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Global Solvability of a Kernel Determination Problem in 2D Heat Equation with Memory

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Abstract. The inverse problem of determining two dimensional kernel in the integro-differential heat equation is considered in this paper. The kernel depends on the time variable t and space variable x. Assuming that kernel function is given, the direct initial-boundary value problem with Neumann conditions on the boundary of a rectangular domain is studied for this equation. Using the Green's function, the direct problem is reduced to integral equation of the Volterra-type of the second kind. Then, using the method of successive approximation, the existence of a unique solution of this equation is proved. The direct problem solution on the plane $y = 0$ is used as an overdetermination condition for inverse problem. This problem is replaced by an equivalent auxiliary problem which is more suitable for further study. Then the last problem is reduced to the system of integral equations of the second order with respect to unknown functions. Applying the fixed point theorem to this system in the class of continuous in time functions with values in the Hölder spaces with exponential weight norms, the main result of the paper is proved. It consists of the global existence and uniqueness theorem for inverse problem solution.

Keywords: integro-differential heat equation, inverse problem, Banach theorem, existence, uniqueness.

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1. Introduction and preliminaries

The integro-differential equations with an integral term of convolution type are used in the mathematical modeling of biological phenomena and engineering sciences when it is necessary to take into consideration the history of the processes. In these integro-differential equations the convolution kernel accounts for memory influences. The key point when considering memory effects is that the kernel cannot be considered a known function because there is no way to measure it directly. Kernel can be reconstructed by additional measurements of physical field taken on a suitable subset of the body. Thus, an inverse problem has to be solved. The constitutive

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relations for a linear non-homogeneous heat propagation and diffusion processes in medium with memory contain a time- and space-dependent kernel functions in the integral of time convolution type [1, 2]. The memory effect phenomenon is governed by hyperbolic and parabolic integro–differential equations with time dependent memory kernel when the medium is homogeneous and time-space dependent memory kernel when the medium is heterogeneous. The kernel determination problems in one-dimensional heat conduction equations are widely encountered where memory kernel depends only on time variable. For example, in [3–13] (see also references therein) these problems were studied on the basis of the fixed point argument, and the local/global in time existence and uniqueness of inverse problems were derived. The numerical solutions for this problems were considered and efficient computational algorithms were constructed [14–17].

In this paper, the inverse problems of determining kernels of an integral convolution-type term in the integro-differential heat equation are studied with the use of the solution of the initial-boundary value problem in a rectangular domain given on the boundary $y = 0$. Unlike existing works, here the unknown kernel depends on both time and spatial coordinates. Consider the problem of determining functions $u(x, y, t)$ and $k(x, t)$ from the following equations:

$$u_t - \Delta u = \int_0^t k(x, t-\tau) u(x, y, \tau) d\tau + f(x, y, t), \quad (x, y, t) \in D_T, \tag{1}$$

$$u \mid_{t=0} = \varphi(x, y), \ (x, y) \in \overline{D}, \tag{2}$$

$$u_x \mid_{x=0} = u_x \mid_{x=1} = 0, \ u_y \mid_{y=0} = u_y \mid_{y=1} = 0, \ (x, y) \in \partial D \times [0, T], \tag{3}$$

$$u \mid_{y=0} = h(x, t), \ (x, t) \in [0, 1] \times [0, T], \tag{4}$$

where $\Delta = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}$ is the Laplace operator, $D_T = D \times (0, T]$, $D = \{(x, y) : x \in (0, 1), \ y \in (0, 1)\}$, $T > 0$ is an arbitrary fixed number. In the theory of inverse problems for differential equations, initial-boundary value problem (1)–(3) of determining function $u(x, y, t)$ with Neumann boundary conditions is called the *direct problem.* Function $u(x, y, t) \in C^{2,1}_{x,t}(D_T) \cap C^{1,0}_{x,t}\left(\overline{D}_T\right)$ is regular solution of the direct problem if it satisfies equalities (1)–(3).

Regular solution of (1)–(4) presupposes the fulfilment of the following conditions

$$\varphi_x(0, y) = \varphi_x(1, y) = 0, \quad \varphi_y(x, 0) = \varphi_y(x, 1) = 0, \quad \varphi(x, 0) = h(x, 0).$$

Let us introduce the class $H^l(D)$ of Hölder continuous functions on D with $l \in (0, 1)$. The space $H^{m+l}(D)$ (m is a nonnegative integer) and norms $|\cdot|^l$, $|\cdot|^{m+l}$ are defined in [18, pp. 16–27].The class of j times continuous differentiable with respect to t variable on the segment $[0, T]$ with values in $H^l(D)$ functions is denoted by $C^j\left(H^l(D), \ [0, T]\right)$. For a fixed t, the norm of function $g(x, y, t)$ in the $H^l(D)$ is denoted by $|g|^l(t)$. The norm of function $g(x, y, t)$ in $C^j\left(H^l(D), \ [0, T]\right)$ is defined by the equality

$$\|g\|^l := \sum_{i=0}^{j} \max_{t \in [0, T]} \left| \frac{\partial^i g}{\partial t^i} \right|^l (t).$$

