\phi_4(d_1), w_2) = \prod_{v\in V} d_{\lambda_B}(\phi_5(v, d_1), w_2(v))$ for $\phi_4(d_1)w_2 = w_2^{\{d_1\}}$, $\phi_5(v, d_1)(w_2(v)) = \phi(e/((\phi_6(d_1)v)/d_1)^\psi)(w_2(v))$, where $\phi_6(d_1)(v^{[(d_1\setminus e)]}) = v$ for each $v \in V$. Thus the measure $\lambda_C^{L_{(d_1,f_1)}}(dw)$ is equivalent to $\lambda_C(dw)$ and $d_{\lambda_C}(L_{(d_1,f_1)}, w) = d_{\lambda_D}(L_{d_1}, w_1)d_{\lambda_F}(L_{\beta(d_1,f_1,w_1)}, w_2^{\{d_1\}})d_{\lambda_F}(\phi_4(d_1), w_2)$. □

Corollary 2. *If the conditions of theorem 2 are satisfied and λ_D aand λ_B are left-invariant, $\lambda_B^{\phi(a_1)} = \lambda_B$ for each $a_1 \in A$, then λ_C is left-invariant.*

Remark 2. Examples of metagroups, satisfying the conditions of Theorem 2, are provided in [15, 16] and with the help of theorems given there. Examples of left quasigroups are in [17]. For their construction it is possible to use not only topological quasigroups, but also topological groups b subsequently construct topological left (and symmetrically right) quasigroups, loops and metagroups with the help of smashed and twisted wreath products. In their turn, with the help of Theorems 1, 2 and Corollaries 1, 2 this provides abundant families of locally compact left (or right) quasigroups and loops with left- (or right-) quasiinvariant or invariant measures. It also is possible to use topological isotopies according to the theorem given below.

Theorem 3. *Assume that $G = Q(A)$ and $H = Q_1(B)$ are topological left T_1 quasigroups, which are topologically isotopic: $\gamma A(x_1, x_2) = B(\alpha_1 x_1, \alpha_2 x_2)$ for each x_1 and x_2 in Q, where α_1, α_2 are γ homeomorphisms of topological spaces Q and Q_1. Assume also that λ_G is a measure left relative to $\alpha_2^{-1}\gamma$ quasiinvariant on G. Then λ_G and isotopy $(\alpha_1, \alpha_2, \gamma)$ induces a left- quasiinvariant measure on H.*

Proof. The measure λ_G and γ induce the measure $\lambda_H(W) = \lambda_G(\gamma^{-1}W)$ for each $W \in \mathcal{B}(Q_1)$. Then $\lambda_H(\gamma U) = \lambda_G(U)$ and $\lambda_H^{L_b}(\gamma U) = \lambda_G^{L_{\alpha_1^{-1}b}}(\alpha_2^{-1}\gamma U)$ for each $U \in \mathcal{B}(Q)$ and $b \in Q_1$, where $\alpha_2^{-1}\gamma U = \alpha_2^{-1}(\gamma(U))$. By the conditions of this theorem the measures $\lambda_G(du)$, $\lambda_G^{L_a}(du)$ and $\lambda_G^{\alpha_2^{-1}\gamma}(du)$ are equivalent for each $a \in Q$, where $u \in Q$. Therefore the measures $\lambda_H(dw)$ and $\lambda_H^{L_b}(dw)$ are equivalent for each $b \in Q_1$, since $a = \alpha_1^{-1}b \in Q$, where $w \in Q_1$. Moreover $d_{\lambda_H}(L_b, w) = d_{\lambda_G}(L_{\alpha_1^{-1}b}, \alpha_2^{-1}w) d_{\lambda_G}(\alpha_2^{-1}\gamma, \gamma^{-1}w)$. □

2. Applications of invariant measures and ideals in convolution algebras

Definition 2. Let G be a topological (left) T_1 quasigroup, $\mathbf{M}(G, \mathbf{F})$ be a space of σ-additive measures $\mu : \mathcal{F}_\mu(G) \to \mathbf{F}$ with a finite norm $\|\mu\| = |\mu|(G) < \infty$, where $\mathbf{F} = \mathbf{R}$ or $\mathbf{F} = \mathbf{C}$. For μ_1 and μ_2 belonging to $\mathbf{M}(G, \mathbf{F})$ the convolution is provided by the formula

$$(\mu_1 * \mu_2)(\Omega) = \int_G \int_G \chi_\Omega(xy)\mu_1(dx)\mu_2(dy),$$

where $\chi_\Omega(z)$ is the characteristic function of the subset Ω in G, $\chi_\Omega(z) = 1$ for each $z \in \Omega$, $\chi_\Omega(z) = 0$ for each $z \in G - \Omega$, $\Omega \in \mathcal{F}_\mu(G)$.

