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On the Velocities of Rayleigh Surface Waves Propagating along Boundaries of Generalized Continua

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Abstract. It is demonstrated that mathematical models of gradient-elastic medium and reduced Cosserat medium, in contrast to the model of classical deformable solid, allow one to describe experimentally observed dispersion of Rayleigh surface wave, i.e., relationship between phase velocity of surface wave and frequency. At the same time, according to the model of gradient-elastic half-space, velocity of surface wave cannot exceed the velocity of shear wave but at certain values of frequency it can reach it. According to reduced Cosserat model, velocity of surface wave exceeds the velocity of shear wave as well as velocity of propagation of surface wave in classical half-space and gradient-elastic half-space.

Keywords: gradient-elastic half-space, reduced Cosserat model, surface wave, dispersion, phase velocity, frequency.

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In 1885, English scientist Lord Rayleigh (John William Strutt) theoretically demonstrated that waves can propagate along the flat boundary of a solid elastic half-space with vacuum or with sufficiently rarefied medium (for example, air), and their amplitudes rapidly decrease with depth [1]. These waves called Rayleigh surface waves. They depend on the frequency range and have different applied directions.

It became obvious that Rayleigh waves in the low-frequency range (1–100 Hz) are the main type of waves observed during earthquakes. Therefore, they have been studied in detail in seismology for almost 140 years [2].

The main features of propagation of Rayleigh waves are as follows: absence of dispersion, i.e., the wave speed does not depend on its frequency and it is constant for each material; the speed is slightly less than the speed of the bulk shear wave by a factor $0.87-0.96$; the displacement vector has longitudinal and transverse components, and the transverse component always exceeds the longitudinal component [3].

A series of works by V. V. Krylov [4–10] was devoted to the study of elastic vibrations of the earth generated by trains and motor vehicles. Very high level of ground vibrations generated by high-speed trains moving at a speed higher than the speed of Rayleigh surface waves in the ground was theoretically predicted. For these works V. V. Krylov was honoured with the Rayleigh Medal in 2000 awarded by the Acoustical Institute of Great Britain and often called the Nobel

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Prize in Acoustics. Krylov's theory was experimentally confirmed in 1997–1998 (with his direct participation) on a new high-speed line in Sweden (Gothenburg-Malmo), where on some sections of the route the speed of Rayleigh waves was only 45 m/s, and a train speed of 160 km/h was enough to observe the effect. The discovered effect became known as "ground vibration shock" (by analogy with the well-known sonic boom from a supersonic aircraft). The generation sources became known as "trans-Rayleigh trains" [11].

It should be noted that the existence of critical speeds of load movement along rail guides above which bending waves are generated in the guides was discussed back in the first half of the 1980s [12–14]. However, the critical speeds calculated at that time showed the practical unattainability of the effect of generating bending waves in the guides by a vehicle. It turned out to be easier for the load to overcome the speed of the Rayleigh wave in the soil located under the rail guide and the guide itself with the system of sleepers and ballast acted as an intermediary between the source of wave generation and the environment in which these waves arose.

At present, problems of stability of motion of high-speed objects along rail guides and problems of generation of bending and bending-torsional waves in rail guides are recognized as relevant, and results of their solution serve as methodological and computational support for experiments on high-speed acceleration (or braking) of payloads on rocket tracks citeerofeev15,erofeev16,erofeev17,erofeev18, erofeev19, erofeev20.

The mechanics of a homogeneous isotropic deformable solid excludes the possibility of surface wave propagation with a speed greater than the speed of shear wave. However, along with the classical continuum model generalized continuum models are also quite widely used in the mechanics of deformable solid [21–24].

The study of generation of Rayleigh waves by sources moving along the boundaries of non-classical elastic half-spaces was presented [25]. The purpose of the work was to determine how the velocities of shear waves and Rayleigh surface waves are related for materials described by the equations of mechanics of generalized (non-classical) continua such as Cosserat continuum [26] and its modifications [27, 28]) and the gradient-elastic medium [29–31].

1. Dispersion properties of surface waves in generalized continua

As generalized continua, the gradient-elastic medium and the reduced Cosserat medium are further considered.

