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Univalence of Some Integral Operators Involving Rabotnov Fractional Exponential Function

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Abstract. In this paper, we introduce three new integral operators involving normalized Rabotnov fractional exponential functions $\mathbb{R}_{\alpha,\beta}(z)$. Furthermore, we shall find sufficient conditions for these integral operators. Finally, some special cases are deduced for different values of α and β.

Keywords: analytic functions, integral operators, Rabotnov function.

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1. Introduction and preliminaries

Special functions are used in many applications of physics, engineering, and applied mathematics and statistics, also they play important roles in geometric function theory. The wide use of these functions has attracted many researchers to obtain geometric properties of special functions such as hypergeometric functions, Bessel functions, Struve functions, Mittag–Leffler functions, Wright functions, and some other related functions. We refer to some geometric properties of these functions [2, 7, 34, 35, 42] and references therein.

In 1948, Rabotnov [33] introduced a special function applied in viscoelasticity. This function, known today as the Rabotnov fractional exponential function or briefly Rabotnov function, is defined as follows

$$R_{\alpha,\beta}(z) = z^{\alpha} \sum_{n=0}^{\infty} \frac{(\beta)^n z^{n(1+\alpha)}}{\Gamma((n+1)(1+\alpha))}, \quad (\alpha, \beta, z \in \mathbb{C}\ ). \tag{1.1}$$

Rabotnov function is the particular case of the familiar Mittag–Leffler function [28] widely used in the solution of fractional order integral equations or fractional order differential equations. The relation between the Rabotnov function and Mittag–Leffler function can be written as follows

$$R_{\alpha,\beta}(z) = z^{\alpha} E_{1+\alpha,1+\alpha}(\beta z^{1+\alpha}),$$

where E is Mittag-Leffler function and $\alpha, \beta, z \in \mathbb{C}$. Several properties of Mittag–Leffler function and generalized Mittag–Leffler function can be found in [4, 5, 21, 22].

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Let $\mathcal{A}$ denote the class of the normalized functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1.2}$$

which are analytic in the open unit disk $\Delta = \{z \in \mathbb{C} : |z| < 1\}$. Further, with $\mathcal{S}$ we denote the class of all functions in $\mathcal{A}$ which are univalent (or schlicht) in Δ.

It is clear that the Rabotnov function $R_{\alpha,\beta}(z)$ does not belong to the family $\mathcal{A}$. Thus, it is natural to consider the following normalization of Rabotnov function

$$\begin{aligned} \mathbb{R}_{\alpha,\beta}(z) &= z^{\frac{1}{1+\alpha}} \Gamma(1+\alpha) R_{\alpha,\beta}(z^{\frac{1}{1+\alpha}}) \\ &= z + \sum_{n=1}^{\infty} \frac{\beta^n \Gamma(1+\alpha)}{\Gamma((1+\alpha)(n+1))} z^{n+1}, \ z \in \Delta. \end{aligned} \tag{1.3}$$

Geometric properties including starlikeness, convexity and close-to-convexity for the normalized Rabotnov function $\mathbb{R}_{\alpha,\beta,\gamma}(z)$ were recently investigated by Eker and Ece in [16]. Furthermore, in [1] Amourah et al. introduced a new class of normalized analytic functions and bi-univalent functions associated with the normalized Rabotnov function.

Whilst formula (1.3) holds for complex-valued α, β and $z \in \mathbb{C}$, however in this paper, we shall restrict our attention to the case of real-valued $\alpha > -1$, $\beta \in \mathbb{C}$ and $z \in \Delta$.

Observe that the function $\mathbb{R}_{\alpha,\beta}$ contains many well-known functions as its special case, for example,

$$\begin{cases} \mathbb{R}_{0,-\frac{1}{3}}(z) = ze^{-\frac{z}{3}}, \\ \mathbb{R}_{1,-\frac{1}{4}}(z) = 2\sqrt{z} \sin \frac{\sqrt{z}}{2}, \\ \mathbb{R}_{1,\frac{1}{2}}(z) = \sqrt{2z} \sinh \sqrt{\frac{z}{2}}, \\ \mathbb{R}_{1,1}(z) = \sqrt{z} \sinh \sqrt{z}, \\ \mathbb{R}_{1,2}(z) = \frac{1}{2} \sqrt{2z} \sinh \sqrt{2z}. \end{cases}$$

