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Sommerfeld's Method for Solving the Dynamic Rigid Stamp Indentation Problem

Alexey G. Fatyanov*
Institute of Computational Mathematics and Mathematical Geophysics SB RAS
Novosibirsk, Russian Federation

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Abstract. The work is based on Sommerfeld's ideas in solving the diffraction problem on a mirror segment. On this basis, a new method for solving the dynamic problem for a vibrating rigid stamp is developed. The solution is sought by minimizing a functional. Sommerfeld's method is used to select the only physically correct solution. Namely, the expressions in the minimized functional are reduced to dimensionless form. This allowed us to create a method for calculating wave acoustic fields for arbitrary radius of a rigid stamp. Applied to vibration problems, the solution for a small rigid stamp is obtained in explicit form. This allows stable calculation of vibrating wave fields for teleseismic distances. The program created on this basis allows carrying out calculations even on personal computers with OpenMP parallelization. A result of analytical calculations the distinction of wave fields for a stamp and a distributed source of small dimensions are shown.

Keywords: Sommerfeld method, mixed problem, hard stamp, functional minimization, dimensionality equalization, acoustic waves.

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Introduction

A considerable number of works are devoted to methods of solving mixed problems. Let us note the works [1–6]. In the paper, based on the solution of the diffraction problem [1], a new method for solving the dynamic problem for a vibrating rigid die is developed. This method for solving the dynamic problem for a vibrating rigid stamp, which allows to carry out calculations for teleseismic distances. The acoustic case for an arbitrary layered medium is considered. Following [2, 3, 4, 6] in a cylindrical coordinate system on one part of the daytime surface (z=0) is given a displacement different from zero; on the other — stress equal to zero. Following [1], the solution is sought by minimising the functional. This method of solution construction admits an infinite number of mathematically correct solutions. And only one of them will give physically correct solution. To choose the only solution, the behaviour of the solution in the vicinity of the point of discontinuity of the of boundary conditions (condition on an rib) [3, 4, 5]. But other methods are also known. In [2], it is assumed that the force applied to the stamp is known. On this basis of the only solution is found. In this paper, in order to select the only physicallys correct solution, Sommerfeld's method [1] is used. Namely, in the minimised functional, the expressions are reduced to dimensionless form (dimensionalitys is equalised). As a result, the problem is

*agfat@mail.ru https://orcid.org/0000-0001-6038-6505

reduced to the solution of a system of linear algebraic equations (SLAE). This allowed us to create a method of calculation of wave acoustic fields for arbitrary radius of rigid stamp. SLAE is solved on the basis of the open package of linear algebra, developed at Moscow State University. In this case, the solution of SLAE for significant spatial and temporal scales requires the use of technology of high-performance computing. That is, for calculations it is necessary to use supercomputer. For vibration problems, the size of the seismic source (stamp) is always much smaller than the length of the wavelength. In this case, an explicit received formula for the solution in the spectral domain. For it it is no longer necessary to solve SLAE. And it allows to stably calculate vibrating wave fields for teleseismic distances. The programme created on this basis allows to carry out calculations even on personal computers with OpenMP parallelisation. A result of analytical calculations, the distinction between the wave fields for a stamp and a distributed source of small sizs.

1. Statement of the task

The mathematical statement of the task of modelling P waves is formulated in a cylindrical coordinate system ($0 \leqslant r < \infty$, $0 \leqslant z < \infty$) in the axisymmetric case as follows. Determine the function $u(z, r, t)$ from the equation:

$$\frac{\partial^2 u}{\partial z^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial r^2} = \frac{1}{V^2(z)}\frac{\partial^2 u}{\partial t^2}. \tag{1}$$

In this paper, the problem of wave propagation from a vibrating rigid stamp. Statements of the task for a rigid stamp are given in many works [2, 3, 4, 6]. Of these, the boundary conditions in the axisymmetric case for the wave displacement u in the cylindrical coordinate system (r, z) are set as follows:

$$u/_{z=0} = f(t), \qquad r \leqslant r_0, \tag{2}$$

$$\frac{\partial u}{\partial z}/_{z=0} = 0, \qquad r > r_0. \tag{3}$$

The initial conditions are added to the (1)–(3) formulas

$$u = \frac{\partial u}{\partial t}/_{t=0} = 0. \tag{4}$$

In (1)–(3) r_0 is the radius of the stamp, the velocity $V(z) > 0$ is an arbitrary piecewise function (layered medium). The input impulse $f(t)$ is chosen as a Gaussian function $e^{-(\pi f_0 t/2)^2}\sin(2\pi f_0 t)$, f_0 is its carrier frequency.

