Journal of Siberian Federal University. Mathematics & Physics 2024, 17(5), 684–688

EDN: YYANLK
УДК 512.54

To the Question of the Closure of the Carpet

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Received 10.03.2024, received in revised form 15.04.2024, accepted 17.05.2024

Abstract. For a root system Φ, the set $\mathfrak{A} = \{\mathfrak{A}_r \mid r \in \Phi\}$ of additive subgroups $\mathfrak{A}_r$ over commutative ring K is called a carpet of type Φ if commuting two root elements $x_r(t), t \in \mathfrak{A}_r$ and $x_s(u), u \in \mathfrak{A}_s$, gives a result where each factor lies in the subgroup $\Phi(\mathfrak{A})$ generated by the root elements $x_r(t), t \in \mathfrak{A}_r, r \in \Phi$. The subgroup $\Phi(\mathfrak{A})$ is called a carpet subgroup. It defines a new set of additive subgroups $\overline{\mathfrak{A}} = \{\overline{\mathfrak{A}}_r \mid r \in \Phi\}$, the name of the closure of the carpet $\mathfrak{A}$, which is set by equation $\overline{\mathfrak{A}}_r = \{t \in K \mid x_r(t) \in \Phi(\mathfrak{A})\}$. Ya. Nuzhin wrote down the following question in the Kourovka notebook. *Is the closure $\overline{\mathfrak{A}}$ of a carpet $\mathfrak{A}$ a carpet too? (question 19.61).* The article provides a partial answer to this question. It is proved that the closure of a carpet of type Φ over commutative ring of odd characteristic p is a carpet if 3 does not divide p when Φ of type G_2.

Keywords: commutative ring, Chevalley group, carpet of additive subgroups, K-character.

Citation: E.N. Troyanskaya, To the Question of the Closure of the Carpet, J. Sib. Fed. Univ. Math. Phys., 2024, 17(5), 684–688. EDN: YYANLK.

1. Introduction

Let Φ be an indecomposable root system of rank $l, \Phi(K)$ be an elementary Chevalley group of type Φ over a commutative ring K. The group $\Phi(K)$ is generated by its root subgroups

$$x_r(K) = \{x_r(t) \mid t \in K\},\ r \in \Phi.$$

The subgroups $x_r(K)$ are abelian and for each $r \in \Phi$ and any $t, u \in K$ the following relations hold

$$x_r(t)x_r(u) = x_r(t+u). \tag{1}$$

A set of additive subgroups $\mathfrak{A} = \{\mathfrak{A}_r \mid r \in \Phi\}$ is called a *carpet* of type Φ over the ring K if

$$C_{ij,rs}\mathfrak{A}_r^i\mathfrak{A}_s^j \subseteq \mathfrak{A}_{ir+js},\ \text{ at }\ r, s, ir+js \in \Phi,\ i>0,\ j>0, \tag{2}$$

where $\mathfrak{A}_r^i = \{a^i \mid a \in \mathfrak{A}_r\}$, and the constants $C_{ij,rs} = \pm1, \pm2, \pm3$ are defined by the Chevalley commutator formula

$$[x_s(u), x_r(t)] = \prod_{i,j>0} x_{ir+js}(C_{ij,rs}(-t)^i u^j),\quad r, s, ir+js \in \Phi. \tag{3}$$

This definition of a carpet was introduced by V. M. Levchuk in the article [1]. Each carpet $\mathfrak{A}$ defines a *carpet subgroup*

$$\Phi(\mathfrak{A}) = \langle x_r(\mathfrak{A}_r) \mid r \in \Phi\rangle,$$

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where $\langle M \rangle$ denotes the subgroup generated by the set M from any subgroup. We call the *closure* of the carpet $\mathfrak{A}$ the set $\overline{\mathfrak{A}} = \{\overline{\mathfrak{A}}_r \mid r \in \Phi\}$ that is defined by

$$\overline{\mathfrak{A}}_r = \{t \in K \mid x_r(t) \in \Phi(\mathfrak{A})\},\ r \in \Phi.$$

A carpet $\mathfrak{A}$ is called *closed* if $\Phi(\mathfrak{A}) \cap x_r(K) = x_r(\mathfrak{A}_r), r \in \Phi$. In our notation this is equivalent to the equality $\overline{\mathfrak{A}}_r = \{\mathfrak{A}_r\}$ for all $r \in \Phi$, shortly $\overline{\mathfrak{A}} = \mathfrak{A}$. Examples of non-closed carpets for commutative rings of sufficiently wide classes are given in the articles [2] and [3].