Recently, many mathematicians have set the univalence criteria of several integral operators which preserve the class $\mathcal{S}$ (see for example [8–11, 18, 19]). Baricz and Frasin [6] first used a special function (the Bessel function) to introduce a single family integral operator and studied its univalency conditions. This operator was further studied by Frasin [17], Ularu [39], Arif and Raza [3] and. Recently, some authors have studied the families of one parameter integral operator using certain special functions, such as Mittag–Leffler functions [38], Struve functions [14], Lommel functions [29], Dini functions [15] and generalized Bessel function [36] (see also, [12, 13]).

The main object of this article is to introduce and study the univalence criteria for integral operators that involve Rabotnov function $\mathbb{R}_{\alpha,\beta}$ and defined as follows:

$$\mathbb{F}_{\alpha_i,\beta,\lambda_i,\zeta}(z) = \left\{ \zeta \int_0^z t^{\zeta-1} \prod_{i=1}^{n} \left(\frac{\mathbb{R}_{\alpha_i,\beta}(t)}{t} \right)^{1/\lambda_i} dt \right\}^{1/\zeta}, \tag{1.4}$$

$$\mathbb{F}_{\alpha_i,\beta,\lambda,n}(z) = \left\{ (n\lambda+1) \int_0^z \prod_{i=1}^{n} \left(\mathbb{R}_{\alpha_i,\beta}(t) \right)^{\lambda} dt \right\}^{1/(n\lambda+1)}, \tag{1.5}$$

and

$$\mathbb{F}_{\alpha,\beta,\gamma}(z) = \left\{ \gamma \int_0^z t^{\gamma-1} \left(e^{\mathbb{R}_{\alpha,\beta}(t)} \right)^\gamma dt \right\}^{1/\gamma}, \tag{1.6}$$

where the parameters $\lambda_1, \lambda_1, \dots, \lambda_n, \lambda, \zeta$ and γ are complex numbers such that the integrals in (1.4)–(1.6) exist. Here and throughout in the sequel every many-valued function is taken with the principal branch.

In this paper, we are mainly interested in the above integral operators (1.4)–(1.6). More precisely, we would like to show that by using some inequalities for the normalized Rabotnov function the univalence of the integral operators given by (1.4)–(1.6) which involve Rabotnov functions .

In the proofs of our main results we need the following univalence criteria.

Lemma 1.1 (Pescar [31]). *Let ζ and s be complex numbers such that* $\operatorname{Re}\zeta > 0$ *and* $|s| \leqslant 1$, $s \neq -1$. *If the function $h \in \mathcal{A}$ satisfies the inequality*

$$\left| s\,|z|^{2\zeta} + (1 - |z|^{2\zeta}) \frac{zh''(z)}{\zeta h'(z)} \right| \leqslant 1$$

for all $z \in \Delta$, then the function $F_\zeta \in \mathcal{A}$, defined by

$$F_\zeta(z) = \left\{ \zeta \int_0^z t^{\zeta-1} h'(t) dt \right\}^{1/\zeta}, \tag{1.7}$$

is in the class $\mathcal{S}$, i.e. is univalent in Δ.

Lemma 1.2 (Pascu [30]). *Let $\lambda \in \mathbb{C}$ such that* $\operatorname{Re}\lambda > 0$. *If $h \in \mathcal{A}$ satisfies*

$$\frac{1 - |z|^{2\operatorname{Re}\lambda}}{\operatorname{Re}\lambda} \left| \frac{zh''(z)}{h'(z)} \right| \leqslant 1$$

for all $z \in \Delta$, then for all $\zeta \in \mathbb{C}$ such that $\operatorname{Re}\zeta \geqslant \operatorname{Re}\lambda$, *the function, defined by* (1.7)*, is in the class $\mathcal{S}$.*

Lemma 1.3 (Pescar [32]). *Let $\gamma \in \mathbb{C}$ and $\lambda \in \mathbb{R}$ such that* $\operatorname{Re}\gamma \geqslant 1$, $\lambda > 1$ *and* $2\lambda|\gamma| \leqslant 3\sqrt{3}$. *If $h \in \mathcal{A}$ satisfies the inequality $|zh'(z)| \leqslant \lambda$ for all $z \in \Delta$, then the function $F_\gamma : \Delta \to \mathbb{C}$, defined by*

$$F_\gamma(z) = \left\{ \gamma \int_0^z t^{\gamma-1} (e^{h(t)})^\gamma dt \right\}^{1/\gamma},$$

is in the class $\mathcal{S}$.