2. Study of direct problem

The solution of problem (1)–(3) is equivalent to the following Volterra type integral equation

$$u(x,y,t)=\int_0^1\int_0^1 G(x,y,\xi,\eta,t)\varphi(\xi,\eta)d\xi d\eta+\int_0^t\int_0^1\int_0^1 G(x,y,\xi,\eta,t-\tau)f(\xi,\eta,\tau)d\xi d\eta d\tau+$$
$$+\int_0^t\int_0^1\int_0^1 G(x,y,\xi,\eta,t-\tau)\int_0^\tau k(\xi,\tau-\alpha)u(\xi,\eta,\alpha)d\alpha d\xi d\eta d\tau, \tag{5}$$

where $G(x,y,\xi,\eta,t)$ is the Green function and it is defined as

$$G(x,y,\xi,\eta,t)=1+4\sum_{m,n=1}^{\infty}e^{-\lambda_{mn}t}\cos\pi nx\cos\pi my\cos\pi n\eta\cos\pi m\xi,\ \lambda_{mn}=\pi\sqrt{m^2+n^2}.$$

Lemma 2.1. *Suppose that* $\varphi(x,y)\in H^l(D)$, $f(x,y,t)\in C\Big(H^l(D),[0,T]\Big)$ *and* $k(x,t)\in C\Big(H^l([0,1]),[0,T]\Big)$. *Then there is a unique solution of integral equation (5) such that* $u(x,y,t)\in C^1\Big(H^{l+2}(D),[0,T]\Big)$.

Proof. To prove this Lemma, the method of successive approximations is used. At the first step, the following sequences of functions is constructed

$$\begin{aligned} u_0(x,y,t)&=\int_0^1\int_0^1 G(x,y,\xi,\eta,t)\varphi(\xi,\eta)d\xi d\eta+\int_0^t\int_0^1\int_0^1 G(x,y,\xi,\eta,t-\tau)f(\xi,\eta,\tau)d\xi d\eta d\tau,\\ u_i(x,y,t)&=\int_0^t\int_0^1\int_0^1 G(x,y,\xi,\eta,t-\tau)\int_0^\tau k(\xi,\tau-\alpha)u_{i-1}(\xi,\eta,\alpha)d\alpha d\xi d\eta d\tau,\ i=1,2,\dots .\end{aligned} \tag{6}$$

For brevity, introduce the following notations

$$\varphi_{00}:=|\varphi|^l,\ \ f_0:=\|f\|^l,\ \ k_0:=\|k\|^l.$$

Let us estimate modules of functions $u_i(x,y,t)$. Using the Green's function property $\int\limits_0^1\int\limits_0^1 G(x,y,\xi,\eta,t)d\xi d\eta=1$,one can obtain from (6) for $(x,y,t)\in\overline{D}_T$ that

$$\Big|u_0(x,y,t)\Big|^l\leqslant\Big|\int_0^1\int_0^1 G(x,y,\xi,\eta,t)\varphi(\xi,\eta)d\xi d\eta\Big|^l+$$
$$+\Big|\int_0^t\int_0^1\int_0^1 G(x,y,\xi,\eta,t-\tau)f(\xi,\eta,\tau)d\xi d\eta d\tau\Big|^l\leqslant\varphi_{00}+f_0t,$$

$$\Big|u_i(x,y,t)\Big|^l\leqslant\Big|\int_0^t\int_0^1\int_0^1 G(x,y,\xi,\eta,t-\tau)\int_0^\tau k(\xi,\tau-\alpha)u_{i-1}(\xi,\eta,\alpha)d\alpha d\xi d\eta d\tau\Big|^l\leqslant$$
$$\leqslant k_0^i\Big(\varphi_{00}\frac{t^{2i}}{2i!}+f_0\frac{t^{2i+1}}{(2i+1)!}\Big),\ i=1,2,\dots .$$

Let us define functional series $\sum\limits_{i=0}^{\infty}u_i(x,y,t)$. Using values obtained above, this series can be estimated as follows

$$\sum_{i=0}^{\infty}\Big|u_i(x,y,t)\Big|\leqslant\sum_{i=0}^{\infty}k_0^i\Big(\varphi_{00}\frac{T^{2i}}{2i!}+f_0\frac{T^{2i+1}}{(2i+1)!}\Big),\quad (x,y,t)\in\overline{D}_T.$$

Since the last number series converges, series $\sum\limits_{i=0}^{\infty} u_i(x,y,t)$ converges uniformly and absolutely. Obviously, under conditions of the Lemma the inclusion $u_0(x,y,t) \in C^{2,1}_{x,t}(D_T)$ is satisfied. Consequently, all $u_i(x,y,t)$ have this property, i.e., $u_i(x,y,t) \in C^{2,1}_{x,t}(D_T)$, $i = 1,2,\ldots$. Then, according to the general theory of linear integral equations of Volterra type, $\sum\limits_{i=0}^{\infty} u_i(x,y,t)$ is a regular solution of direct problem (1)–(3).