In addition, we still need a condition for the isolation of a single physically correct solution, which will be discussed below.

2. Analytical method of solution

The solution (1)–(4) is constructed by using finite integral transformations in terms of of time t and lateral variable r:

$$u(z, r, t) = \frac{1}{2T}\sum_{j=-\infty}^{\infty} u(z, r, \omega_j)\exp(-\omega_j \cdot t). \tag{5}$$

$$u(z,r,\omega_j) = \frac{2}{a^2}\sum_{n=1}^{\infty} u(z,k_n\omega_j)J_0(k_nr)/J_1^2(k_na). \tag{6}$$

In (5) and (6) $\omega_j = j\cdot\pi/T,\ k_n = x_n/a$. Where x_n are the roots of the equation $J_0(x) = 0$. T and a are the boundaries of the computational domain.

In the following, irrelevant indices will be omitted to shorten the notation. Also, for the sake of clarity, we first consider the case of a homogeneous half-space. In this case, the equation (1) using (5)–(6) will turn into an ordinary differential equation:

$$\frac{d^2u(z,k_n,\omega)}{dz^2} = (k_n^2-\omega^2/V^2)u(z,k_n,\omega). \tag{7}$$

From (7) we elementarily obtain

$$u(z,k_n,\omega) = u(0,k_n,\omega)\exp(-\nu_nz) = C_n\exp(-\nu_nz), \tag{8}$$

$$\frac{du(z,k_n,\omega)}{dz} = -\nu_nC_n\exp(-\nu_nz). \tag{9}$$

The formulas (8) and (9) give expressions for the displacement u and "stress" $\dfrac{du}{dz}$ as a function of the as yet unknown coefficients C_n, $\nu_n = \sqrt{k_n^2-\omega^2/V^2}$. In the following, for the sake of clarity, "stress" will not be taken in quotes. To satisfy the conditions at infinity in (8)–(9), the principle of limiting absorption is used. For this purpose instead of ω^2 we take $\omega^2 + i\cdot\varepsilon\cdot\omega$, where ε is a small value [7].

Given (8)–(9), the boundary conditions (2)–(3) in the spectral region (k,ω) will look as follows:

$$\sum_{m=1}^{\infty} C_m\beta_mJ_0(k_mr) = F(\omega), \qquad r \leqslant r_0, \tag{10}$$

$$-\sum_{m=1}^{\infty}\nu_mC_m\beta_mJ_0(k_mr) = 0, \qquad r > r_0. \tag{11}$$

In (10)–(11) $F(\omega)$ is the spectrum of the function $f(t)$ and the abbreviation is introduced: $\beta_m = 2/\left\langle a^2J_1^2(k_ma)\right\rangle$.

To find the unknown coefficients of C_m, following Sommerfeld [1] we consider the quadratic errors corresponding to (10) and (11):

$$\int_0^{r_0}\left|F-\sum_m C_m\beta_m J_0(k_mr)\right|^2 rdr \quad\text{and}\quad \int_{r_0}^{a}\left|\sum_m \nu_mC_m\beta_m J_0(k_mr)\right|^2 rdr. \tag{12}$$

The sum of both errors in (12) should reach a minimum at the appropriate choice of C_m. Differentiating over C_n^* we obtain a system of linear algebraic equations (SLAE):

$$\sum_m (a_{n,m}+\nu_n^*\nu_mb_{n,m})\beta_mC_m = F(\omega)\int_0^{r_0} rJ_0(k_nr)dr = F(\omega)g_n. \tag{13}$$

In (13) and hereafter, an asterisk denotes a complex-conjugate quantity. The matrices $a_{n,m}$ and $b_{n,m}$ using Green's formula [1] are calculated exactly by explicit formulas.

$$a_{nn} = \int_0^{r_0} rJ_0(k_nr)J_o(k_mr)rdr = \frac{r_0^2}{2}\left[J_0^2(k_nr_0)+J_1^2(k_nr_0)\right],$$

$$a_{nm} = r_0 \frac{k_m J_0(k_n r_0) J_1(k_m r_0) - k_n J_0(k_m r_0) J_1(k_n r_0)}{k_m^2 - k_n^2}, \quad n \neq m. \tag{14}$$

Similarly for $b_{n,m}$.