Theorem 4. *Assume that G is a topological left T_1 quasigroup, $\mu_j \in \mathbf{M}(G, \mathbf{F})$, $j \in \{1, 2\}$. Then $\|\mu_1 * \mu_2\| \leqslant \|\mu_1\|\|\mu_2\|$ and $\mu_1 * \mu_2 \in \mathbf{M}(G, \mathbf{F})$.*

Proof. For each $\Omega \in \mathcal{B}(G)$ by virtue of the Fubini theorem

$$\left|\int_G\int_G \chi_\Omega(xy)\mu_1(dx)\mu_2(dy)\right| \leqslant \int_G \left|\int_G \chi_\Omega(xy)\mu_1(dx)\right||\mu_2|(dy) \leqslant$$

$$\leqslant \int_G \left(\int_G \chi_\Omega(xy)|\mu_1|(dx)\right)|\mu_2|(dy) = (|\mu_1| * |\mu_2|)(\Omega) \leqslant \|\mu_1\|\|\mu_2\|,$$ since $x \setminus \Omega \in \mathcal{B}(G)$ for each $x \in G$. Therefore

$$(\mu_1 * \mu_2)(\bigcup_{j=1}^\infty \Omega_j) = \sum_{j=1}^\infty (\mu_1 * \mu_2)(\Omega_j)$$

for each disjoint Ω_j in $\mathcal{B}(G)$, $\Omega_{j_1} \cap \Omega_{j_2} = \emptyset$ for each $j_1 \neq j_2$ in $\mathbf{N}$, with $\bigcup_{j=1}^\infty \Omega_j = \Omega$, since $\chi_\Omega(z) = \sum_{j=1}^\infty \chi_{\Omega_j}(z)$ for each $z \in G$. Then $\mu_1 * \mu_2$ has an extension on $\mathcal{F}_{\mu_1 * \mu_2}(G) \supset \mathcal{B}(G)$. □

Proposition 1. *Assume that G is a compact left T_1 quasigroup, $\lambda : \mathcal{F}_\lambda(G) \to [0,1]$ is a left-invariant measure, $\lambda(G) = 1$. Then $\mathbf{F}\lambda$ is a one-dimensional left ideal in the algebra $(\mathbf{M}(G,\mathbf{F}), +, *)$.*

Proof. For each $\Omega \in \mathcal{F}_\lambda(G)$ and $\mu \in \mathbf{M}(G,\mathbf{F})$ there is the equality $\mu * \lambda(\Omega) = \lambda(\Omega)\mu(G)$, since $\mu * \lambda(\Omega) = \int_G \lambda(x \setminus \Omega)\mu(dx)$. □

Theorem 5. *Let G be a locally compact T_1 quasigroup, $\lambda : \mathcal{F}_\lambda(G) \to [0,\infty]$ (and $\lambda_1 : \mathcal{F}_{\lambda_1}(G) \to [0,\infty]$) be a nontrivial left- (or right- correspondingly) invariant measure, $\mu \in \mathbf{M}(G,\mathbf{F})$; the measure μ generates an one-dimensional left (or right) ideal in the algebra $(\mathbf{M}(G,\mathbf{F}), +, *)$. Then G is compact, the measure μ generates the one-dimensional (two-sided) ideal, $\mu(dx) = \frac{\gamma}{\alpha(x)}\lambda(dx)$ with a constant $\gamma \ne 0$ in $\mathbf{F}$, $\alpha : G \to \mathbf{F}$ is a continuous bounded function, $\alpha(xy) = \alpha(x)\alpha(y)$ for each x and y in G, $\lambda(dx) = \beta\lambda_1(dx)$ with a positive constant β.*

Proof. If $\mathbf{F}\mu$ is the one-dimensional left ideal in $\mathbf{M}(G,\mathbf{F})$, then $\nu * \mu = \alpha_\nu \mu$ with $\alpha_\nu \in \mathbf{F}$ for each $\nu \in \mathbf{M}(G,\mathbf{F})$. In particular, $\delta_a * \mu = \alpha(a)\mu$ with $\alpha(a) \in \mathbf{F}$ for each $a \in G$. Moreover

$\|\delta_a * \mu\| = |\alpha(a)|\|\mu\| \leqslant \|\delta_a\|\|\mu\| = \|\mu\|$ by Theorem 4, consequently, $\alpha(a)$ is the bounded function. For each $a \in G$ and $f \in C_0(G,\mathbf{F})$ there is the equality