Let us show that equation (5) has a unique solution. For this, let us assume the opposite, that is, integral equation (5) has two different solutions $u^1(x,y,t)$ and $u^2(x,y,t)$ with the same data:

$$u^i(x,y,t) = \int_0^1\int_0^1 G(x,y,\xi,\eta,t)\varphi(\xi,\eta)d\xi d\eta + \int_0^t\int_0^1\int_0^1 G(x,y,\xi,\eta,t-\tau)f(\xi,\eta,\tau)d\xi d\eta d\tau+$$
$$+\int_0^t\int_0^1\int_0^1 G(x,y,\xi,\eta,t-\tau)\int_0^\tau k(\xi,\tau-\alpha)u^i(\xi,\eta,\alpha)d\alpha d\xi d\eta d\tau,\ i=1,2.$$

The difference of these functions is defined by $Z(x,y,t) = u^1(x,y,t) - u^2(x,y,t)$:

$$Z(x,y,t) = \int_0^t\int_0^1\int_0^1 G(x,y,\xi,\eta,t-\tau)\int_0^\tau k(\xi,\tau-\alpha)Z(\xi,\eta,\alpha)d\alpha d\xi d\eta d\tau. \tag{7}$$

Let us denote the modular supremum of function $Z(x,y,t)$ on $(x,y)\in D$ for each $t\in[0,T]$ as

$$\widetilde{Z}(t) = \sup_{(x,y)\in D}\Big|Z(x,y,t)\Big|,\quad t\in[0,T].$$

It follows from integral equation (7) that

$$\widetilde{Z}(t) \leqslant k_0 T\int_0^t \widetilde{Z}(\tau)d\tau.$$

According to the Gronuolla–Bellman inequality, the last integral inequality has only $\widetilde{Z}(t)\equiv 0$ solution. It means that $Z(x,y,t) = 0$ or $u^1(x,y,t) = u^2(x,y,t)$ in domain D_T. The lemma is proved. □

3. Auxiliary problem

Suppose that functions in problem (1)–(4) are sufficiently smooth. The degree of smoothness for each function will be determined later.

The following assertion is true.

Lemma 3.1. *Problem (1)–(4) is equivalent to the following auxiliary problem for functions* $\omega(x,y,t)$, $k(x,t)$:

$$\omega_t - \Delta\omega = k(x,t)\varphi_{yy}(x,y) + f_{tyy}(x,y,t) + \int_0^t k(x,t-\tau)\omega(x,y,\tau)d\tau,\quad (x,y,t)\in D_T, \tag{8}$$

$$\omega\,|_{t=0} = \Delta\varphi_{yy}(x,y) + f_{yy}(x,y,0),\ (x,y)\in D, \tag{9}$$

$$\omega_x\,|_{x=0} = \omega_x\,|_{x=1} = 0,\ \omega_y\,|_{y=0} = \omega_y\,|_{y=1} = 0,\ \partial D\times[0,T], \tag{10}$$

$$\begin{aligned}\omega\,|_{y=0} = h_{tt}(x,t) - h_{xxt}(x,t) - k(x,t)\varphi(x,0) - f_t(x,0,t) -\\ -\int_0^t k(x,t-\tau)h_t(x,0,\tau)d\tau,\ (x,t)\in[0,1]\times[0,T],\end{aligned} \tag{11}$$

where $\omega(x,y,t) := u_{tyy}(x,y,t)$.

Proof. Upon differentiating equations (1)–(4) with respect to t and setting $\vartheta(x,y,t) := u_t(x,y,t)$, one can obtain the following equivalent problem for functions ϑ, k

$$\vartheta_t - \Delta\vartheta = k(x,t)\varphi(x,y) + f_t(x,y,t) + \int_0^t k(x,t-\tau)\vartheta(x,y,\tau)d\tau, \quad (x,y,t) \in D_T, \tag{12}$$

$$\vartheta\mid_{t=0} = \Delta\varphi(x,y) + f(x,y,0), \ \ (x,y) \in D, \tag{13}$$

$$\vartheta_x\mid_{x=0} = \vartheta_x\mid_{x=1} = 0, \ \ \vartheta_y\mid_{y=0} = \vartheta_y\mid_{y=1} = 0, \ \ (x,y) \in \partial D \times [0,T], \tag{14}$$

$$\vartheta\mid_{y=0} = h_t(x,t), \ \ (x,t) \in [0,1] \times [0,T]. \tag{15}$$

Here, it is assumed that

$$\Delta\varphi_x(0,y) + f_x(0,y,0) = \Delta\varphi_x(1,y) + f_x(1,y,0),$$

$$\Delta\varphi_y(x,0) + f_y(x,0,0) = \Delta\varphi_y(x,1,0) + f_y(x,1,0), \ \Delta\varphi(x,0) = h_t(x,0).$$