In this case, the time frequency ω is included as a parameter in the SLAE (13).

$$g_n = \int_0^{r_0} r J_0(k_n r) dr = r_0 J_1(k_n r_0)/k_n. \tag{15}$$

The equality (11) can be multiplied by any real number X different from zero. In this case, when minimising the functional, instead of the SLAE (13) we obtain:

$$\sum_m (a_{n,m}/\mathrm{X}^2 + \nu_n^* \nu_m b_{n,m}) \beta_m C_m = F(\omega) g_n/\mathrm{X}^2 = F(\omega) f_n. \tag{16}$$

SLAE (16) has an infinite number of mathematically correct solutions for different X. And only one will give a physically correct solution [5]. B At present, the only physically correct solution is chosen on the basis of the asymptotics of the solution in the vicinity of the stamp edge (rib) [3, 4, 5].

However, other methods are also known. In [2], it is assumed that the the force applied to the stamp known. Based on this, received correct solution. In this paper, the single solution is determined based on the method of Sommerfeld method [1]. At consideration of diffraction on a part of a mirror, he brings the corresponding quantities to a dimensionless form. Since ν_n has dimension inverse to the metre, then X must have a dimension in metres. From (15) and (16), consider the expression g_n/X^2. Require, that at $r_0 \to 0 \quad g_n/\mathrm{X}^2 \to 1$. That is, so that at the point stamp ($r_0 = 0$) there is a concentrated impact. Since $J_1(\alpha) \approx \alpha/2$ when α is small, we obtain that $\mathrm{X} = r_0/\sqrt{2}$. The value X will have the dimension in metres. Thus the dimensions in (10) and (11) will coincide. In this case

$$f_n = g_n/\mathrm{X}^2 = 2J_1(k_n r_0)/k_n r_0. \tag{17}$$

The expression (17) coincides with the source of the normal force uniformly distributed over the area of the circle on a flat day surface [8].

For a stamp of arbitrary sizes, the SLAE (16) is solved using the software open source software developed at Moscow State University.

For vibration problems, the size of the seismic source (stamp) is always is much smaller than the wavelength. In this case, following [1], an approximate explicit formula for the solution in the spectral region is obtained. Let $r_0 \ll \lambda$. Here λ is the wavelength. This condition is known fulfilled for the radiating platform of the vibrator.

It is known, for example, from [1] that at small ρ

$$J_0(\rho) \sim 1, \quad J_1(\rho) \sim \rho/2. \tag{18}$$

In (18), terms above the first order of smallness are discarded.

Consider the SLAE (16) at small r_0. Using (18) we obtain

$$a_{nn}/X^2 = \frac{2}{r_0^2} \frac{r_0^2}{2} \left[J_0^2(k_n r_0) + J_1^2(k_n r_0) \right] \sim 1, \quad a_{nm}/X^2 \sim 1. \tag{19}$$

Given (19), the SLAE of (16) will take the form:

$$\sum_m \beta_m C_m + \nu_n^* \nu_n C_n = F(\omega) f_n. \tag{20}$$

After the transformation $\beta_m C_m = x_m$ (20) will take the elementary form:

$$\frac{\nu_n \nu_n^*}{\beta_n} x_n + \sum_m x_m = F(\omega) f_n. \tag{21}$$

We find the solution to (21) as follows. Assume

$$x_m = c(F(\omega) f_m \beta_m / \nu_m \nu_m^*). \tag{22}$$

Substituting (22) into (21) we determine c.

