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On One Integral Representation of the Potential Type

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Abstract. In this article, we consider some integral representation of the potential type (Cauchy–Fantappiè) for a smooth function defined on the boundary of a bounded multidimensional domain. Derivatives of this integral representation are found and their boundary behavior is studied. An analogue of the Bochner–Martinelli formula for smooth functions is proved.

Keywords: Bochner–Martinelli integral, bounded domain, boundary behavior.

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The method of integral representations is one of the main constructive methods in the study of holomorphic functions of several complex variables (see, for example, [1–4]). One such representation is the classical Bochner–Martinelli representation. Although it does not have a holomorphic kernel, its versatility has allowed it to be used in matters of analytical continuation of functions and other analytic objects. It has been studied in detail in the monograph [5].

The integral representation considered in the paper is close to the Bochner–Martinelli representation. The aim of the work is to study the properties of this integral representation for holomorphic functions (Cauchy–Fantappiè type), the kernel of which consists of derivatives of the fundamental solution of the Laplace equation.

We consider n-dimensional complex space $\mathbb{C}^n$, $n > 1$ with variables $z = (z_1, \ldots, z_n)$. Let us introduce the vector module $|z| = \sqrt{z_1^2 + \ldots + z_n^2}$ and the differential forms $dz = dz_1 \wedge \ldots \wedge dz_n$ and $d\bar{z} = d\bar{z}_1 \wedge \ldots \wedge d\bar{z}_n$ and also $dz[k] = dz_1 \wedge \ldots \wedge dz_{k-1} \wedge dz_{k+1} \wedge \ldots \wedge dz_n$.

We shall consider bounded domains $D \subset \mathbb{C}^n$ with a smooth boundary ∂D of class $\mathcal{C}^1$, that is $D = \{z \in \mathbb{C}^n : \ \rho(z) < 0\}$, where ρ is real-valued function of class $\mathcal{C}^1$ on some neighborhood of the closure of domain D, and the differential $d\rho \neq 0$ on ∂D. Let us denote the "complex" guiding cosines

$$\rho_k = \frac{1}{|\operatorname{grad}\rho|}\frac{\partial\rho}{\partial z_k}, \qquad \rho_{\bar{k}} = \frac{1}{|\operatorname{grad}\rho|}\frac{\partial\rho}{\partial\bar{z}_k}, \qquad k = 1, \ldots, n.$$

Consider the Bochner-Martinelli kernel, which is an exterior differential form $U(\zeta, z)$ of type $(n, n-1)$ (see, for example, [5, Ch. 1]), given by

$$U(\zeta, z) = \frac{(n-1)!}{(2\pi i)^n}\sum_{k=1}^{n}(-1)^{k-1}\frac{\bar{\zeta}_k - \bar{z}_k}{|\zeta - z|^{2n}}\, d\bar{\zeta}[k] \wedge d\zeta.$$

This kernel plays an important role in multidimensional complex analysis (see, for example, [1–6]). It is a closed differential form of type $(n, n-1)$. For $n = 1$ this kernel turns into a Cauchy kernel.

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Let $g(\zeta, z)$ be the fundamental solution to the Laplace equation:

$$g(\zeta, z) = -\frac{(n-2)!}{(2\pi i)^n} \frac{1}{|\zeta - z|^{2n-2}}, \quad n > 1,$$

then

$$U(\zeta, z) = \sum_{k=1}^{n} (-1)^{k-1} \frac{\partial g}{\partial \zeta_k} d\bar{\zeta}[k] \wedge d\zeta.$$

For the function $f \in \mathcal{C}^1(\partial D)$ we introduce the Bochner-Martinelli integral (integral operator)

$$M[f](z) = \int_{\partial D} f(\zeta) U(\zeta, z), \quad z \notin \partial D,$$

and also the single-layer potential (integral operator)

$$\Phi[f](z) = -i^n 2^{n-1} \int_{\partial D} f(\zeta) g(\zeta, z)\, d\sigma(\zeta) = \frac{(n-2)!}{2\pi^n} \int_{\partial D} f(\zeta) \frac{d\sigma}{|\zeta - z|^{2n-2}}, \quad z \notin \partial D,$$

where $d\sigma$ is the Lebesgue surface measure on ∂D.