Vector equations of the dynamics of studied media are written with respect to displacements as follows

- for a gradient-elastic medium

$$\rho\ddot{\mathbf{u}} - (\lambda+\mu) grad\, div\, \mathbf{u} - \mu\Delta\mathbf{u} + 4\mu L^2 \Delta(\Delta\mathbf{u} + \widetilde{\nu} grad\, div\, \mathbf{u}) = 0 \quad (1)$$

  where $\mathbf{u}$ — vector of displacements, λ and μ — Lame elastic constants, L — the ratio of the curvature modulus to the shear modulus, $\widetilde{\nu}$ — dimensionless constant;

- for the reduced Cosserat medium

$$(\lambda+2\mu)\nabla(\nabla\cdot\mathbf{u}) - \mu\nabla * (\nabla * \mathbf{u}) - J\frac{\partial^2}{\partial t^2}\nabla * (\nabla * \mathbf{u}) = \rho\frac{\partial^2 \mathbf{u}}{\partial t^2}, \quad (2)$$

  where J — constant which characterize the inertial properties of a macrovolume, ρ — density of the material.

By introducing a scalar φ and vector ψ potentials, solutions of equations (1) and (2) are sought. In this case, displacement vector $\mathbf{u}$ can be written in the form

$$\mathbf{u} = \nabla\varphi + \nabla * \psi. \tag{3}$$

Note that for a plane problem the vector potential has only one non-zero component which is denoted by ψ. Then, two equations are obtained from equations (1) and (2):

- for a gradient-elastic medium

$$\Delta\varphi - \frac{1}{c_1^2}\frac{\partial^2\varphi}{\partial t^2} = 0, \quad \Delta(1 - L^2\Delta)\psi - \frac{1}{c_2^2}\frac{\partial^2\psi}{\partial t^2} = 0, \tag{4}$$

- for the reduced Cosserat medium

$$\Delta\varphi - \frac{1}{c_1^2}\frac{\partial^2\varphi}{\partial t^2} = 0, \quad \Delta\psi + G\Delta\frac{\partial^2\psi}{\partial t^2} - \frac{1}{c_2^2}\frac{\partial^2\psi}{\partial t^2} = 0, \tag{5}$$

where $G = J/\mu$.

The solution of equations (4) and (5) is sought in the form of harmonic waves propagating in the direction of the axis . Moreover, one should select only those solutions which correspond to a decrease in wave amplitudes with depth. Then one can obtained

- for a gradient-elastic medium

$$\varphi = Ae^{\zeta y + i(\omega t - kx)}, \quad \psi = B_1 e^{\eta_1 y + i(\omega t - kx)} + B_2 e^{\eta_2 y + i(\omega t - kx)}, \tag{6}$$

- for the reduced Cosserat medium

$$\varphi = Ae^{\zeta y + i(\omega t - kx)}, \quad \psi = Be^{\eta y + i(\omega t - kx)}. \tag{7}$$

Taking into account the absence of stresses at the boundary $y = 0$, the following dispersion equation for a gradient-elastic medium are obtained

$$\begin{gathered}16(1-\beta\varsigma)(1+\alpha-\varsigma)[1+2\alpha+2\sqrt{\alpha(1+\alpha-\varsigma)}] = \\ = (2-\varsigma)^2[(1-3\alpha^2)^2+\alpha(3-\alpha)(1+\alpha-\varsigma)+(1-\alpha^2)(1+\alpha-\varsigma)^2+\alpha(1+\alpha-\varsigma)^3+ \\ +2(1-3\alpha^2)(3-\alpha)\sqrt{\alpha(1+\alpha-\varsigma)}+2(1-3\alpha^2)(1-\alpha)(1+\alpha-\varsigma)- \\ -2(1-3\alpha^2)(1+\alpha-\varsigma)\sqrt{\alpha(1+\alpha-\varsigma)}+2(3-\alpha)(1-\alpha)(1+\alpha-\varsigma)\sqrt{\alpha(1+\alpha-\varsigma)}- \\ -2(3-\alpha)(1+\alpha-\varsigma)^2\alpha-2(1-\alpha)(1+\alpha-\varsigma)^2\sqrt{\alpha(1+\alpha-\varsigma)}].\end{gathered} \tag{8}$$

Here $\varsigma = c_R^2 = \dfrac{\omega^2}{k^2c_2^2}$, $\alpha = L^2k^2$, $\beta = \dfrac{1-2\nu}{2-2\nu}$, ν — Poisson ratio. When $L = 0$, equation eqrefeq:8 is reduced to the dispersion equation of the Rayleigh surface wave in the classical case [3]. Analysis of equation (8) showed that for gradient-elastic medium, the dispersion properties of the Rayleigh surface wave in the "wave number – frequency" plane are described by two curves. The first curve (the lower one) comes from the origin of coordinates. The origin of the second curve is shifted upward along the frequency axis. In this case the surface wave has two modes, and each mode has dispersion since velocities of both modes depend on frequency. When frequency increases the speed of each mode of the surface wave increases, and as frequency tends to infinity ($\omega \to \infty$) the velocity of the lower mode goes from below to the horizontal asymptote