Furthermore, we need the following result which is mainly based on [16] (see also, [23]).

Lemma 1.4. *Let $\alpha > -1$ and $\beta \in \mathbb{C}$ with $0 < |\beta| < (1+\alpha)\ln 2$. Then the function $\mathbb{R}_{\alpha,\beta} : \Delta \to \mathbb{C}$ defined by* (1.3) *satisfies the following inequalities:*

$$\left| \frac{z\mathbb{R}'_{\alpha,\beta}(z)}{\mathbb{R}_{\alpha,\beta}(z)} - 1 \right| \leqslant \frac{\frac{|\beta|}{1+\alpha} e^{\frac{|\beta|}{1+\alpha}}}{2 - e^{\frac{|\beta|}{1+\alpha}}}, \quad z \in \Delta, \tag{1.8}$$

and

$$\left| z\mathbb{R}'_{\alpha,\beta}(z) \right| \leqslant \frac{(1+\alpha+|\beta|)\, e^{\frac{|\beta|}{1+\alpha}}}{1+\alpha}, \quad z \in \Delta. \tag{1.9}$$

2. Univalence of some integral operators involving Rabotnov functions

Our first main result is an application of Lemma 1.1 and contains sufficient conditions for an integral operator of the type (1.4).

Theorem 2.1. *Let $\beta \in \mathbb{C}$ and $\alpha_1, \alpha_2, \ldots, \alpha_n > -1$ with $\alpha_i > \dfrac{|\beta|}{\ln 2} - 1$ for all $i \in \{1, 2, \ldots, n\}$ and consider the normalized Rabotnov functions $\mathbb{R}_{\alpha_i,\beta}$ defined by*

$$\mathbb{R}_{\alpha_i,\beta}(z) = z^{1/(1+\alpha_i)}\Gamma(1+\alpha_i)R_{\alpha_i,\beta}(z^{1/(1+\alpha_i)}). \tag{2.1}$$

Let $\alpha = \min\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$, $\zeta \in \mathbb{C}$ with $\operatorname{Re}\zeta > 0$, $s \in \mathbb{C}$ with $s \neq -1$ and let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be nonzero complex numbers. Moreover, suppose that these numbers satisfy the following inequality

$$|s| + \left(\frac{\frac{|\beta|}{1+\alpha}e^{\frac{|\beta|}{1+\alpha}}}{2 - e^{\frac{|\beta|}{1+\alpha}}}\right)\sum_{i=1}^{n}\frac{1}{|\zeta\lambda_i|} \leqslant 1.$$

Then the function $\mathbb{F}_{\alpha_i,\beta,\lambda_i,\zeta}$ defined by (1.4) is in $\mathcal{S}$, i.e. is univalent in Δ.

Proof. Define the function $\mathbb{F}_{\alpha_i,\beta,\lambda_i}$ by

$$\mathbb{F}_{\alpha_i,\beta,\lambda_i}(z) = \int_0^z \prod_{i=1}^{n}\left(\frac{\mathbb{R}_{\alpha_i,\beta}(t)}{t}\right)^{1/\lambda_i} dt. \tag{2.2}$$