Let us define the differential form μ_f for the function $f \in \mathcal{C}^1(\partial D)$ as follows [5, Ch. 1]:

$$\mu_f = \sum_{k=1}^{n} (-1)^{n+k-1} \frac{\partial f}{\partial \bar{\zeta}_k} d\zeta[k] \wedge d\bar{\zeta}.$$

In the monograph [5], the problem of holomorphicity of the harmonic function $f \in \mathcal{C}^1(\bar{D})$ satisfying condition (23.5) in [5] of the following form is posed

$$\mu_f \big|_{\partial D} = \sum_{k>l} a_{k,l}(z) df \wedge d\bar{z}[k,l] \wedge dz \big|_{\partial D}, \tag{1}$$

where $a_{k,l}$ are some smooth functions on ∂D. Here, the differential form $d\bar{z}[k,l]$ is obtained from the differential form $d\bar{z} = d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_n$ by removing the differentials $d\bar{z}_k$, $d\bar{z}_l$.

This is related to the problem of holomorphicity of functions represented by the Bochner–Martinelli integral (see [5, Ch. 15]) (in this case, all functions $a_{kl} = 0$). Some special cases of this problem are considered in [5, Ch. 23]. In [5, Ch. 23], the problem 1 is rewritten in integral form.

Recall Green's formula (in complex form) for the function f (corollary 1.2 of [5]).

Theorem 1 (Green's formula)**.** *Let D be a bounded domain with a piecewise smooth boundary, the function f is harmonic in D and $f \in \mathcal{C}^1(\bar{D})$, then*

$$\int_{\partial D} f(\zeta) U(\zeta, z) - \int_{\partial D} g(\zeta, z) \mu_f = \begin{cases} f(z), & z \in D, \\ 0, & z \notin \bar{D}. \end{cases} \tag{2}$$

From the equality of (1) and Green's formula (2), we obtain that

$$f(z) = \int_{\partial D} f(\zeta) U(\zeta, z) - \int_{\partial D} g(\zeta, z) \sum_{k>l} a_{k,l}(\zeta) df \wedge d\bar{\zeta}[k,l] \wedge d\zeta, \qquad z \in D. \tag{3}$$

Applying the Stokes and the Green's formula (2), in [5, Ch. 23] it is shown that the equality (3) for functions $f \in \mathcal{C}^1(\bar{D})$ and harmonic in D is equivalent to the condition

$$f(z) = \int_{\partial D} f(\zeta) U(\zeta, z) + \int_{\partial D} f(\zeta) \sum_{k>l} d(a_{k,l}(\zeta) g(\zeta, z)) \wedge d\bar{\zeta}[k,l] \wedge d\zeta, \qquad z \in D. \tag{4}$$

The first integral is a Bochner–Martinelli integral (integral operator) of the function f, i.e.

$$M[f](z) = \int_{\partial D} f(\zeta)U(\zeta, z), \quad z \notin \partial D,$$

and the second integral (integral operator) is denoted by

$$G[f](z) = \int_{\partial D} f(\zeta) \sum_{k>l} d(a_{k,l}(\zeta)g(\zeta, z)) \wedge d\bar{\zeta}[k, l] \wedge d\zeta, \qquad z \notin \partial D.$$

For $n = 1$ this integral disappears.

Let us introduce the kernel of the second integral operator

$$W(\zeta, z) = \sum_{k>l} d(a_{k,l}(\zeta)g(\zeta, z)) \wedge d\bar{\zeta}[k, l] \wedge d\zeta,$$

we obtain that for holomorphic functions f an integral representation of the Cauchy–Fantappiè type is valid (see, for example, [4, Ch. 26])

$$f(z) = \int_{\partial D} f\big(\zeta)(U(\zeta, z) + W(\zeta, z)\big), \quad z \in D. \tag{5}$$

Thus, the problem (1) transforms into the problem of holomorphicity of the harmonic function f satisfying the equality (5) in the domain D (see [5, Ch. 23]).

Let us denote the operator $M + G$ by

$$Q[f](z) = M[f](z) + G[f](z) = \int_{\partial D} f\big(\zeta)(U(\zeta, z) + W(\zeta, z)\big), \quad z \notin \partial D. \tag{6}$$

In this paper, we will study the properties of this integral with the kernel $U(\zeta, z) + W(\zeta, z)$, calculate its derivatives and their boundary behavior.

1. Derivatives of the integral operator

Let the domain D have a boundary of the class $\mathcal{C}^2$ (i.e., the function ρ is twice smooth in a neighborhood of the closure of the domain D). The function $f \in \mathcal{C}^2(\partial D)$, and the functions $a_{k,l} \in \mathcal{C}^2(\partial D)$, $k, l = 1, \ldots, n$.