$$c = \frac{f_n}{f_n + s} = \frac{1}{1 + \frac{1}{f_n} s}. \tag{23}$$

In (23)

$$s = \sum_m \frac{\beta_m}{\nu_m \nu_m^*} f_m. \tag{24}$$

Since from (17) and (18)

$$\frac{1}{f_n} = \frac{k_n r_0}{2} \frac{1}{J_1(k_n r_0)} \sim 1. \tag{25}$$

In (25), terms above the first order of smallness are also discarded.
Taking into account (25) we obtain

$$c = c(\omega) = \frac{1}{1 + s}. \tag{26}$$

Since $C_m = x_m / \beta_m$ then from (22–26) taking into account (8) we obtain the the solution for a stamp of small dimensions. On the day surface $z = 0$ the solution looks as follows:

$$u(0, k_n, \omega) = c \frac{F(\omega)}{\nu_n \nu_n^*} f_n. \tag{27}$$

Verification of the accuracy of the formula (27) was performed by comparing it with the solution of the SLAE (16) in the physical domain. For transition to the physical domain formulas (5) and (6) were used. The result was a match with an accuracy of three digits.

At present, the formulations often used for vibration problems are, when the stress distribution on the day surface is given. In [9] it is stated that such a problem is solved much easier than the mixed problem. И there is no need to solve the mixed problem at small sizes of the source.

Let us set a uniform stress distribution on the day surface at $0 < r \leqslant r_0$:

$$\frac{\partial u}{\partial z}/_{z=0} = \frac{2}{r_0^2} f(t), \qquad r \leqslant r_0; \quad \frac{\partial u}{\partial z}/_{z=0} = 0, \qquad r > r_0. \tag{28}$$

The solution of the problem (1), (28), (4) is well known [8]. In the notation of this paper, it is as follows:

$$u(0, k_n, \omega) = \frac{F(\omega)}{\nu_n} f_n. \tag{29}$$

Thus, the solution for a rigid stamp (27) is fundamentally different from the solution for a radiation source in the form of a distributed force (29).

In this approach it is quite simple to take into account the layering of the medium. For this purpose second-order ordinary differential equation (7) by introducing a auxiliary function $\alpha(z)$ such that $\dfrac{du}{dz} = -\alpha(z)u$ in each layer reduces to a first order equation [10].

$$\frac{d\alpha}{dz} - \alpha^2 = -\nu^2. \tag{30}$$

The nonlinear equation (30) has an explicit solution. Let the medium consists of N layers. And all of them are located on a half-space. In this case, the recalculation of the auxiliary function α_p from the layer with index p to the layer with index $p-1$. Index $p-1$ is made by the formula:

$$\alpha_{p-1} = \nu_p \frac{\alpha_p + \nu_p th(\nu_p(h_p - h_{p-1}))}{\nu_p + \alpha_p th(\nu_p(h_p - h_{p-1}))}. \tag{31}$$

In (31) $h_p - h_{p-1}$ is the power of the layer with index k.

The process starts with the layer with index N. In this case, $\alpha_N = \sqrt{k_n^2 - \omega^2/V_{N+1}^2}$, where V_{N+1} is the velocity in the half-space. Finally, using the differential sweep method (31). α_0 is found [10]. And then the solution for a rigid stamp of small size will be given by the formula (27), in which ν is replaced by α_0.

3. Results of the analytical solution

Fig. 1 gives the wavefield for the rigid stamp at z=0. On the vertical axis is the time, milliseconds (increases down); on the horizontal axis is the dis tance, kilometres. The initial distance is 1 kilometre and the final distance is 5 kilometre. The radius of the rigid stamp r_0=. 1 metre. A homogeneous half-space is considered. The velocity in the half-space V=1 km/sec. The pulse $f(t)$ in the source is taken as a Gaussian function with a carrier frequency of 50 hertz. Fig. 1 (A) shows the displacement, and Fig. 1 (B) — stress. It can be seen from Fig. 1 that the displacement occurs and the the stress is zero. Thus it is numerically shown that the condition (3) is fulfilled.