First observe that, since for all $i \in \{1, 2, \ldots, n\}$ we have $\mathbb{R}_{\alpha_i,\beta} \in \mathcal{A}$, i.e. $\mathbb{R}_{\alpha_i,\beta}(0) = \mathbb{R}'_{\alpha_i,\beta}(0) - 1 = 0$, clearly $\mathbb{F}_{\alpha_i,\beta,\lambda_i} \in \mathcal{A}$, i.e. $\mathbb{F}_{\alpha_i,\beta,\lambda_i}(0) = \mathbb{F}'_{\alpha_i,\beta,\lambda_i}(0) - 1 = 0$. From (2.2), we see that

$$\mathbb{F}'_{\alpha_i,\beta,\lambda_i}(z) = \prod_{i=1}^{n}\left(\frac{\mathbb{R}_{\alpha_i,\beta}(t)}{t}\right)^{1/\lambda_i}$$

and

$$\frac{z\mathbb{F}''_{\alpha_i,\beta,\lambda_i}(z)}{\mathbb{F}'_{\alpha_i,\beta,\lambda_i}(z)} = \sum_{i=1}^{n}\frac{1}{\lambda_i}\left(\frac{z\mathbb{R}'_{\alpha_i,\beta}(z)}{\mathbb{R}_{\alpha_i,\beta}(z)} - 1\right).$$

Now, by using the inequality (1.8) for each α_i, β, where $i \in \{1, 2, \ldots, n\}$, we obtain

$$\begin{aligned}\left|\frac{z\mathbb{F}''_{\alpha_i,\beta,\lambda_i}(z)}{\mathbb{F}'_{\alpha_i,\beta,\lambda_i}(z)}\right| &= \sum_{i=1}^{n}\frac{1}{|\lambda_i|}\left|\frac{z\mathbb{R}'_{\alpha_i,\beta}(z)}{\mathbb{R}_{\alpha_i,\beta}(z)} - 1\right| \\ &\leqslant \sum_{i=1}^{n}\frac{1}{|\lambda_i|}\left(\frac{\frac{|\beta|}{1+\alpha_i}e^{\frac{|\beta|}{1+\alpha_i}}}{2 - e^{\frac{|\beta|}{1+\alpha_i}}}\right) \\ &\leqslant \sum_{i=1}^{n}\frac{1}{|\lambda_i|}\left(\frac{\frac{|\beta|}{1+\alpha}e^{\frac{|\beta|}{1+\alpha}}}{2 - e^{\frac{|\beta|}{1+\alpha}}}\right)\end{aligned}$$

for all $z \in \Delta$ and $\alpha_i > \dfrac{|\beta|}{\ln 2} - 1$, $\beta \in \mathbb{C}$ for all $i \in \{1, 2, \ldots, n\}$. Here we used that the function $\psi : (\dfrac{|\beta|}{\ln 2} - 1, \infty) \to \mathbb{R}$, defined by

$$\psi(x) = \frac{\frac{|\beta|}{1+x}e^{\frac{|\beta|}{1+x}}}{2 - e^{\frac{|\beta|}{1+x}}},$$

is decreasing. Therefore, for $i \in \{1, 2, \ldots, n\}$ we have

$$\frac{\frac{|\beta|}{1+\alpha_i} e^{\frac{|\beta|}{1+\alpha_i}}}{2 - e^{\frac{|\beta|}{1+\alpha_i}}} \leqslant \frac{\frac{|\beta|}{1+\alpha} e^{\frac{|\beta|}{1+\alpha}}}{2 - e^{\frac{|\beta|}{1+\alpha}}}.$$

Now, by using the triangle inequality and the hypothesis, we obtain

$$\left| s\,|z|^{2\zeta} + (1 - |z|^{2\zeta}) \frac{z\mathbb{F}''_{\alpha_i,\beta_i,\lambda_i}(z)}{\zeta\mathbb{F}'_{\alpha_i,\beta_i,\lambda_i}(z)} \right| \leqslant |s| + \frac{\frac{|\beta|}{1+\alpha} e^{\frac{|\beta|}{1+\alpha}}}{2 - e^{\frac{|\beta|}{1+\alpha}}} \sum_{i=1}^{n} \frac{1}{|\zeta\lambda_i|} \leqslant 1,$$

which in view of Lemma 1.1 implies that $\mathbb{F}_{\alpha_i,\beta,\lambda_i,\zeta} \in \mathcal{S}$. With this the proof is complete. □

Choosing $\lambda_1 = \lambda_2 = \cdots = \lambda_n = \lambda$ in Theorem 2.1, we have the following result.