We introduce, as in the article [8], the following differential operators

$$L_m(f) = \frac{\partial f}{\partial \zeta_m} - \rho_m \sum_{k=1}^{n} \rho_k \frac{\partial f}{\partial \bar{\zeta}_k},$$

$$K_m(f) = i^n 2^{n-1} \sum_{s,k=1}^{n} \left[\rho_k \frac{\partial}{\partial \zeta_s}\left(\rho_m \rho_{\bar{k}} \frac{\partial f}{\partial \bar{\zeta}_s}\right) - \rho_m \frac{\partial}{\partial \zeta_k}\left(\rho_m \rho_{\bar{k}} \frac{\partial f}{\partial \bar{\zeta}_s}\right)\right],$$

accordingly,

$$L_{\bar{m}}(f) = \frac{\partial f}{\partial \bar{\zeta}_m} - \rho_{\bar{m}} \sum_{k=1}^{n} \rho_k \frac{\partial f}{\partial \bar{\zeta}_k},$$

$$K_{\bar{m}}(f) = i^n 2^{n-1} \sum_{s,k=1}^{n} \left[\rho_k \frac{\partial}{\partial \zeta_s}\left(\rho_{\bar{m}} \rho_{\bar{k}} \frac{\partial f}{\partial \bar{\zeta}_s}\right) - \rho_{\bar{m}} \frac{\partial}{\partial \zeta_k}\left(\rho_{\bar{m}} \rho_{\bar{k}} \frac{\partial f}{\partial \bar{\zeta}_s}\right)\right].$$

Then, according to Corollary 1 of [8], we get

$$\frac{\partial M[f]}{\partial z_m} = M\big[L_m(f)\big] - \Phi\big[K_m(f)\big], \tag{7}$$

$$\frac{\partial M[f]}{\partial \bar{z}_m} = M\big[L_{\bar{m}}(f)\big] - \Phi\big[K_{\bar{m}}(f)\big]. \tag{8}$$

These formulas are derived from the formulas of the classical potential theory [7] and formulas from [5, Ch.1].

Similarly, we introduce the operators

$$\tilde{L}_m(f) = -f\rho_m,$$
$$\tilde{K}_m(f) = +i^n 2^{n-1}\sum_{k=1}^{n}\Big[\rho_k \frac{\partial}{\partial \zeta_m}\big(f\rho_{\bar{k}}\big) - \rho_m \frac{\partial}{\partial \zeta_k}\big(f\rho_{\bar{k}}\big)\Big],$$

accordingly,

$$\tilde{L}_{\bar{m}}(f) = -f\rho_{\bar{m}},$$
$$\tilde{K}_{\bar{m}}(f) = i^n 2^{n-1}\sum_{k=1}^{n}\Big[\rho_k \frac{\partial}{\partial \zeta_{\bar{m}}}\big(f\rho_{\bar{k}}\big) - \rho_{\bar{m}} \frac{\partial}{\partial \zeta_k}\big(f\rho_{\bar{k}}\big)\Big].$$

Then, according to Corollary 1 of [8], we get

$$\frac{\partial \Phi[f]}{\partial z_m} = M\big[\tilde{L}_m(f)\big] - \Phi\big[\tilde{K}_m(f)\big], \tag{9}$$

$$\frac{\partial \Phi[f]}{\partial \bar{z}_m} = M\big[\tilde{L}_{\bar{m}}(f)\big] - \Phi\big[\tilde{K}_{\bar{m}}(f)\big]. \tag{10}$$

Lemma 1. *Let D be a bounded domain with a boundary of the class $\mathcal{C}^2$, a function f is harmonic in D and $f \in \mathcal{C}^2(\overline{D})$, and $a_{k,l} \in \mathcal{C}^2(\partial D)$, $k,l = 1,\ldots,n$, then $G[f] = -\Phi[h]$, where*

$$h(\zeta) = \sum_{k>l}(-1)^{k+l}a_{k,l}(\zeta)\Big(\frac{\partial f}{\partial \bar{\zeta}_k}\rho_{\bar{l}} - \frac{\partial f}{\partial \bar{\zeta}_l}\rho_{\bar{k}}\Big).$$

Proof. It follows from formulas (3) and (4) that

$$\int_{\partial D} f(\zeta)\sum_{k>l} d(a_{k,l}(\zeta)g(\zeta,z)) \wedge d\bar{\zeta}[k,l] \wedge d\zeta = -\int_{\partial D} g(\zeta,z)\sum_{k>l} a_{k,l}(\zeta)df \wedge d\bar{\zeta}[k,l] \wedge d\zeta.$$