$\int_G {}_af(x)\mu(dx) = \alpha(a)\int_G f(z)\mu(dz)$,

since $\delta_a * \mu(\Omega) = \mu^{L_a^{-1}}(\Omega) = \alpha(a)\mu(\Omega)$ for each $\Omega \in \mathcal{F}_\mu(G)$, where $C_0(G,\mathbf{F})$ denotes the set of all continuous functions $f : G \to \mathbf{F}$ such that for each $0 < \epsilon < \infty$ there exists a compact subset K in G with $|f(x)| < \epsilon$ for each $x \in G - K$, where $\mathbf{F} = \mathbf{R}$ or $\mathbf{F} = \mathbf{C}$, ${}_af(x) := f(ax)$. From Theorem 5.2 in [17] it follows that $\alpha(a)$ is the continuous function, since

$\int_G ({}_af(x) - {}_bf(x))\mu(dx) = (\alpha(a) - \alpha(b))\int_G f(x)\mu(dx)$

for each a and b in G. Since $\mathbf{F}\mu$ is the one-dimensional left ideal, then $\exists \Omega \in \mathcal{F}_\lambda(G), \mu(\Omega) \ne 0$, consequently, α is the nonzero function.

From the proof above it follows that $\mu^{L_a^{-1}}(dx)/\mu(dx) = \alpha(a)$, consequently, μ is the left-quasiinvariant measure. In view of the Riesz Theorem 7.2.8 in [4] and Remark 4.4 the measures $|\mu|$, λ and λ_1 are regular. In view of Theorem 4 in [17] $\mu \ll \lambda$ a function exists $h \in \mathbf{L}^1(G,\lambda,\mathbf{F})$ such that $\mu(dx) = h(x)\lambda(dx)$. Therefore $h(a \setminus x) = \alpha(a)h(x)$, since λ is left-invariant. Since $a(a \setminus x) = x$ is the quasigroup G, then $\mu(dx) = \frac{\gamma}{\alpha(x)}\lambda(dx)$, where $\gamma \ne 0$ is the constant in $\mathbf{F}$. Therefore $\alpha(az) = \alpha(a)\alpha(z)$ for each a and z in G. Thus the measure $\alpha(x)\mu(dx)$ is left-invariant. The function $\alpha(x)$ is bounded, consequently, $\lambda(G) < \infty$. Then $M(f) := \int_G f(x)\lambda(dx)$ is the left-invariant mean on $\mathbf{L}^1(G,\lambda,\mathbf{F})$, since λ is left-invariant. Moreover $M(\mathbf{L}^1(G,\lambda,\mathbf{F})) \ne 0$.

It remains to prove that G is compact. Assume the contrary, that G is locally compact noncompact left T_1 quasigroup, M is the left-invariant mean on $\mathbf{L}^1(G,\lambda,\mathbf{F})$. We take an open subset V in G such that its closure $cl_G V$ is compact. We choose $b_1 \in G$, $b_2 \in G - b_1V$ with $b_2V \subset G - b_1V$, further by induction $b_n \in G - \bigcup_{j=1}^{n-1} b_jV$ with $b_nV \subset G - \bigcup_{j=1}^{n-1} b_jV$, since G is noncompact. The function $f_n = \sum_{j=1}^n \hat{L}_{b_j}^{-1}\chi_V$ belongs to $\mathbf{L}^1(G,\lambda,\mathbf{F})$, since $cl_G V$ is compact, $n \in \mathbf{N}$, where $\hat{L}_bf(x) = f(bx)$, $\hat{L}_b^{-1}f(x) = f(b \setminus x)$ for each $x \in G$. Then for $x \in G$ the condition $b_j \setminus x \in V$ is equivalent to $x \in b_jV$. On the other hand, $(b_jV) \cap (b_kV) = \emptyset$ for each $j \ne k$ according to the choice of $b_1, \dots, b_n$ in G. Therefore either $f_n(x) = 1$ or $f_n(x) = 0$, consequently, $\|\frac{1}{n}f_n\| = \frac{1}{n}$ and $M(\chi_V) = 0$, since $M(\hat{L}_bf) = M(f)$ is equivalent to $M(\hat{L}_b^{-1}f) = M(f)$ for each

$b \in G$, $f \in \mathbf{L}^1(G, \lambda, \mathbf{F})$. Thus, $M(\mathbf{L}^1(G, \lambda, \mathbf{F})) = \{0\}$. This gives the contradiction, consequently, G is compact.