Next, the wave fields for a stamp and a distributed source are given. Moreover, the rigid stamp and the distributed source have small sizes. For the simplest model of a layer on a half-space is taken for comparison. Fig. 2 is given the wave field with a distributed source. A layer on a half-space is taken. The velocity in the layer is 1 km/sec and in the half-space is 2 km/sec. The thickness of the layer is 1 km. The distributed source has a size of 1 metre. In Fig. 3 shows the wave field in the case for a 1 metre size rigid stamp. The other parameters are the same as in Fig. 2. In Figs. 2 and 3, P — direct wave, PP — reflected wave, PPP — multiple wave. It can be seen from Fig. 2 that in the case of supercritical reflection, for example, the reflected wave becomes larger than the direct wave. This is consistent with wave theory [11]. In the case of the rigid stamp in Figure 3, the wave dynamics strongly changes. Thus, the wave fields are different for a rigid stamp and a distributed source of small sizes.

Conclusions

In this paper, based on Sommerfeld's ideas, a new method of solving the dynamic problem for a rigid stamp. It allows to carry out calculations of acoustic waves for teleseismic distances.

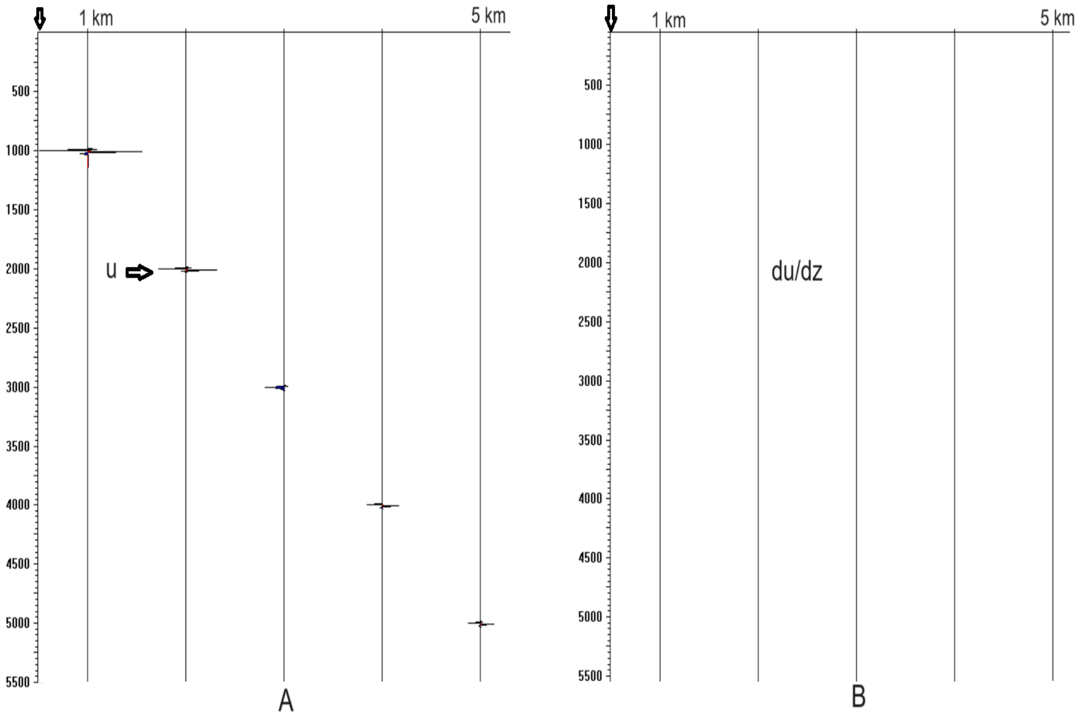

Fig. 1. Wave fields for a rigid stamp. Half-space. The radius of the stamp is 1 metre. Displacement field (A). Stress field (B)