Therefore, transforming the differential form $df \wedge d\bar{\zeta}[k,l] \wedge d\zeta$, we get

$$df \wedge d\bar{\zeta}[k,l] \wedge d\zeta = \Big((-1)^{l-1}\frac{\partial f}{\partial \bar{\zeta}_l}d\bar{\zeta}[k] + (-1)^k\frac{\partial f}{\partial \bar{\zeta}_k}d\bar{\zeta}[l]\Big) \wedge d\zeta =$$
$$= (-1)^{l-1}\frac{\partial f}{\partial \bar{\zeta}_l}2^{n-1}i^n(-1)^{k-1}\rho_{\bar{k}}d\sigma + (-1)^k\frac{\partial f}{\partial \bar{\zeta}_k}2^{n-1}i^n(-1)^{l-1}\rho_{\bar{l}}\,d\sigma =$$
$$= 2^{n-1}i^n\Big((-1)^{l+k}\frac{\partial f}{\partial \bar{\zeta}_l}\rho_{\bar{k}} + (-1)^{k+l-1}\frac{\partial f}{\partial \bar{\zeta}_k}\rho_{\bar{l}}\Big)d\sigma = 2^{n-1}i^n(-1)^{k+l-1}\Big(\frac{\partial f}{\partial \bar{\zeta}_k}\rho_{\bar{l}} - \frac{\partial f}{\partial \bar{\zeta}_l}\rho_{\bar{k}}\Big)d\sigma,$$

where $d\sigma$ is the Lebesgue surface measure on ∂D. Then

$$G[f] = 2^{n-1}i^n\int_{\partial D}\sum_{k>l}(-1)^{k+l}a_{k,l}(\zeta)\Big(\frac{\partial f}{\partial \bar{\zeta}_k}\rho_{\bar{l}} - \frac{\partial f}{\partial \bar{\zeta}_l}\rho_{\bar{k}}\Big)g(\zeta,z)\,d\sigma(\zeta).$$

Therefore, from the form of the integral operator Φ, we get that $G[f] = -\Phi[h]$.

□

We formulate a theorem on the form of partial derivatives of the function f.

Theorem 2. *Let D be a bounded domain with a twice smooth boundary and a function f is harmonic in D and $f \in \mathcal{C}^1(\overline{D})$ and $a_{k,l} \in \mathcal{C}^1(\partial D)$, $k, l = 1, \dots, n$, then*

$$\frac{\partial f}{\partial z_m} = \frac{\partial Q[f]}{\partial z_m} = M\big[L_m(f) + \widetilde{L}_m(h)\big] - \Phi\big[K_m(f) + \widetilde{K}_m(h)\big],$$
$$\frac{\partial f}{\partial \bar{z}_m} = \frac{\partial Q[f]}{\partial \bar{z}_m} = M\big[L_{\bar{m}}(f) + \widetilde{L}_{\bar{m}}(h)\big] - \Phi\big[K_{\bar{m}}(f) + \widetilde{K}_{\bar{m}}(h)\big].$$

Proof. From Lemma 1 and formula (6) we get that

$$Q[f] = M[f] + G[f] = M[f] - \Phi[h].$$

Now, using formulas (7)–(10), we obtain expressions for partial derivatives of the function f, and hence the operator $Q[f]$. Then

$$\frac{\partial f}{\partial z_m} = \frac{\partial Q[f]}{\partial z_m} = \frac{\partial M[f]}{\partial z_m} - \frac{\partial \Phi[h]}{\partial z_m} =$$
$$= M\big[L_m(f)\big] + \Phi\big[K_m(f)\big] + M\big[\widetilde{L}_m(h)\big] - \Phi\big[\widetilde{K}_m(h)\big] =$$
$$M\big[L_m(f) + \widetilde{L}_m(h)\big] - \Phi\big[K_m(f) + \widetilde{K}_m(h)\big].$$

Similarly

$$\frac{\partial f}{\partial \bar{z}_m} = \frac{\partial Q[f]}{\partial \bar{z}_m} = \frac{\partial M[f]}{\partial \bar{z}_m} - \frac{\partial \Phi[h]}{\partial \bar{z}_m} =$$
$$= M\big[L_{\bar{m}}(f)\big] - \Phi\big[K_{\bar{m}}(f)\big] + M\big[\widetilde{L}_{\bar{m}}(h)\big] - \Phi\big[\widetilde{K}_{\bar{m}}(h)\big] =$$
$$M\big[L_{\bar{m}}(f) + \widetilde{L}_{\bar{m}}(h)\big] - \Phi\big[K_{\bar{m}}(f) + \widetilde{K}_{\bar{m}}(h)\big].$$

□

The boundary behavior of the potential of a simple layer and the Bochner-Martinelli integral operator is well known (see, for example, [7], [5, Ch. 1], [6, Ch. 1]). Therefore, from these properties of potentials, we obtain the statement

Theorem 3. *If $\partial D \in \mathcal{C}^\infty$ and $f \in \mathcal{C}^\infty(\partial D)$, $a_{k,l} \in \mathcal{C}^\infty(\partial D)$, $k, l = 1, \dots, n$, then the integral $Q[f](z)(z \in D,\ z \in \mathbb{C}^n \setminus \overline{D})$ continues on $\overline{D}$ and on $\mathbb{C}^n \setminus D$, respectively, as an infinitely differentiable function.*

For the Bochner–Martinelli integral, this property is noted in [8].