Corollary 2.2. *Let the numbers $\zeta, s, \alpha, \alpha_1, \alpha_2, \ldots, \alpha_n$ be as in Theorem 2.1 and let λ be a nonzero complex number. Moreover, suppose that the functions $\mathbb{R}_{\alpha_i,\beta} \in \mathcal{A}$ are as in Theorem 2.1 and the following inequality*

$$|s| + \frac{n}{|\lambda\zeta|} \left(\frac{\frac{|\beta|}{1+\alpha} e^{\frac{|\beta|}{1+\alpha}}}{2 - e^{\frac{|\beta|}{1+\alpha}}} \right) \leqslant 1$$

is valid. Then, the function $\mathbb{F}_{\alpha_i,\beta,\lambda,\zeta}$ defined by

$$\mathbb{F}_{\alpha_i,\beta,\lambda,\zeta}(z) = \left\{ \zeta \int_0^z t^{\zeta-1} \prod_{i=1}^{n} \left(\frac{\mathbb{R}_{\alpha_i,\beta}(t)}{t} \right)^{1/\lambda} dt \right\}^{1/\zeta},$$

is in $\mathcal{S}$, i.e. is univalent in Δ.

Taking $n = 1$ in Corollary 2.2, we immediately obtain the following result.

Corollary 2.3. *Let $\beta \in \mathbb{C}$ with $\alpha > \frac{|\beta|}{\ln 2} - 1$, $\zeta \in \mathbb{C}$ with $\operatorname{Re}\zeta > 0$, $s \in \mathbb{C}$ with $s \neq -1$ and let $\lambda \neq 0$ be a complex number. Moreover, suppose that these numbers satisfy the following inequality*

$$|s| + \frac{1}{|\lambda\zeta|} \left(\frac{\frac{|\beta|}{1+\alpha} e^{\frac{|\beta|}{1+\alpha}}}{2 - e^{\frac{|\beta|}{1+\alpha}}} \right) \leqslant 1.$$

Then the function $\mathbb{F}_{\alpha,\beta,\lambda,\zeta}$ defined by

$$\mathbb{F}_{\alpha,\beta,\lambda,\zeta}(z) = \left\{ \zeta \int_0^z t^{\zeta-1} \left(\frac{\mathbb{R}_{\alpha,\beta}(t)}{t} \right)^{1/\lambda} dt \right\}^{1/\zeta},$$

is in $\mathcal{S}$.

In particular we have the following univalent functions in Δ:

Example 2.4. *(i) If $|s| + \frac{e^{\frac{1}{3}}}{3\left(2 - e^{\frac{1}{3}}\right)|\lambda\zeta|} \leqslant 1$, then the function $\mathbb{F}_{0,-\frac{1}{3},\lambda,\zeta}$ defined by*

$$\mathbb{F}_{0,-\frac{1}{3},\lambda,\zeta}(z) = \left\{ \zeta \int_0^z t^{\zeta-1} \left(e^{-\frac{t}{3}} \right)^{1/\lambda} dt \right\}^{1/\zeta},$$

is univalent in Δ.

(*ii*) *If* $|s| + \dfrac{e^{\frac{1}{8}}}{8\left(2-e^{\frac{1}{8}}\right)|\lambda\zeta|} \leqslant 1$, *then the function* $\mathbb{F}_{1,-\frac{1}{4},\lambda,\zeta}$ *defined by*

$$\mathbb{F}_{1,-\frac{1}{4},\lambda,\zeta}(z) = \left\{\zeta \int_0^z t^{\zeta-1}\left(\frac{2\sin\frac{\sqrt{t}}{2}}{\sqrt{t}}\right)^{1/\lambda} dt\right\}^{1/\zeta},$$

is univalent in Δ.

(*iii*) *If* $|s| + \dfrac{e^{\frac{1}{4}}}{4\left(2-e^{\frac{1}{4}}\right)|\lambda\zeta|} \leqslant 1$, *then the function* $\mathbb{F}_{1,\frac{1}{2},\lambda,\zeta}$ *defined by*

$$\mathbb{F}_{1,\frac{1}{2},\lambda,\zeta}(z) = \left\{\zeta \int_0^z t^{\zeta-1}\left(\frac{\sqrt{2}\sinh\sqrt{\frac{t}{2}}}{\sqrt{t}}\right)^{1/\lambda} dt\right\}^{1/\zeta},$$

is univalent in Δ.