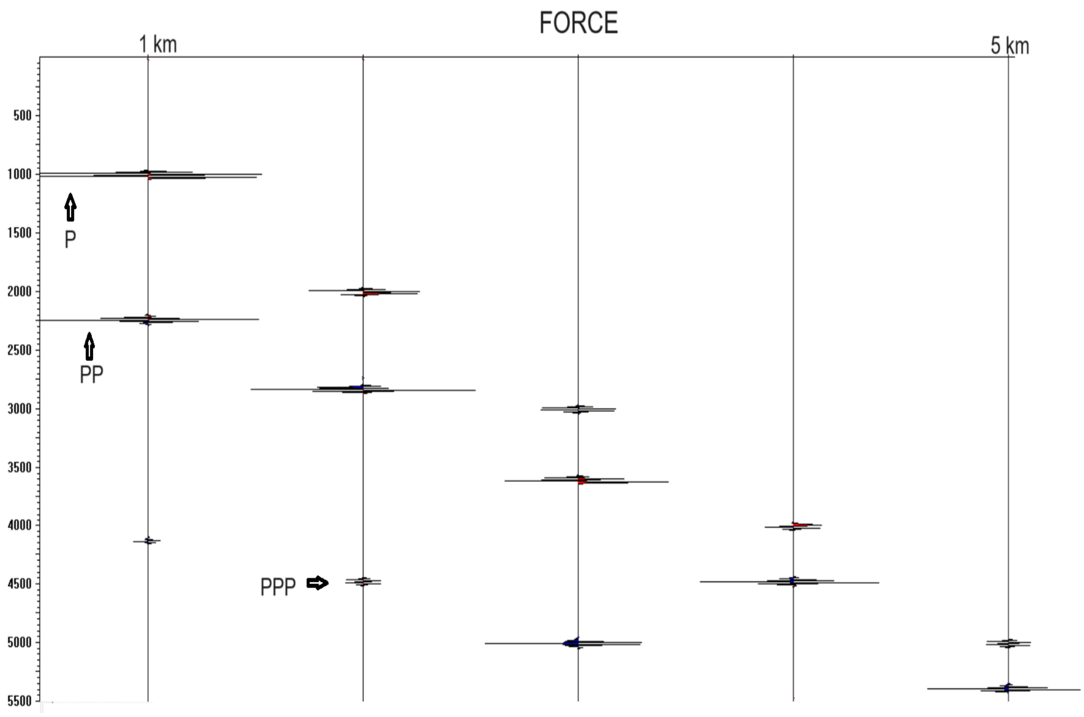

Fig. 2. Wave field for a distributed source of 1 metre. The layer on a half-space