In view of proposition 1 $\mathbf{F}\lambda$ is the left ideal, and symmetrically $\mathbf{F}\lambda_1$ is the right ideal in $(\mathbf{M}(G, \mathbf{F}), +, *)$. Therefore $\lambda_1 * \lambda = p\lambda_1 = q\lambda$, where p and q are positive constants, consequently, a positive constant exists β such that $\lambda(dx) = \beta\lambda_1(dx)$. □

Conclusion

The results of this article can be used for the subsequent studies of the topological quasigroups and loops structure, homogeneous spaces and noncommutative manifolds related with quasigroups and loops [11, 12]. Besides applications of left- (or right-)invariant or quasiinvariant measures on quasigroups and loops mentioned in the introduction it is interesting to mention possible applications to the mathematical control of ruling simultaneously functioning programmed robots [18, 22], since they frequently are based on topologically-algebraic binary systems and measures. Other very important applications are: representation theory of quasigroups and loops, harmonic analysis on quasigroups and loops [1–3, 5–8], mathematical physics, etc.

Smashed products of topological left quasigroups and smashed twisted wreath products of topological metagroups were use in this article. On the other side, smooth quasigroups were constructed in [23] with the help of a generalization of the Lie group construction method such that a composition law was dependent on transformation parameters and on transformed variables. In particular, this was used on hypersurfaces described in the first class restrictions in a phase space.

It is possible to formulate a question for subsequent studies. Whether will coincide an extended measure from the loop on an enveloping (by Sabinin) group with the Haar measure of the enveloping loop? There different extension may be, while transformations of measures on the loop and on the group are different, since loops are nonassociative.

3. Appendix

For convenience of readers definitions and a notation are recalled in this appendix from works [15–17].

Definition 3. A topological left quasigroup prescribed by the conditions given below is denoted by $A^{\xi_1,\xi_2,\phi_1,\phi_2,\phi_3}B$ and it is called a smashed product of topological left quasigroups (with smashing mappings $\xi_1, \xi_2, \phi_1, \phi_2, \phi_3$).

Conditions. Let

(i) (A, τ_A) and (B, τ_B) be topological left quasigroups,

(ii) $\xi_i : A \times B \times A \to B$ and $A \times B \ni (a, b) \mapsto \phi_j(a)b \in B$ be (jointly) continuous mappings for each $i \in \{1, 2\}$ and $j \in \{1, 2, 3\}$,

(iii) $\mu : (A \times B)^2 \to A \times B$ be a mapping such that

(iv) $\mu((a_1, b_1), (a_2, b_2)) = ((a_1a_2), [(\xi_1(a_1, b_1, a_2)b_1^{(a_2)})\xi_2(a_1, b_1, a_2)]^{\{a_1\}}b_2^{a_1})$ for each a_1, a_2 in A; b_1, b_2 in B, where

(v) $b_2^{a_1} := \phi_1(a_1)b_2$,

(vi) $b_1^{(a_2)} := \phi_2(a_2)b_1$,

(vii) $b_2^{\{a_1\}} := \phi_3(a_1)b_2$,

$\phi_j : A \to \mathcal{A}(B)$, where $\mathcal{A}(B)$ denotes a family of all homeomorphisms from B onto B,

$(viii)$ the Cartesian product $C = A \times B$ is suppliedwith the Tychonoff product topology τ_C and the mapping μ.

Remark 3. Let A and B be topological metagroups, $\mathcal{C}$ be a commutative group such that

(i) $\mathcal{C}_m(A) \hookrightarrow \mathcal{C}$, $\mathcal{C}_m(B) \hookrightarrow \mathcal{C}$, $\mathcal{C} \hookrightarrow \mathcal{C}(A)$ and $\mathcal{C} \hookrightarrow \mathcal{C}(B)$,

where $\mathcal{C}_m(A)$ denotes a minimal closed subgroup in $\mathcal{C}(A)$ containing $t_A(a, b, c)$ for each a, b and c in A.

An equivalence relation Ξ is considered on $A \times B$:

(ii) $(\gamma v, b)\Xi(v, \gamma b)$ and $(\gamma v, b)\Xi\gamma(v, b)$ and $(\gamma v, b)\Xi(v, b)\gamma$

for each v in A, b in B, γ in $\mathcal{C}$.

