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Another Proof of Puiseux's Theorem on Algebraic Functions

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Abstract. The paper gives a simpler proof of Puiseux's theorem on the algebraic function for polynomials of a special form.

Keywords: Newton diagram, Puiseux series, singular point, Weierstrass polynomial.

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Introduction

Isaac Newton, in a letter to Oldenburg [1], outlined the idea of an algorithm for finding a solution to an algebraic equation $F(z,w)=0$ in the form of a series with a fractional exponent of the variable z. Now it is called the Newton diagram method. It should be noted that Newton did not consider the question of the convergence of the resulting series. For the first time, the fact that the solutions obtained by the Newton diagram method converge in a certain neighborhood of zero was proved by Victor Puiseux in [2]. This fact is called Puiseux's theorem.

The next stage in the development of interest in this issue were works using techniques equivalent to resolving the singularities of algebraic curves in modern terminology. Namely, for an algebraic curve V, defined by the equation $F(w,z)=0$, are constructed a non-singular curve $\bar{V}$ and a map $\phi:\bar{V}\to V$ such that the restriction $\phi:\bar{V}\setminus\phi^{-1}(V_{sing})\to V\setminus V_{sing}$ is a birational isomorphism. The map ϕ is a composition of blow-ups of singular points ($\sigma-$processes). For a non-singular curve $\bar{V}$ identification of regular branches in a neighborhood of points from $\phi^{-1}(V_{sing})$ is possible due to the implicit function theorem. These branches are mapped by ϕ into solutions of the equation $F(z,w)=0$, which are given by convergent series. A thorough presentation of this approach to finding solutions to the equation $F(z,w)=0$, in the language of modern algebraic geometry is given in the [3, 5, chapter 2], as well as in [4, section 8.4].

Puiseux's theorem can also be obtained from other considerations, for example, from the expansion of the polynomial $F(z,w)$ into the product of irreducible Weierstrass polynomials with respect to the variable z. By considering each irreducible polynomial separately, it is possible to construct a local parameterization of the branch of the curve it defines. Each of the formal solutions of the equation $F(z,w)=0$ coincides with one of the obtained parameterizations, and is thus convergent. A detailed proof of this fact can be found in the monograph [4, section 8.3].

For some classes of equations, the proof of Puiseux's theorem can be obtained without using the constructions discussed above. This paper presents one such class of equations whose coefficients are convergent Puiseux series. It is shown that all convergent solutions can be obtained immediately from the Newton diagram of the original equation, and, in particular, intermediate resolutions of singularities can be omitted. Thus, the proposed method is of interest for assessing the theoretical complexity of solving equations of the form $F(z,w)=0$.

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1. Newton's diagram and statement of the main theorem

In this paper we consider an arbitrary equation of two complex variables

$$F(z,w) = \sum_{(\beta,\alpha)\subset A\subset\mathbb{N}^2} a_{\beta\alpha} z^\alpha w^\beta = \sum_\beta A_\beta(z) w^\beta = 0, \tag{1}$$

where $A_\beta(z) \in \mathbb{C}[z]$.

Definition 1. *A Puiseux series in one variable is a formal algebraic expression of the form*

$$f(z) = \sum_{n=n_0}^{+\infty} a_n z^{\frac{n}{m}},$$

where n_0 is an integer, m is natural (for $m = 1$ the result is a Laurent series), coefficients a_n taken from some ring R.

Definition 2. *Newton's diagram $N(F)$ of equation (1) is the set of compact faces of the unbounded polyhedron c.h. $(\cup P_\beta)$, where $P_\beta = \{(\beta, s) : s \geqslant \alpha\}$ (c.h. denotes the convex hull).*

Let us give a brief description of Newton's algorithm.
It is necessary to find all solutions $w = w(z)$ of equation (1) in the form of Puiseux's series:

$$F(z, w(z)) = 0.$$

The strategy for finding solutions $w(z)$ is the following:
Let $w(z) = cz^\sigma + \widetilde{w}(z)$, where $\widetilde{w}$ is a series of monomials of degree $> \sigma$ and let $\sigma = \dfrac{p}{q}$. Then

$$F(z, w(z)) = \sum_{(\beta,\alpha)\in A} a_{\beta\alpha} z^\alpha (cz^\sigma + \widetilde{w})^\beta = \sum_{(\beta,\alpha)\in A} (a_{\beta\alpha} c^\beta z^{\alpha+\frac{p}{q}\beta} + o(z^{\sigma\beta+\alpha})).$$