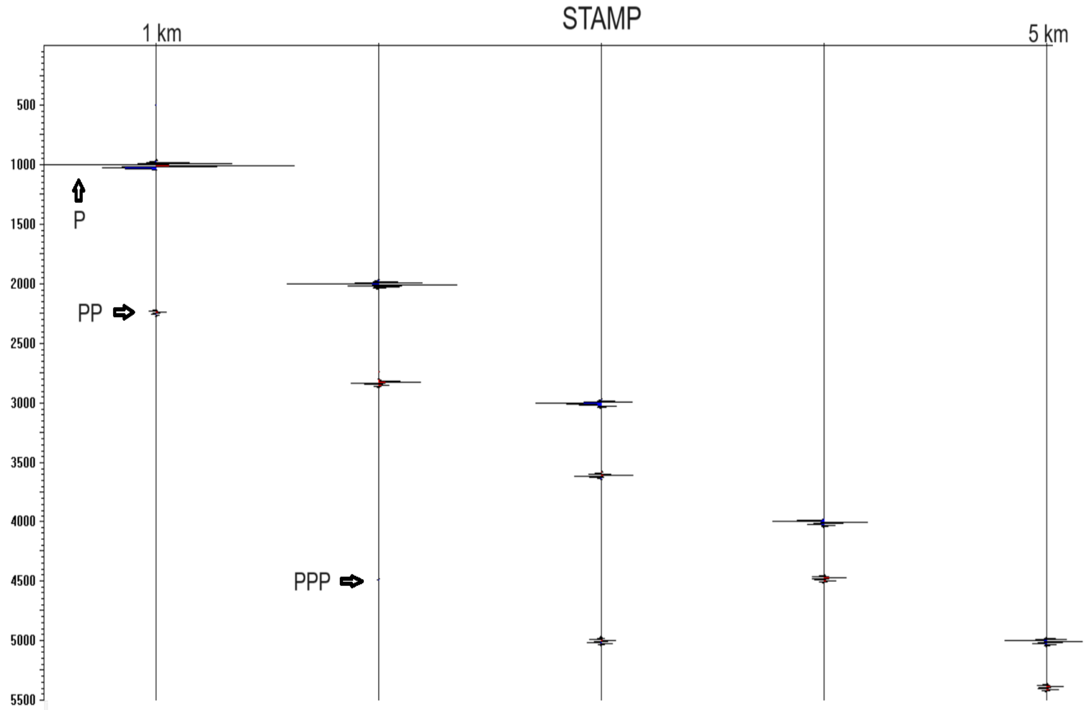

Fig. 3. Wavefield for a 1 metre rigid stamp. The layer on a half-space

The method is based on minimisation of a functional. The functional includes the displacement at the foot of the stamp and the stress outside the stamp. Knowing the law of stress distribution under the plate of the seismic source is not necessary for this method. In order to choose the only physically correct solution of this diffraction problem, the Sommerfeld method is used. Namely, in the minimised functional, the expressions are reduced to a dimensionless form (dimensionalitys is equalised). From the minimisation of the functional in the standard way, a system of linear algebraic equations (SLAE) is obtained. For its solution is used open source software developed at the MSU.

Applied to vibration problems, when the size of the stamp is much smaller than the wavelength, an explicit formula for the solution is obtained. In this case, there is no need to solve SLAE. Therefore, the created programme allows to calculate vibration wave fields for teleseismic distances even on personal computers with OpenMP parallelisation. As a result of analytical calculations, a distinction was found between the wave fields for a rigid stamp and a distributed source in the case of their small sizes.

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References

[1] A.Sommerfeld, Partielle differentialgleichungen der physik, Leipzig, 1948.

[2] S.G.Mikhlin, Application of integral equations, Moscow, 1947 (in Russian).

[3] W.Z.Parton, P.I.Perlin, Methods of mathematical theory of elasticity, Moscow, 1981 (in Russian).

[4] V.A.Babeshko, The method of generalised factorisation in spatial dynamic mixed problems of elasticity theory, Moscow, 1984 (in Russian).

[5] R.Mittra, S.Lee, Analytical techniques in the theory of guided waves, New York, 1971.

[6] A.V.Razin, A.L.Sobisevich, Geoacoustics of layered media, Moscow, 2012 (in Russian).

[7] V.A.Babeshko, I.S.Chichinin, Problems of vibrational research of the Earth, Moscow, 1977, 53–61, (in Russian).

[8] V.V.Gushchin, V.P.Dokuchaev, Y.M.Zaslavsky, V.D.Konyukhova, Earth exploration by non-explosive seismic sources, Moscow, 1981, 113–126 (in Russian).

[9] I.S.Chichinin, Vibratory radiation of seismic waves, Moscow, 1984 (in Russian).

[10] A.G.Fatianov, A semi-analytic method for solving direct dynamical problems in layered media, *Dokl. AN SSSR*, **310**(1990), no. 2, 323–327.

[11] K.Aki, P.G.Richards, Quantitative seismology, San Francisco, 1983.

Метод Зоммерфельда решения динамической задачи о вдавливании жесткого штампа

Алексей Г. Фатьянов
Институт вычислительной математики и математической геофизики СО РАН
Новосибирск, Российская Федерация

Аннотация. Работа основана на идеях Зоммерфельда при решении задачи дифракции на сегменте зеркала. На этой основе развит новый метод решения динамической задачи для вибрирующего жесткого штампа. Решение ищется с помощью минимизации функционала. Для выбора единственного физически верного решения используется метод Зоммерфельда. А именно, в минимизируемом функционале выражения приводятся к безразмерному виду. Это позволило создать метод расчета волновых акустических полей для произвольного радиуса жесткого штампа. Применительно к вибрационным задачам получено решение для малого жесткого штампа в явном виде. Это позволяет устойчиво вычислять вибрационные волновые поля на телесейсмические расстояния. Созданная на этой основе программа позволяет проводить расчеты даже на персональных компьютерах с распараллеливанием OpenMP. В результате аналитических расчетов показано отличие волновых полей для штампа и распределенного источника малых размеров.

Ключевые слова: метод Зоммерфельда, смешанная задача, жесткий штамп, минимизация функционала, выравнивание размерностей, акустические волны.