Hence it follows that if (u,k) is a solution of problem (1)–(4) then (12)–(15) has a solution (ϑ,k) with the same k. Let us prove the converse. Let (ϑ,k) satisfy relations (12)–(15) then

$$u(x,y,t) = \int_0^t \vartheta(x,y,\tau)d\tau + \varphi(x,y).$$

Let us show that relation (1) holds. It follows from (12)—(15) that

$$u_t - \Delta u - \int_0^t k(x,\tau)u(x,y,t-\tau)d\tau - f(x,y,t) =$$

$$= \vartheta(x,y,t) - \int_0^t \Delta\vartheta(x,y,\tau)d\tau - \Delta\varphi(x,y) - \int_0^t k(x,\tau)\int_0^{t-\tau}\vartheta(x,y,\alpha)d\alpha d\tau - \int_0^t k(x,\tau)\varphi(x,y)d\tau -$$

$$-f(x,y,t) = \int_0^t \vartheta_\tau(x,y,\tau)d\tau + \Delta\varphi(x,y) + f(x,y,0) - \int_0^t \Delta\vartheta(x,y,\tau)d\tau - \Delta\varphi(x,y) -$$

$$-\int_0^t k(x,\tau)\int_0^{\tau}\vartheta(x,y,\tau-\alpha)d\alpha d\tau - \int_0^t k(x,\tau)\varphi(x,y)d\tau - \int_0^t f_\tau(x,y,\tau)d\tau - f(x,y,0) =$$

$$= \int_0^t \Big[\vartheta_\tau - \Delta\vartheta - \int_0^\tau k(x,\alpha)\vartheta(x,y,\tau-\alpha)d\alpha - k(x,\tau)\varphi(x,y) - f_\tau(x,y,\tau)\Big]d\tau = 0.$$

This completes the proof of equivalence of problems (1)–(4) and (12)–(15).

Now consider the second auxiliary problem. It can be obtained from problem (12)–(15) for function $p(x,y,t) := \vartheta_y(x,y,t)$:

$$p_t - \Delta p = k(x,t)\varphi_y(x,y) + f_{ty}(x,y,t) + \int_0^t k(x,t-\tau)p(x,y,\tau)d\tau, \quad (x,y,t) \in D_T, \tag{16}$$

$$p\mid_{t=0} = \Delta\varphi_y(x,y) + f_y(x,y,0), \quad (x,y) \in D, \tag{17}$$

$$p_x\mid_{x=0} = p_x\mid_{x=1} = 0, \quad p_y\mid_{y=0} = p_y\mid_{y=1} = 0, \quad \partial D \times [0,T], \tag{18}$$

$$\begin{aligned} p_y\mid_{y=0} = {} & h_{tt}(x,t) - h_{xxt}(x,t) - k(x,t)\varphi(x,0) - f_t(x,0,t) - \\ & - \int_0^t k(x,t-\tau)h_t(x,0,\tau)d\tau, \quad (x,t) \in [0,1] \times [0,T]. \end{aligned} \tag{19}$$

It is assumed that

$$\Delta\varphi_{xy}(0,y)+f_{xy}(0,y,0)=\Delta\varphi_{xy}(1,y)+f_{xy}(1,y,0),$$

$$\Delta\varphi_{yy}(x,0)+f_{yy}(x,0,0)=\Delta\varphi_{yy}(x,1)+f_{yy}(x,1,0),$$

$$\Delta\varphi_{yy}(x,0)+f_{yy}(x,0,0)=h_{tt}(x,0)-h_{xxt}(x,0)-k(x,0)\varphi(x,0)-f_t(x,0,0).$$

This follows from (12—(15), and it can be proved by complete analogy with the previous case.

Therefore, if problem (12)–(15) has solution (ϑ,k), then problem (16)–(19) has solution (p,k) with the same k. Moreover, $p(x,y,t)=\vartheta_y(x,y,t)$. Conversely, let (p,k) satisfy relations (16)–(19).