Let (iii) $\phi : A \to \mathcal{A}(B)$ be (jointly) continuous (single valued) mappings,

where $\mathcal{A}(B)$ denotes a family of all homeomorphisms from B onto B, satisfying conditions (iv)–$(viii)$ given below. If $a \in A$ and $b \in B$, then it will be shortly written b^a instead of $\phi(a)b$, where $\phi(a) : B \to B$. Let also

$\eta_{A,B,\phi} : A \times A \times B \to \mathcal{C}$, $\kappa_{A,B,\phi} : A \times B \times B \to \mathcal{C}$

and $\xi_{A,B,\phi} : ((A \times B)/\Xi) \times ((A \times B)/\Xi) \to \mathcal{C}$

are (single valued jointly) continuous mappings shortly written by η, κ and ξ such that

(iv) $(b^u)^v = b^{vu}\eta(v, u, b)$, $e^u = e$, $b^e = b$;

(v) $\eta(v, u, \gamma b) = \eta(v, u, b)$;

(vi) $(cb)^u = c^u b^u \kappa(u, c, b)$;

(vii) $\kappa(u, \gamma c, b) = \kappa(u, c, \gamma b) = \kappa(u, c, b)$;

$(viii)$ $\kappa(u, \gamma, b) = \kappa(u, b, \gamma) = e$;

$\xi((\gamma u, c), (v, b)) = \xi((u, c), (\gamma v, b)) = \xi((u, c), (v, b))$

$\xi((\gamma, e), (v, b)) = e$ and $\xi((u, c), (\gamma, e)) = e$

for each u and v in A, b, c in B, γ in $\mathcal{C}$, where e denotes the unit element in $\mathcal{C}$, A and B.

Let D be a topological metagroup, A be a submetagroup in D, V be a transversal set for A in D. Then, as it is known,

(ix) $\forall a \in D, \ \exists_1 s \in A, \ \exists_1 b \in V, \ a = sb$.

Then b in the decomposition (ix) is denoted by $b = \tau(a) = a^\tau$ and $s = \psi(a) = a^\psi$, where τ and ψ are shortened notations for $\tau_{A,D,V}$ and $\psi_{A,D,V}$. Thus (single-valued) mappings exist

(x) $\tau : D \to V$ and $\psi : D \to A$.

Assume that mappings τ and ψ are continuous.

(xi) If $b = a^\tau$, then e/b is also denoted by $a^{e/\tau}$, and $b \setminus e$ is also denoted by $a^{\tau\setminus e}$.

We put

(xii) $(a^\tau)^{[c]} := (a^\tau c)^\tau$ for each a and c in D.

Remark 4. Let B and D be topological metagroups, A be a submetagroup in D, V be the transversal set for A in D. Assume also that conditions (i)-$(viii)$ are satisfied in remark3 for A and B. Then a topological metagroup exists

(i) $F = B^V$, where $B^V = \prod_{v\in V} B_v$, $B_v = B$ for each $v \in V$.

We put $T_h f = f^h$ for each $f \in F$ in $h : V \to A$. The we define

$\hat{S}_d(T_h f J) = T_{hS_d^{-1}} f S_d J$,

where $J : V \times F \to B$, $J(f, v) = fJv$, $S_d Jv = Jv^{[d\setminus e]}$ for each $d \in D$, $f \in F$ and $v \in V$. Let also for each $f \in F$, $d \in D$

(ii) $f^{\{d\}} = \hat{S}_d(T_{g_d} f E)$,

where

(iii) $s(d,v) = e/(v/d)^\psi$, $g_d(v) = s(d,v)$,
$fEv = f(v)$ for each $v \in V$
(see also (ix) and (xii) in Remark 3).

Definition 4. Assume that the conditions of Remarks 3 and 4 are satisfied, and on $C = D \times F$ a binary operation is provided by the formula:

(i) $(d_1, f_1)(d, f) = (d_1 d, \xi((d_1^\psi, f_1), (d^\psi, f)) f_1 f^{\{d_1\}})$,

where $\xi((d_1^\psi, f_1), (d^\psi, f))(v) = \xi((d_1^\psi, f_1(v)), (d^\psi, f(v)))$ for each d and d_1 in D, f in f_1 in F, $v \in V$, where $C = D \times F$ is supplied with the Tychonoff product topology. Then the topological loop C supplied with this multiplication is called a smashed twisted wreath product of D with F. It is denoted by $C = D\Delta^{\phi,\eta,\kappa,\xi} F$.

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Меры на сокрушающих произведениях квазигрупп и их алгебры

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Аннотация. Исследуются квазиинвариантные и инвариантные меры на сокрушающих и скрученных венечных произведениях квазигрупп. Также изучается квазиинвариантность мер относительно изотопий. Найдены специфические особенности для квазигрупп по сравнению с группами. Изучаются пространства мер. Алгебры сверток в общем случае оказываются неассоциативными из-за неассоциативности квазигрупп. Исследуются идеалы топологических алгебр сверток.

Ключевые слова: квазигруппа, мера, алгебра, свертка, топология, инвариантность.