(*iv*) *If* $|s| + \dfrac{e^{\frac{1}{2}}}{2\left(2-e^{\frac{1}{2}}\right)|\lambda\zeta|} \leqslant 1$, *then the function* $\mathbb{F}_{1,1,\lambda,\zeta}$ *defined by*

$$\mathbb{F}_{1,1,\lambda,\zeta}(z) = \left\{\zeta \int_0^z t^{\zeta-1}\left(\frac{\sinh\sqrt{t}}{\sqrt{t}}\right)^{1/\lambda} dt\right\}^{1/\zeta},$$

is univalent in Δ.

The following result contains another sufficient conditions for integrals of the type (1.5) to be univalent in the unit disk Δ. The key tools in the proof are Lemma 1.2 and the inequality (1.8).

Theorem 2.5. *Let* $\beta \in \mathbb{C}$ *and* $\alpha_1, \alpha_2, \ldots, \alpha_n > -1$ *with* $\alpha_i > \dfrac{|\beta|}{\ln 2} - 1$ *for all* $i \in \{1, 2, \ldots, n\}$ *and consider the normalized Rabotnov function* $\mathbb{R}_{\alpha_i,\beta}$ *defined by* (2.1)*. Let* $\alpha = \min\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$, $\lambda \in \mathbb{C}$ *with* $\operatorname{Re}\lambda > 0$, *and suppose that these numbers satisfy the following inequality*

$$|\lambda| \leqslant \frac{1}{n}\left(\frac{\frac{|\beta|}{1+\alpha}e^{\frac{|\beta|}{1+\alpha}}}{2-e^{\frac{|\beta|}{1+\alpha}}}\right)\operatorname{Re}\lambda.$$

Then the function $\mathbb{F}_{\alpha_i,\beta,\lambda,n}$ *defined by (1.5)is in* $\mathcal{S}$, *i.e. is univalent in* Δ.

Proof. Let us consider the auxiliary function $\mathbb{F}_{\alpha_i,\beta_i,\lambda}$ defined by

$$\mathbb{F}_{\alpha_i,\beta,\lambda}(z) = \int_0^z \prod_{i=1}^{n}\left(\frac{\mathbb{R}_{\alpha_i,\beta}(t)}{t}\right)^{\lambda} dt.$$

Observe that $\mathbb{F}_{\alpha_i,\beta,\lambda} \in \mathcal{A}$, i.e. $\mathbb{F}_{\alpha_i,\beta,\lambda}(0) = \mathbb{F}'_{\alpha_i,\beta,\lambda}(0) - 1 = 0$. On the other hand, by using (1.8) and the fact that for all $i \in \{1, 2, \ldots, n\}$

$$\frac{\frac{|\beta|}{1+\alpha_i}e^{\frac{|\beta|}{1+\alpha_i}}}{2-e^{\frac{|\beta|}{1+\alpha_i}}} \leqslant \frac{\frac{|\beta|}{1+\alpha}e^{\frac{|\beta|}{1+\alpha}}}{2-e^{\frac{|\beta|}{1+\alpha}}},$$

we obtain that for all $z \in \Delta$

$$\frac{1-|z|^{2\operatorname{Re}\lambda}}{\operatorname{Re}\lambda}\left|\frac{z\mathbb{F}''_{\alpha_i,\beta,\lambda}(z)}{\mathbb{F}'_{\alpha_i,\beta,\lambda}(z)}\right| \leqslant \frac{|\lambda|}{\operatorname{Re}\lambda}\sum_{i=1}^{n}\left|\frac{z\mathbb{R}'_{\alpha_i,\beta}(z)}{\mathbb{R}_{\alpha_i,\beta}(z)}-1\right| \leqslant \frac{n|\lambda|}{\operatorname{Re}\lambda}\left(\frac{\frac{|\beta|}{1+\alpha}e^{\frac{|\beta|}{1+\alpha}}}{2-e^{\frac{|\beta|}{1+\alpha}}}\right) \leqslant 1.$$