Hence it follows that

$$\vartheta(x,y,t)=\int_0^y p(x,z,t)dz+h_t(x,t),$$

$$\begin{aligned}
&\vartheta_t-\Delta\vartheta-k(x,t)\varphi(x,y)-f_t(x,y,t)-\int_0^t k(x,t-\tau)\vartheta(x,y,\tau)d\tau=\\
&=\int_0^y p_t(x,z,t)dz+h_{tt}(x,t)-\int_0^y \Delta p(x,z,t)dz-p_y(x,0,t)-h_{xxt}(x,t)-\\
&\quad-\int_0^y k(x,t)\varphi_z(x,z)dz-k(x,t)\varphi(x,0)-\int_0^y f_{tz}(x,z,t)dz-f_t(x,0,t)-\\
&\quad-\int_0^y\int_0^t k(x,t-\tau)\vartheta_z(x,z,\tau)d\tau dz-\int_0^t k(x,t-\tau)h_t(x,0,\tau)d\tau=\\
&=\int_0^y\Big[p_t-\Delta p-k(x,t)\varphi_z(x,z)-f_{tz}(x,z,t)-\int_0^t k(x,t-\tau)\vartheta_z(x,z,\tau)d\tau\Big]dz-\\
&-p_y(x,0,t)+h_{tt}(x,t)-h_{xxt}(x,t)-f_t(x,0,t)-k(x,t)\varphi(x,0)-\int_0^t k(x,t-\tau)h_t(x,0,\tau)d\tau=0
\end{aligned}$$

Then the equivalence of problems (12)–(15) and (16)—(19) is proved. In similar way, one can show that problem (16)–(19) is equivalent to problem (10)–(13) for function $\omega:=p_y(x,y,t)$. This implies the equivalence of problems (1)–(4) and (8)–(11). The lemma is proved. □

4. Study of inverse problem (8)–(11)

The solution of problem (8)–(10) is equivalent to the following Volterra type integral equation

$$\begin{aligned}
\omega(x,y,t)=&\int_0^1\int_0^1 G(x,y,\xi,\eta,t)\Big(\Delta\varphi_{yy}(\xi,\eta)+f_{yy}(\xi,\eta)\Big)d\xi d\eta+\\
&+\int_0^t\int_0^1\int_0^1 G(x,y,\xi,\eta,t-\tau)f_{tyy}(\xi,\eta,\tau)d\xi d\eta d\tau+\\
&+\int_0^t\int_0^1\int_0^1 G(x,y,\xi,\eta,t-\tau)k(\xi,t-\tau)\varphi_{yy}(\xi,\eta)d\xi d\eta d\tau+\\
&+\int_0^t\int_0^1\int_0^1 G(x,y,\xi,\eta,t-\tau)\int_0^\tau k(\xi,\tau-\alpha)\omega(\xi,\eta,\alpha)d\alpha d\xi d\eta d\tau.
\end{aligned} \tag{20}$$

Let $\varphi(x,0)\neq 0$, for all $x\in[0,1]$. Using equation (20) and additional conditions (11), one can obtain the following integral equation with respect to function $k(x,t)$

$$\begin{aligned}
k(x,t)=&\frac{1}{\varphi(x,0)}\Big[h_{tt}(x,t)-h_{xxt}(x,t)-f_t(x,0,t)\Big]-\frac{1}{\varphi(x,0)}\int_0^t k(x,t-\tau)h_t(x,0,\tau)d\tau-\\
&-\frac{1}{\varphi(x,0)}\int_0^1\int_0^1 G(x,0,\xi,\eta,t)\Big(\Delta\varphi_{yy}(\xi,\eta)-f_{yy}(\xi,\eta)\Big)d\xi d\eta-\\
&-\frac{1}{\varphi(x,0)}\int_0^t\int_0^1\int_0^1 G(x,0,\xi,\eta,t-\tau)f_{tyy}(\xi,\eta,\tau)d\xi d\eta d\tau-\\
&-\frac{1}{\varphi(x,0)}\int_0^t\int_0^1\int_0^1 G(x,0,\xi,\eta,t-\tau)k(\xi,t-\tau)\varphi_{yy}(\xi,\eta)d\xi d\eta d\tau-\\
&-\frac{1}{\varphi(x,0)}\int_0^t\int_0^1\int_0^1 G(x,0,\xi,\eta,t-\tau)\int_0^\tau k(\xi,\tau-\alpha)\omega(\xi,\eta,\alpha)d\alpha d\xi d\eta d\tau.
\end{aligned}\tag{21}$$

The main result of this work is the following assertion. **

Theorem 4.1. *Assume that* $\varphi(x,y)\in H^{l+4}(D)$, $|\varphi(x,0)|\geqslant\varphi_0=const>0$, $f(x,y,t)\in C^1\Big(H^{l+2}(D);[0,T]\Big)$, $h(x,t)\in C^2\Big(H^{l+2}([0,1]);[0,T]\Big)$. *In addition, all the above matching conditions with respect to the specified functions are fulfilled. Then for any fixed* $T>0$, *there exists a unique solution of integral equations (20), (21) and* $\omega(x,y,t)\in C\Big(H^{l+2}(D);[0,T]\Big)$, $k(x,t)\in C\Big(H^{l}([0,1]);[0,T]\Big)$.