$C_R = \sqrt{2}\, c_2$, where $c_2 = \sqrt{\dfrac{\mu}{\rho}}$. The speed of the upper mode of the surface wave increases from the classical values for Rayleigh waves $(0,87\breve{\ }0,96)c_2$, reaches a maximum, and then goes to the horizontal asymptote $C_R = \sqrt{2}\, c_2$ when $(\omega \to \infty)$. The value of the maximum velocity increases with increasing Poisson ratio. For example, for materials with Poisson ratio close to 0.5 the surface wave velocity at the peak of the curve is 4% greater than the maximum surface wave velocity for materials with Poisson ratio close to 0.2. Analysis of the second equation of system (4) shows that volume shear wave in gradient-elastic medium also has dispersion. This follows from the non-linear relationship between frequency and the wave number $\omega^2 = c_2^2 k^2(1 + L^2k^2)$. It allows one to calculate the phase velocity

$$V_\phi^2 = \frac{\omega^2}{k^2} = c_2^2(1 + L^2k^2). \tag{9}$$

It is easy to see from (9) that for any non-zero value of the wave number (or frequency) $V_{ph} > c_2$ and, consequently, c_2 is not the true velocity of the dispersive shear wave but it serves only as its lower limit. Therefore, the Rayleigh wave velocity C_R should be compared not with c_2 but with V_{ph}.

This result demonstrates that velocity of the surface wave cannot exceed the phase velocity of the shear wave, reaching it at certain frequency values.

Similarly, taking into account the absence of stresses at the boundary $y = 0$, one can obtain dispersion equation for reduced Cosserat medium

$$\eta \left[\eta^3 - 8\eta^2 + \left(24 - 16\frac{\varsigma}{1 - \frac{J}{\mu}\omega^2}\right)\eta - 16\left(2 - \frac{1}{1 - \frac{J}{\mu}\omega^2} - \varsigma\right)\right] = 0. \tag{10}$$

where $\varsigma = \dfrac{c_2^2}{c_1^2}$, $\eta = \dfrac{c_R^2}{c_2^2}$.

Analysis of equation (10) showed that here, too, unlike the classical case [3], the Rayleigh surface wave has dispersion. In the "wave number – frequency" plane, there are two dispersion branches: lower and upper. With increasing frequency, the speed of the surface wave related to the lower dispersion branch decreases and at infinity the square of the speed of the surface wave $c_R^2 \to 0.7\, c_2^2$. The velocity of the surface wave related to the upper dispersion branch increases with increasing frequency. For dimensionless frequencies $\omega > 9$ this growth becomes unlimited. Consequently, the upper dispersion branch describes wave processes in the interval of dimensionless frequencies $0 < \omega < 9$, then the process ceases to be wave-like.

Analysis of the second equation of system (5) shows that bulk shear wave in the reduced Cosserat medium also has dispersion. This follows from the non-linear relationship between frequency and the wave number $\omega^2 = \dfrac{k^2c_2^2}{1 + \frac{Gk^2}{c_2^2}}$ which allows one to calculate the phase velocity

$$V_{ph\tau}^2 = \frac{\omega^2}{k^2} = \frac{c_2^2}{1 + \frac{Gk^2}{c_2^2}}. \tag{11}$$

Comparing the frequency dependence of the velocity of the surface wave related to the upper dispersion branch and the phase velocity of the shear wave given in (11), one can see that velocity of the surface wave in the entire frequency range exceeds the phase velocity of the shear wave which converges to c_2 for $\omega \to 0$ and decreases monotonically to zero when $C_R^2 \to 0.8c_2^2$ for $\omega \to \infty$.

The frequency dependencies of $V_{ph\tau 1}^2$ (the square of the phase velocity of the shear wave in a gradient-elastic medium) and $V_{ph\tau 1}^2$ (the square of the phase velocity of the shear wave in the reduced Cosserat medium) are shown in Fig. 1. It is evident from the graphs that phase velocity of the shear wave in the gradient-elastic medium exceeds phase velocity of the shear wave in the reduced Cosserat medium in the entire frequency range.