Now, since $\operatorname{Re}(n\lambda+1) > \operatorname{Re}\lambda$ and the function $\mathbb{F}_{\alpha_i,\beta,\lambda,n}$ can be rewritten in the form

$$\mathbb{F}_{\alpha_i,\beta,\lambda,n}(z) = \left\{(n\lambda+1)\int_0^z t^{n\lambda}\prod_{i=1}^{n}\left(\frac{\mathbb{R}_{\alpha_i,\beta}(t)}{t}\right)^{\lambda}dt\right\}^{1/(n\lambda+1)},$$

applying Lemma 1.2, we have $\mathbb{F}_{\alpha_i,\beta,\lambda,n}(z) \in \mathcal{S}$,which completes the proof of Theorem 2.5. □

Now, by choosing $n = 1$ in Theorem 2.5 we obtain the following result.

Corollary 2.6. *Let $\beta \in \mathbb{C}$ with $\alpha > \dfrac{|\beta|}{\ln 2} - 1$, and consider the normalized Rabotnov function $\mathbb{R}_{\alpha,\beta}$ defined by (2.1). Moreover, let $\lambda \in \mathbb{C}$ such that $\operatorname{Re}\lambda > 0$ and*

$$|\lambda| \leqslant \left(\frac{\frac{|\beta|}{1+\alpha}e^{\frac{|\beta|}{1+\alpha}}}{2-e^{\frac{|\beta|}{1+\alpha}}}\right)\operatorname{Re}\lambda.$$

Then the function $\mathbb{F}_{\alpha,\beta,\lambda}$ defined by

$$\mathbb{F}_{\alpha,\beta,\lambda}(z) = \left\{(\lambda+1)\int_0^z (\mathbb{R}_{\alpha,\beta}(t))^{\lambda}\,dt\right\}^{1/(\lambda+1)},$$

is univalent in Δ.

In particular we have the following univalent functions in Δ:

Example 2.7. *(i) If $|\lambda| \leqslant \frac{e^{\frac{1}{3}}}{3\left(2-e^{\frac{1}{3}}\right)}\operatorname{Re}\lambda$, then the function $\mathbb{F}_{0,-\frac{1}{3},\lambda}$ defined by*

$$\mathbb{F}_{0,-\frac{1}{3},\lambda}(z) = \left\{(\lambda+1)\int_0^z \left(te^{-\frac{t}{3}}\right)^{1/\lambda}dt\right\}^{1/(\lambda+1)},$$

is univalent in Δ.

(ii) If $|\lambda| \leqslant \dfrac{e^{\frac{1}{8}}}{8\left(2-e^{\frac{1}{8}}\right)}\operatorname{Re}\lambda$, then the function $\mathbb{F}_{1,-\frac{1}{4},\lambda}$ defined by

$$\mathbb{F}_{1,-\frac{1}{4},\lambda}(z) = \left\{(\lambda+1)\int_0^z \left(2\sqrt{t}\sin\frac{\sqrt{t}}{2}\right)^{1/\lambda}dt\right\}^{1/(\lambda+1)},$$

is univalent in Δ.

(iii) If $|\lambda| \leqslant \dfrac{e^{\frac{1}{4}}}{4\left(2-e^{\frac{1}{4}}\right)}\operatorname{Re}\lambda$, then the function $\mathbb{F}_{1,\frac{1}{2},\lambda}$ defined by

$$\mathbb{F}_{1,\frac{1}{2},\lambda}(z) = \left\{(\lambda+1)\int_0^z \left(\sqrt{2t}\sinh\sqrt{\frac{t}{2}}\right)^{\lambda}dt\right\}^{1/(\lambda+1)},$$

is univalent in Δ.

(*iv*) *If* $|\lambda| \leqslant \dfrac{e^{\frac{1}{2}}}{2\left(2-e^{\frac{1}{2}}\right)} \operatorname{Re}\lambda$, *then the function* $\mathbb{F}_{1,1,\lambda}$ *defined by*

$$\mathbb{F}_{1,1,\lambda}(z) = \left\{(\lambda+1)\int_0^z \left(\sqrt{t}\sinh\sqrt{t}\right)^{\lambda} dt\right\}^{1/(\lambda+1)},$$

is univalent in Δ.

Finally, by applying Lemma 1.3 and the inequality (1.9), we easily get the following result.