2. Integral representation for smooth functions

For the integral representation of (5), an analogue of the Bochner–Martinelli formula for smooth functions is valid (see, for example, [5, Ch. 1]).

Theorem 4. *Let D be a bounded domain with a smooth boundary and a function f of class $\mathcal{C}^1(\bar{D})$, then*

$$f(z) = \int\limits_{\partial D} f(\zeta) U(\zeta, z) - \int\limits_{D} \bar{\partial} f(\zeta) \wedge U(\zeta, z), \qquad z \in D, \tag{11}$$

where the operator $\bar{\partial} = \sum\limits_{k=1}^{n} \dfrac{\partial}{\partial \bar{\zeta}_k} d\bar{\zeta}_k$, and the integral of the domain in (11) converges absolutely.

We will now prove an analogue of this formula for our operator Q.

Theorem 5. *Let D be a bounded domain with a smooth boundary and a function f of class $\mathcal{C}^1(\bar{D})$ and $a_{k,l} \in \mathcal{C}^1(\bar{D})$, $k, l = 1, \dots, n$, then*

$$f(z) = \int\limits_{\partial D} f(\zeta)\big(U(\zeta,z) + W(\zeta,z)\big) - \int\limits_{D} \bar\partial f(\zeta) \wedge \big(U(\zeta,z) + W(\zeta,z)\big), \qquad z \in D, \tag{12}$$

and the integral of the domain in (12) *converges absolutely.*

Proof. Since the theorem is true for the operator $U(\zeta,z)$ (Theorem 1.3 in [5]), it remains to show that

$$\int\limits_{\partial D} f(\zeta) W(\zeta,z) - \int\limits_{D} \bar\partial f(\zeta) \wedge W(\zeta,z) = 0, \qquad z \in D.$$

Let $z \in D$, by $B(z,\varepsilon)$ denote a ball of radius $\varepsilon > 0$ centered at z, and its boundary by $S(z,\varepsilon)$. For sufficiently small ε, using the Stokes formula, we obtain

$$\int\limits_{D} \bar\partial f(\zeta) \wedge W(\zeta,z) = \int\limits_{D\setminus B(z,\varepsilon)} \bar\partial f(\zeta) \wedge W(\zeta,z) + \int\limits_{B(z,\varepsilon)} \bar\partial f(\zeta) \wedge W(\zeta,z) =$$

$$= \int\limits_{\partial D} f(\zeta) W(\zeta,z) - \int\limits_{S(z,\varepsilon)} f(\zeta) W(\zeta,z) + \int\limits_{B(z,\varepsilon)} \bar\partial f(\zeta) \wedge W(\zeta,z).$$

According to Green's formula (2), for the modulus of the integral, we get

$$\left| \int\limits_{S(z,\varepsilon)} f(\zeta) W(\zeta,z) \right| = \left| \int\limits_{S(z,\varepsilon)} g(\zeta,z) \mu_f \right| \leqslant \frac{(n-2)!}{(2\pi)^n \varepsilon^{2n-2}} \int\limits_{S(z,\varepsilon)} |\mu_f| \leqslant C\varepsilon,$$

then $\lim\limits_{\varepsilon\to+0} \int\limits_{S(z,\varepsilon)} f(\zeta) W(\zeta,z) = 0$.

Since the singularity of the integral $\int\limits_{B(z,\varepsilon)} \bar\partial f(\zeta) \wedge W(\zeta,z)$ is equal to $(2n-1) < 2n$, then $\lim\limits_{\varepsilon\to+0} \int\limits_{B(z,\varepsilon)} \bar\partial f(\zeta) \wedge W(\zeta,z) = 0$. From here we get the necessary equality.

□

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Об одном интегральном представлении типа потенциала

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Аннотация. Цель работы состоит в исследовании свойств одного интегрального представления для голоморфных функций (типа Коши–Фантаппье), ядро которого состоит из производных фундаментального решения уравнения Лапласа.

Ключевые слова: интеграл Бохнера-Мартинелли, ограниченная область, граничное поведение.