The frequency dependences of C_{R1}^2 (the square of the Rayleigh wave velocity in a gradient-elastic half-space), C_{R2}^2 (the squared Rayleigh wave velocity in a reduced Cosserat medium (lower dispersion branch)), C_{R3}^2 (the Rayleigh wave velocity in the classical isotropic elastic half-space) are shown in Fig. 2. It is evident from the graphs that at low frequencies the maximum propagation velocity of Rayleigh waves is observed in the reduced Cosserat medium. At the same time, when frequency increases the Rayleigh wave velocity in the reduced medium decreases, and the Rayleigh wave velocity in the gradient-elastic half-space increases and exceeds the wave velocity in the Cosserat medium over the entire frequency range. It is also evident from the graphs that Rayleigh wave velocity in the classical half-space exceeds the wave velocity in the Cosserat medium when frequency increases. At the same time, the surface wave velocity in the classical medium is independent of frequency. Therefore, waves in this medium do not have dispersion. Considering the upper dispersion branch of the Rayleigh wave velocity, one can see that velocity of the surface wave in the reduced Cosserat medium exceeds the surface wave velocities in the classical medium and in the gradient-elastic medium.

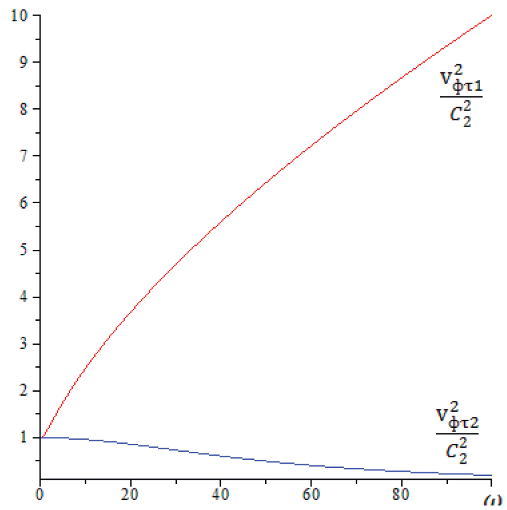

Fig. 1. Frequency dependences of phase velocities of shear waves

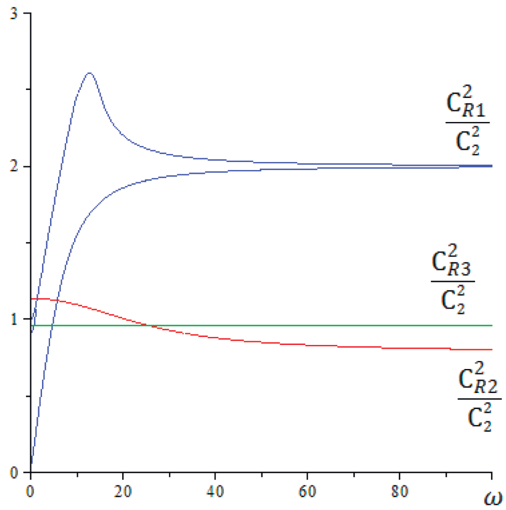

Fig. 2. Frequency dependences of velocities of surface waves

Conclusion

It is shown that velocity of the surface wave propagating along the free boundary of the gradient-elastic half-space is a function of frequency, i.e., the wave has dispersion, and it can exceed the velocity of the bulk shear wave calculated as the square root of the ratio of the shear modulus to the density of material. However, in the medium under consideration the shear wave also has dispersion and the value of the specified velocity is only the lower limit of its phase velocity. Thus, in a gradient-elastic medium the phase velocity of the surface wave cannot exceed the phase velocity of the bulk shear wave but at certain values of the wave number it can reach it. Rayleigh surface waves propagating along the free boundary of the half-space of the Cosserat medium (reduced model) also have dispersion. In the "phase velocity – frequency" plane for such waves there are lower and upper dispersion branches. When frequency increases the phase velocity of the wave related to the lower dispersion branch decreases. The phase velocity of the wave related to the upper dispersion branch increases when frequency increases. The phase velocity of the surface wave exceeds the phase velocity of the bulk shear wave in the entire frequency range.

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О скоростях поверхностных волн Рэлея, распространяющихся вдоль границ обобщенных континуумов

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Аннотация. Показано, что математические модели градиентно-упругой среды и редуцированной среды Коссера, в отличие от модели классического деформируемого твердого тела, позволяют описать наблюдаемую экспериментально дисперсию поверхностной волны Рэлея, т.е. зависимость фазовой скорости от поверхностной волны частоты. При этом, согласно модели градиентно-упругого полупространства, скорость поверхностной волны не может превосходить скорости сдвиговой волны, но при определенных значениях частоты может ее достигать. Согласно же редуцированной модели Коссера скорость поверхностной волны превышает скорость сдвиговой волны, а также скорость распространения поверхностной волны в классическом полупространстве и градиентно-упругом полупространстве.

Ключевые слова: градиентно-упругое полупространство, редуцированная модель Коссера, поверхностная волна, дисперсия, фазовая скорость, частота.