Theorem 2.8. *Let* $\beta \in \mathbb{C}$ *with* $\alpha > \dfrac{|\beta|}{\ln 2} - 1$ *and consider the normalized Rabotnov function* $\mathbb{R}_{\alpha,\beta}$ *defined by* (2.1). *If* $\operatorname{Re}\gamma \geqslant 1$ *and*

$$|\gamma| \leqslant \frac{3\sqrt{3}\,(1+\alpha)\,e^{-\left(\frac{|\beta|}{1+\alpha}\right)}}{2\,(1+\alpha+|\beta|)},$$

then the function $\mathbb{F}_{\alpha,\beta,\gamma}$ *defined by (1.6)is univalent in* Δ.

Example 2.9. (*i*) *If* $\gamma \in \mathbb{C}$ *such that* $\operatorname{Re}\gamma \geqslant 1$ *and* $|\gamma| \leqslant \dfrac{9\sqrt{3}e^{-\left(\frac{1}{3}\right)}}{8}$, *then* $\mathbb{F}_{0,-\frac{1}{3},\gamma}$ *defined by*

$$\mathbb{F}_{0,-\frac{1}{3},\gamma}(z) = \left\{\gamma\int_0^z t^{\gamma-1}\left(e^{\left(te^{-\frac{t}{3}}\right)}\right)^{\gamma} dt\right\}^{1/\gamma},$$

is univalent in Δ.

(*ii*) *If* $\gamma \in \mathbb{C}$ *such that* $\operatorname{Re}\gamma \geqslant 1$ *and* $|\gamma| \leqslant \dfrac{12\sqrt{3}e^{-\left(\frac{1}{12}\right)}}{9}$, *then* $\mathbb{F}_{1,-\frac{1}{4},\gamma}$ *defined by*

$$\mathbb{F}_{1,-\frac{1}{4},\gamma}(z) = \left\{\gamma\int_0^z t^{\gamma-1}\left(e^{\left(2\sqrt{t}\sin\frac{\sqrt{t}}{2}\right)}\right)^{\gamma} dt\right\}^{1/\gamma},$$

is univalent in Δ.

(*iii*) *If* $\gamma \in \mathbb{C}$ *such that* $\operatorname{Re}\gamma \geqslant 1$ *and* $|\gamma| \leqslant \dfrac{6\sqrt{3}e^{-\left(\frac{1}{4}\right)}}{5}$, *then* $\mathbb{F}_{1,\frac{1}{2},\gamma}$ *defined by*

$$\mathbb{F}_{1,\frac{1}{2},\gamma}(z) = \left\{\gamma\int_0^z t^{\gamma-1}\left(e^{\left(\sqrt{2t}\sinh\sqrt{\frac{t}{2}}\right)}\right)^{\gamma} dt\right\}^{1/\gamma},$$

is univalent in Δ.

(*iv*) *If* $\gamma \in \mathbb{C}$ *such that* $\operatorname{Re}\gamma \geqslant 1$ *and* $|\gamma| \leqslant \sqrt{3}e^{-\left(\frac{1}{2}\right)}$, *then* $\mathbb{F}_{1,1,\gamma}$ *defined by*

$$\mathbb{F}_{1,1,\gamma}(z) = \left\{\gamma\int_0^z t^{\gamma-1}\left(e^{\left(\sqrt{t}\sinh\sqrt{t}\right)}\right)^{\gamma} dt\right\}^{1/\gamma},$$

is univalent in Δ.

Conclusions

In this present paper, we have introduced three new integral operators involving normalized Rabotnov fractional exponential functions $\mathbb{R}_{\alpha,\beta}(z)$ and find sufficient conditions for these integral operators. In particular, we obtain simple sufficient conditions for some integral operators which involve the exponential and hyperbolic sine functions.

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Однолистность некоторых интегральных операторов, включающих дробную показательную функцию Работнова

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Аннотация. В этой статье мы вводим три новых интегральных оператора, включающих нормализованные дробно-экспоненциальные функции Работнова $\mathbb{R}_{\alpha,\beta}(z)$. Кроме того, мы найдем достаточные условия для этих интегральных операторов. Наконец, выводятся некоторые особые случаи для различных значений α и β.

Ключевые слова: аналитические функции, интегральные операторы, функция Работнова.