In order for $F(z, w(z)) \equiv 0$, it is necessary that the quantity

$$\alpha + \frac{p}{q}\beta = \frac{1}{q}(\alpha q + p\beta)$$

reaches a minimum on A in at least two points, i.e. on some edge $\tau \subset N(F)$. So, the condition on $\sigma = \dfrac{p}{q}$ is as follows:

(1) $N(F)$ has an edge τ with the slope σ, i.e. with the directing vector (q, p).
And the condition on c:

(2) c is a nonzero solution to the equation

$$\sum_{(\beta,\alpha)\in\tau} a_{\alpha\beta} c^\beta = 0.$$

The number of such roots (taking into account multiplicity) is equal to the length of the projection τ onto the β axis.

Let us formulate the main result of this work.

Theorem. Let the equation $F(z, w) = 0$ be such that each edge of its Newton diagram does not contain integer points other than the vertices. Then each of its solutions, obtained using Newton's algorithm, is a convergent Puiseux series.

2. Auxiliary statements

We precede the proof of the theorem with several auxiliary statements.

Lemma 1. *Let $G \subset \mathbb{C}$ be a bounded domain with piecewise smooth boundary and let $f \in \mathcal{O}(\bar{G})$ have a unique zero $a \in G$ in $\bar{G}$ of multiplicity 1. Then for any $\varphi \in \mathcal{O}(\bar{G})$ the following formula holds:*

$$\frac{1}{2\pi i}\int_{\partial G}\varphi\frac{df}{f} = \varphi(a) \tag{2}$$

Proof. It follows immediately from Cauchy's theorem and the residue formula for a meromorphic function at a simple pole:

$$\frac{1}{2\pi i}\int_{\partial G}\varphi\frac{df}{f} = res_{z=a}\frac{\varphi f'}{f} = \frac{\varphi(a)f'(a)}{f'(a)} = \varphi(a).$$

Note: If in the lemma we assume that f has a finite number of simple zeros $a_1, a_2, \ldots, a_N \in G$ in $\bar{G}$, then by the residue theorem and formula (2) we get

$$\frac{1}{2\pi i}\int_{\partial G}\varphi\frac{df}{f} = \sum_{i=1}^{N}\varphi(a_j). \tag{3}$$

In particular, when $\varphi \equiv 1$, we obtain a formula known from the complex analysis course

$$\frac{1}{2\pi i}\int_{\partial G}\frac{df}{f} = N.$$

Let us now assume that a is a zero of f of multiplicity μ, i.e. in a neighborhood U of point a

$$f(z) = (z-a)^{\mu}\psi(z), \qquad \psi(a) \neq 0.$$

Then, for any sufficiently small complex ξ, the function $f(z) - \xi$ has in U exactly μ simple roots $z_j(\xi)$, tending to a as $\xi -> 0$. Indeed, let us make the biholomorphic change $(\xi - z)\psi^{\frac{1}{\mu}}(z) = w$ (here is a branch of the radical $\psi^{\frac{1}{mu}}(z)$ can be chosen in U since $\psi(a) \neq 0$). Then the function $f(z) - \xi$ takes the form $w^{\mu} - \xi$, which shows that it has μ simple roots tending to zero as $\xi -> 0$.

According to (1) and (2)

$$\frac{1}{2\pi i}\int_{\partial U}\varphi(z)\frac{df(\xi)}{f(\xi)} = \frac{1}{2\pi i}\lim_{\xi->0}\int_{\partial U}\varphi(z)\frac{d[f(\xi)-\xi]}{f(\xi)-\xi} = \lim_{\xi->0}\sum_{j=1}^{\mu}\varphi(z_j(\xi)) = \mu\varphi(a).$$

From here, using the residue theorem, we get

Theorem 1 (on logarithmic residue). *Let $G \subset \mathbb{C}$ be a bounded domain with piecewise smooth boundary and $f \in \mathcal{O}(\bar{G})$ has a finite number of zeros $a_j \in G$ of multiplicities in $\bar{G}$ μ_j. Then for any $\varphi \in \mathcal{O}(\bar{G})$*

$$\frac{1}{2\pi i}\int_{\partial G}\varphi\frac{df}{f} = \sum_{j}\mu_j\varphi(a_j).$$

In particular, for $\varphi \equiv 1$ the integral on the left is equal to the number of zeros of the function f, taking into account their multiplicities.