Proof. The system of equation (20), (21) is closed system of integral equations with respect to functions $\omega(x,y,t)$ and $k(x,t)$. Let us write this system in the form of a non-linear operator equation

$$\psi=A\psi,\tag{22}$$

where $\psi=(\psi_1,\psi_2,)=(\omega(x,y,t),k(x,t))^*$, $*$ is the symbol of transposition. The operator in (22) has the form $A\psi=\Big[(A\psi)_1,(A\psi)_2\Big]$;

$$\begin{aligned}
(A\psi)_1=&\psi_{01}(x,y,t)+\int_0^t\int_0^1\int_0^1 G(x,y,\xi,\eta,t-\tau)\psi_2(\xi,t-\tau)\varphi_{yy}(\xi,\eta)d\xi d\eta d\tau+\\
&+\int_0^t\int_0^1\int_0^1 G(x,y,\xi,\eta,t-\tau)\int_0^\tau \psi_2(\xi,\tau-\alpha)\psi_1(\xi,\eta,\alpha)d\alpha d\xi d\eta d\tau.
\end{aligned}\tag{23}$$

$$\begin{aligned}
(A\psi)_2=&\psi_{02}(x,t)-\frac{1}{\varphi(x,0)}\int_0^t \psi_2(x,t-\tau)h_t(x,0,\tau)d\tau-\\
&-\frac{1}{\varphi(x,0)}\int_0^t\int_0^1\int_0^1 G(x,0,\xi,\eta,t-\tau)\psi_2(\xi,t-\tau)\varphi_{yy}(\xi,\eta)d\xi d\eta d\tau-\\
&-\frac{1}{\varphi(x,0)}\int_0^t\int_0^1\int_0^1 G(x,0,\xi,\eta,t-\tau)\int_0^\tau \psi_2(\xi,\tau-\alpha)\psi_1(\xi,\eta,\alpha)d\alpha d\xi d\eta d\tau.
\end{aligned}\tag{24}$$

The following designations are used in equations (23), (24)

$$\psi_{01}(x,y,t)=\int_0^1\int_0^1 G(x,y,\xi,\eta,t)\Big(\Delta\varphi_{yy}(\xi,\eta)+f_{yy}(\xi,\eta)\Big)d\xi d\eta+$$

$$+\int_0^t\int_0^1\int_0^1 G(x,y,\xi,\eta,t-\tau)f_{tyy}(\xi,\eta,\tau)d\xi d\eta d\tau,$$

$$\psi_{02}(x,t)=\frac{1}{\varphi(x,0)}\Big[h_{tt}(x,t)-h_{xxt}(x,t)-f_t(x,0,t)\Big]-$$

$$-\frac{1}{\varphi(x,0)}\int_0^1\int_0^1 G(x,0,\xi,\eta,t)\Big(\Delta\varphi_{yy}(\xi,\eta)-f_{yy}(\xi,\eta)\Big)d\xi d\eta-$$

$$-\frac{1}{\varphi(x,0)}\int_0^t\int_0^1\int_0^1 G(x,0,\xi,\eta,t-\tau)f_{tyy}(\xi,\eta,\tau)d\xi d\eta d\tau.$$

Let $C_\sigma\left(H^l\left(D\right),\ [0,\,T]\right)$ be the Banach space of continuous with respect to t variable on the segment $[0,\,T]$ with values in $H^l\left(D\right)$ functions with the family of weighted norms $\|\cdot\|_\sigma^l$, $\sigma\geqslant 0$

$$\|\psi\|_\sigma^l=\max_{t\in[0,T]}e^{-\sigma t}|\psi_i|^l,\ i=1,2. \tag{25}$$

Obviously, C_σ with $\sigma=0$ is the usual space of continuous in t on $[0,\,T]$ with values in $H^l\left(D\right)$ functions with the ordinary norm (see Introduction). In what follows it is denoted by $\|\cdot\|^l$. Because

$$e^{-\sigma t}\|\psi\|^l\leqslant\|\psi\|_\sigma^l\leqslant\|\psi\|^l \tag{26}$$

norms $\|\cdot\|_\sigma^l$ and $\|\cdot\|^l$ are equivalent for any $t\in[0,T]$. Parameter σ will be defined later.