Consider a function in $(\zeta, z) \in \mathbb{C}^2$ holomorphic at the origin and having a Taylor expansion of the form

$$\Phi(\zeta, z) = zP(\zeta, z) + \sum_{i+j>d} a_{ij}z^i\zeta^j, \tag{4}$$

where $d \geqslant 2$, P is a homogeneous polynomial of degree $d-1$, and $P(\zeta, 0) \neq 0$, that is, P has a monomial of the form $a\zeta^{d-1}$.

Theorem 2 (A. P. Yuzhakov). *The equation $\Phi(\zeta, z) = 0$ has a solution (branch) of the form*

$$z = z(\zeta) = \sum_{k\geqslant 2} c_k \zeta^k.$$

Proof. Let us choose the weight with respect to which the monomial $az\zeta^{d-1}$ has minimal degree in expansion (3). As such a weight we can take $\left(\frac{3}{2}, 1\right)$. Since each monomial $z^i\zeta^j$ with respect to this weight has a degree

$$\frac{3}{2}i + j = \frac{1}{2}i + (i + j),$$

and it is easy to see that on the Newton diagram it reaches its minimum value at a single point $(i, j) = (1, d - 1)$.

Let us denote $\theta(\zeta, z) = \Phi(\zeta, z) - az\zeta^{d-1}$. Then on the skeleton $|z| = r^{\frac{3}{2}}, |\zeta| = r$ we have
$|az\zeta^{d-1}| = |a|r^{d+\frac{1}{2}}$;
$|\theta(\zeta, z)| = r^{\frac{1}{2}+d+\epsilon}\alpha(r)$,
where $\epsilon > 0$ and $\alpha(r)$ is bounded. Consequently, for a sufficiently small r on the set $\left\{|z| = r^{\frac{3}{2}}\right\} \times \left\{\frac{r}{2} \leqslant |\zeta| \leqslant r\right\}$ there is an inequality

$$|az\zeta^{d-1}| > |\theta(\zeta, z)| \tag{5}$$

Considering $\Phi(\zeta, z)$ as a function of z in the circle $|z| \leqslant r^{\frac{3}{2}}$ with parameter ζ from the ring $K = \left\{\frac{r}{2} \leqslant |\zeta| \leqslant r\right\}$, according to Rouche's principle, we obtain that it has a single zero in the indicated circle $z = z(\zeta)$.

By the logarithmic residue formula $\big($applied to $G = \left\{|z| < r^{\frac{3}{2}}\right\}$, $\phi(z) = z\big)$:

$$z(\zeta) = \frac{1}{2\pi i} \int\limits_{|z|=r^{\frac{3}{2}}} \frac{z\Phi'_z(\zeta, z)}{\Phi(\zeta, z)} dz.$$

As an integral over the compact set $|z| = r^{\frac{3}{2}}$ of a continuous integrand that holomorphically depends on the parameter ζ from the ring K, the function $z(\zeta)$ is holomorphic in this ring.

Let $z(\zeta) = \sum\limits_{k=-\infty}^{+\infty} c_k \zeta^k$ be the Laurent expansion for $z(\zeta)$, convergent at least in the ring K. The coefficient c_k is represented by the integral

$$c_k = \frac{1}{2\pi i} \int\limits_{|\zeta|=\rho} \frac{z(\zeta)}{\zeta^{k+1}} d\zeta = \frac{1}{(2\pi i)^2} \int\limits_{\substack{|\zeta|=\rho \\ |z|=r^{\frac{3}{2}}}} \frac{z\Phi'_z(\zeta, z)}{\zeta^{k+1}\Phi(\zeta, z)} dz d\zeta, \tag{6}$$

where $\frac{r}{2} \leqslant |\rho| \leqslant r$.

Recall that

$$\Phi(\zeta, z) = az\zeta^{d-1} + \theta(\zeta, z),$$

where θ is a series in $z^i\zeta^j$, for which $i + j \geqslant d$.

Due to the inequality (4) on the skeleton $|\zeta| = \rho, |z| = r^{\frac{1}{2}}$ there is an expansion into a series of geometric progression

$$\frac{1}{\Phi} = \frac{1}{az\zeta^{d-1}(1 + \frac{\theta}{az\zeta^{d-1}})} = \sum_{l=0}^{\infty} (-1)^l \frac{\theta^l}{(az\zeta^{d-1})^{l+1}},$$

convergent uniformly on the skeleton. The integrand expression will then expand into the series

$$\sum_{l=0}^{\infty}(-1)^l \frac{\zeta^{-k-1} z \Phi'_z \theta^l}{(az\zeta^{d-1})^{l+1}}.$$

The degree of the numerator is equal to $-k-1+d+dl = -k-1+d(l+1)$, and the degree of the denominator is equal to $d(l+1)$. Therefore, the integral of each term is equal to zero if $-k-1 > -2$, that is, if $k < 1$.