Consider space C_σ with $\sigma\geqslant 0$. Let us introduce the ball $S_\sigma(\psi,\|\psi_0\|^l):=\{\psi:\|\psi-\psi_0\|_\sigma^l\leqslant\|\psi_0\|^l\}$ of radius $|\psi_0\|^l$ centred at the point ψ_0, where vector function ψ_0 has components ψ_{0i}, $i=1,2$ and $\|\psi_0\|^l=\max\limits_{i=1,2}|\psi_{0i}|^l$. Obviously, the estimate $\|\psi\|_\sigma^l\leqslant\|\psi\|_\sigma^l+\|\psi_0\|^l\leqslant 2\|\psi_0\|^l$ holds for a function $\psi\in S_\sigma(\psi_0,\|\psi_0\|^l)$. Let $\psi\in S_\sigma(\psi_0,\|\psi_0\|^l)$. Let us prove that operator A is contracting operator on set $\psi\in S_\sigma(\psi_0,\|\psi_0\|^l)$ for an appropriately chosen $\sigma>0$. First let us show that if $\sigma>0$ is chosen appropriately then operator A maps ball $S_\sigma(\psi_0,\|\psi_0\|^l)$ into the same ball, i.e., $A\psi\in S_\sigma(\psi_0,\|\psi_0\|^l)$.

Indeed, using relations (20), (21) for the norm of differences and denoting $\varphi_1:=|\varphi|^{l+4}$, $h_0:=|h|^{l+2}$ for $(x,t)\in[0,1]\times[0,T]$, one can obtain

$$\|(A\psi)_1-\psi_{01}\|_\sigma^l=\max_{t\in[0,t]}\Big|((A\psi)_1-\psi_{01})\Big|^l e^{-\sigma t}\leqslant\max_{t\in[0,t]}\Big|\int_0^t\int_0^1\int_0^1 G(x,y,\xi,\eta,t-\tau)\times$$

$$\times e^{-\sigma t}\psi_2(\xi,t-\tau)e^{-\sigma(t-\tau)}\varphi_{yy}(\xi,\eta)d\xi d\eta d\tau+$$

$$+\int_0^t\int_0^1\int_0^1 G(x,y,\xi,\eta,t-\tau)\int_0^\tau e^{-\sigma(\tau-\alpha)}\psi_2(\xi,\tau-\alpha)e^{-\sigma(\alpha)}\psi_1(\xi,\eta,\alpha)e^{-\sigma(t-\tau)}d\alpha d\xi d\eta d\tau\Big|^l\leqslant$$

$$\leqslant\|\psi_2\|_\sigma^l\varphi_1\frac{1}{\sigma}+2\|\psi_2\|_\sigma^l\|\psi_1\|_\sigma^l\frac{T}{\sigma}\leqslant 2\|\psi_0\|^l(\varphi_1+4\|\psi_0\|^lT)\frac{1}{\sigma},$$

$$\|(A\psi)_2-\psi_{02}\|_\sigma^l=\max_{t\in[0,T]}\Big|((A\psi)_2-\psi_{02})\Big|^l e^{-\sigma t}\leqslant\varphi_0^{-1}\Big[\|\psi_2\|_\sigma^l h_0\frac{1}{\sigma}+\|\psi_2\|_\sigma^l\varphi_1\frac{1}{\sigma}+$$

$$+2\|\psi_2\|_\sigma^l\|\psi_1\|_\sigma^l\frac{T}{\sigma}\Big]\leqslant 2\|\psi_0\|^l\varphi_0^{-1}\Big[h_0+\varphi_1+4\|\psi_0\|^lT\Big]\frac{1}{\sigma}.$$

Let $\sigma\geqslant\sigma_0$, where

$$\sigma_0=2\max\{\varphi_1+4\|\psi_0\|^lT,h_0+\varphi_1+4\|\psi_0\|^lT\}.$$

Then operator A maps $S_\sigma(\psi_0,\|\psi_0\|^l)$ into itself, i.e., $A\psi\in S_\sigma(\psi_0,\|\psi_0\|^l)$.

Let us show the fulfilment of the second property of construction map for operator A. First, one should note that inequalities for $\psi^{(1)} = \left(\psi_1^{(1)},\ \psi_2^{(1)}\right) \in S_\sigma\left(\psi_0, \|^l\psi_0\|^l\right)$, $\psi^{(2)} = \left(\psi_1^{(2)}, \psi_2^{(2)},\right) \in S_\sigma\left(\psi_0, \|\psi_0\|^l\right)$.

$$\left|\psi_2^{(1)}\psi_1^{(1)} - \psi_2^{(2)}\psi_1^{(2)}\right|^l = \left|\left(\psi_2^{(1)} - \psi_2^{(2)}\right)\psi_1^{(1)} + \psi_2^{(2)}\left(\psi_1^{(1)} - \psi_1^{(2)}\right)\right|^l \leqslant$$

$$\leqslant 2\left|\psi^{(1)} - \psi^{(2)}\right|^l \max\left(\left|\psi_1^{(1)}\right|^l,\ \left|\psi_2^{(2)}\right|^l\right) \leqslant 4\|\psi_0\|^l\left|\psi^{(1)} - \psi^{(2)}\right|^l.$$

holds. Then one can obtain

$$\left\|((A\psi)^{(1)} - (A\psi)^{(2)})_1\right\|_\sigma^l = \max_{t\in[0,T]}\left|((A\psi)^{(1)} - (A\psi)^{(2)})_1\right|^l e^{-\sigma t} \leqslant$$

$$\leqslant \max_{t\in[0,T]}\Big|\int_0^t\int_0^1\int_0^1 G(x,y,\xi,\eta,t-\tau)e^{-\sigma t}\Big(\psi_2^{(1)}(\xi,t-\tau) - \psi_2^{(2)}(\xi,t-\tau)\Big)\times$$

$$\times e^{-\sigma(t-\tau)}\varphi_{yy}(\xi,\eta)d\xi d\eta d\tau + \int_0^t\int_0^1\int_0^1 G(x,y,\xi,\eta,t-\tau)\int_0^\tau\Big(e^{-\sigma(\tau-\alpha)}\psi_2^{(1)}(\xi,\tau-\alpha)\times$$

$$\times e^{-\sigma(\alpha)}\psi_1^{(1)}(\xi,\eta,\alpha) - \psi_2^{(2)}(\xi,\tau-\alpha)\psi_1^{(2)}(\xi,\eta,\alpha)e^{-\sigma(t-\tau)}\Big)d\alpha d\xi d\eta d\tau\Big|^l \leqslant$$

$$\leqslant |\psi^{(1)} - \psi^{(2)}|^l\varphi_1\frac{1}{\sigma} + 8\|\psi_0\|^l|\psi^{(1)} - \psi^{(2)}|^l\frac{T}{\sigma} \leqslant |\psi^{(1)} - \psi^{(2)}|^l\Big(\varphi_1 + 8\|\psi_0\|^l T\Big)\frac{1}{\sigma}.$$

$$\left\|((A\psi)^{(1)} - (A\psi)^{(2)})_2\right\|_\sigma^l = \max_{t\in[0,T]}\left|((A\psi)^{(1)} - (A\psi)^{(2)})_2\right|^l e^{-\sigma t} \leqslant$$

$$\leqslant \varphi_0^{-1}\Big[h_0|\psi^{(1)} - \psi^{(2)}|^l\frac{1}{\sigma} + |\psi^{(1)} - \psi^{(2)}|^l\varphi_1\frac{1}{\sigma} + 8\|\psi_0\|^l|\psi^{(1)} - \psi^{(2)}|^l\frac{T}{\sigma}\Big] \leqslant$$

$$\leqslant |\psi^{(1)} - \psi^{(2)}|^l\varphi_0^{-1}\Big[h_0 + \varphi_1 + 8\|\psi_0\|^l T\Big]\frac{1}{\sigma}.$$

Let $\sigma \geqslant \sigma^*$, where

$$\sigma^* = \max\Big\{\varphi_1 + 8\|\psi_0\|^l T, \varphi_0^{-1}\Big[h_0 + \varphi_1 + 8\|\psi_0\|^l T\Big]\Big\}.$$

Then operator A is contracting operator on $S_\sigma(\psi_0, \|\psi_0\|^l)$. It follows from the Banach fixed-point theorem that (22) is solvable and has a unique solution in $S_\sigma(\psi_0, \|\psi_0\|^l)$ for any fixed $T > 0$.

Since $\omega =: \psi_1$ then

$$u_{yyt}(x,y,t) = \psi_1(x,y,t). \tag{27}$$

Function $u(x,y,t)$ is determined from equation (27) as follows

$$u(x,y,t) = h(x,t) + \varphi(x,y) - \varphi(x,0) + \int_0^y\int_0^t(y-\eta)\omega(x,\eta,\tau)d\tau d\eta.$$

Thus, the solution of inverse problem (1)–(4) is found. □

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Глобальная разрешимость задачи определения ядра в двумерном уравнении теплопроводности с памятью

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Аннотация. В статье исследуется обратная задача определения двумерного ядра интегрального члена, зависящего от временной переменной t и первой компоненты пространственной переменной (x, y) в интегро-дифференциальном уравнении теплопроводности. Для этого уравнения при заданном ядра изучается прямая начально-краевая задача с условиями Неймана на границе прямоугольной области. С помощью функции Грина эта задача сводится к интегральному уравнению вольтерровского типа второго рода, а затем методом последовательных приближений доказывается существование единственного решения. В обратной задаче в качестве условия переопределения используется решение прямой задачи на плоскости $y = 0$. Обратная задача заменяется эквивалентной вспомогательной задачей, более удобной для дальнейшего исследования. Далее эта задача сводится к системе интегральных уравнений второго рода относительно неизвестных функций. Применяя к этой системе теорему о неподвижной точке в классе непрерывных по времени со значениями в пространствах Гёльдера функций с экспоненциальной весовой нормой, доказывается основной результат статьи, состоящий в глобальной теореме существования и единственности решения обратной задачи.

Ключевые слова: интегро-дифференциальное уравнение, обратная задача, Теорема Банаха, существование, единственность.