Thus, $\forall k < 1$ the Laurent coefficient $c_k = 0$, thereby $z(\zeta)$ is holomorphic at zero, and $z(0) = 0$. It is easy to show (taking into account the form of $\theta = z^2 p' + \theta'$, where p' is homogeneous of degree $d-2$, and $ord\theta' \geqslant d+1$), so $c_1 = 0$.

3. Proof of Puiseux's theorem

Now we prove the main theorem of two-dimensional algebraic geometry.

Let $F(z,w)$ be a polynomial of two variables whose Newton diagram $N(F)$ has an edge with ends $(\alpha, p+\beta)$ and $(q+\alpha, \beta)$.

We also assume that the edge has no other integer points, so F has the form

$$(az^p + bw^q)w^\alpha z^\beta + \sum_{ip+jq>\alpha p+(p+\beta)q} a_{ij} w^i z^j.$$

The selected two terms can be normalized so that $a = 1, b = -1$:

$$(z^p - w^q)w^\alpha z^\beta + \sum_{ip+jq>\alpha p+(p+\beta)q} a_{ij} w^i z^j.$$

The change $z^p = \xi^q,\ z = \xi^{\frac{q}{p}}$ gives

$$(z^p - w^q)w^\alpha \xi^{\frac{\beta q}{p}} + \sum_{ip+jq>\alpha p+(p+\beta)q} a_{ij} w^i \xi^{\frac{pj}{q}}.$$

But $\xi^q - w^q = (\xi - w)(\xi^{q-1} + \xi^{q-2}w + \cdots + w^{q-1})$, which means function F will look like

$$(\xi - w)P(\xi, w) + \sum_{ip+jq>\alpha p+(p+\beta)q} a_{ij} w^i \xi^{\frac{pj}{q}}.$$

After the change $\xi - w = u$ we get

$$uP(\xi, \xi - u) + \sum_{ip+jq>\alpha p+(p+\beta)q} a_{ij} w^i \xi^{\frac{pj}{q}}.$$

But according to Yuzhakov's theorem there is a solution (holomorphic) $u = u(\xi)$,
therefore, $w = \xi - u(\xi) = z^{\frac{p}{q}} +$ series in powers of $z^{\frac{1}{q}}$.

4. Comparison with the singularity method

Let us illustrate with an example when the proven theorem leads to the goal faster than the technique of resolving the singularities of algebraic curves. Thus, the given result can be considered as an interesting fact for assessing the theoretical complexity of solving an equation of the form $F(z,w) = 0$.

Consider an equation of the form

$$G(z,w) = az^{\alpha}w^{\beta} + \sum_{i+j>\alpha+\beta} a_{ij}z^{i}w^{j} = 0$$

and satisfying the conditions of Theorem 1. This function has a singular point $(0,0)$ of order $\alpha+\beta$. Recall that a σ-process centered at the point $(0,0)$ is (for the case of a plane curve) a transformation which in the affine part of the projective plane is a mapping $\phi : \mathbb{C}^2 \to \mathbb{C}^2$, which in coordinates has the form: $(u,v) \to (u,uv)$. After substituting $z = u$, $w = vu$ we get:

$$u^{\alpha+\beta}(av^{\beta} + \sum_{i+j>d} u^{i+j-(\alpha+\beta)}v^{j}) = u^{\alpha+\beta}\widetilde{G}(u,v) = 0,$$

from which it is clear that the point $(0,0)$ remains singular for the function $\widetilde{G}(u,v)$. This is due to the fact that the tangent cone at the point $(0,0)$ for the curve $G(z,w)$ has multiple components (a component $z = 0$ of multiplicity α and a component $w = 0$ of multiplicity β). According to the construction of the resolution, it is necessary to continue blow up the singular point, i.e. at least more than one step is required. At the same time, the use of Theorem 1 immediately allows us to obtain a convergent solution.

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Еще одно доказательство теоремы Пьюизо об алгебраической функции

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Аннотация. В работе дано более простое доказательство теоремы Пьюизо об алгебраической функции для многочленов специального вида.

Ключевые слова: диаграмма Ньютона, ряд Пьюизо, особая точка, полином Вейерштрасса.