This article received the following question from Ya. Nuzhin from the Kourovka notebook.

A) *Is the closure $\overline{\mathfrak{A}}$ of the carpet $\mathfrak{A}$ a carpet too?* [4, question 19.61]

From the conditions of carpet (2) a statement follows. If $t \in \mathfrak{A}_r, u \in \mathfrak{A}_s$, then each factor from the right side of the formula (3) lies in the carpet subgroup $\Phi(\mathfrak{A})$. On the other hand, for the arbitrary subgroup M of the Chevalley group $\Phi(K)$ the set $\mathfrak{M} = \{\mathfrak{M}_r \mid r \in \Phi\}$, defined by the

$$\mathfrak{M}_r = \{t \in K \mid x_r(t) \in M\},\ r \in \Phi$$

is not always a carpet [5, page 528]. However, for the types A_l, D_l and E_l, the set $\mathfrak{M}$ defined by the subgroup M is a carpet, as for this type formula (3) has the form $[x_r(t), x_s(u)] = x_{r+s}(\pm tu)$. Therefore, for types A_l, D_l and E_l closure of the carper is always a carpet. Thus, the question A) is relevant only for $\Phi = B_l, C_l, F_4, G_2$. The main result of the article is

Theorem 1. *The closure $\overline{\mathfrak{A}}$ of a carpet $\mathfrak{A}$ of type Φ over a ring of odd characteristic p is a carpet if 3 does not divide p when Φ of type G_2.*

2. Preliminary results

The Chevalley group $\Phi(K)$ is increased to the extended group $\hat{\Phi}(K)$ by all diagonal elements $h(\chi)$, where χ is the K-character of the integer root lattice $\mathbb{Z}\Phi$, that is, a homomorphism of the additive group $\mathbb{Z}\Phi$ into the multiplicative group K^* of the field K. Of course, the following equalities hold

$$\chi(a+b) = \chi(a)\chi(b), \quad a, b \in \Phi,$$
$$\chi(-a) = \chi(a)^{-1}, \quad a \in \Phi,$$

which will be used frequently.

Lemma 1. [6, Sec. 7.1] *Any K-character χ is uniquely determined by values on the fundamental roots and for any $r \in \Phi, t \in K$*

$$h(\chi)x_r(t)h^{-1}(\chi) = x_r(\chi(r)t). \tag{4}$$

The next lemma follows from the definition of a carpet and a carpet subgroup.

Lemma 2. *Let $\mathfrak{M} = \{\mathfrak{M}_r \mid r \in \Phi\}$ — a set of additive subgroups of the ring K, the subgroup M of the Chevalley group $\Phi(K)$ is generated by the subgroups $x_r(\mathfrak{M}_r)$, $r \in \Phi$, and $M \cap x_r(K) = x_r(\mathfrak{M}_r)$. A set $\mathfrak{M}$ is a carpet if and only if for any $r, s \in \Phi$ with the condition that $r + s \in \Phi$, each factor from the right side of the commutator formula for elements $x_r(t)$ and $x_s(t)$, where $t \in \mathfrak{M}_r,\ u \in \mathfrak{M}_s$, lies to M.*

The article by Ya. Nuzhin gives examples of a subgroup M of the Chevalley group $\Phi(K)$ of types B_2, G_2 such that the set $\mathfrak{M}$ defined as in Lemma 2, is not a carpet [5, examples 1-2].

Lemma 3. *Each diagonal element $h(\chi)$ normalizes any subgroup of the Chevalley group that is generated by the root elements if for all $r \in \Phi$ the value $\chi(r)$ lies in the simple subring generated by 1.*

3. Proof of the Theorem 1

Any two roots r, s with the condition that $r+s$ is a root lie in a root system of type A_2, B_2 or G_2. Therefore, by Lemma 2, it is enough to prove Theorem 1 for these types of rank 2. As already noted in the introduction, for the type A_2 the closure of any carpet is a carpet. That means that only the types B_2 and G_2 remain.

Let $r, s, r+s$ be the roots of the root system Φ of type B_2 or G_2, $\mathfrak{A} = \{\mathfrak{A}_r \mid r \in \Phi\}$ is a carpet of type Φ, $\overline{\mathfrak{A}} = \{\overline{\mathfrak{A}}_r \mid r \in \Phi\}$ the closure of a carpet,

$$M = < x_r(\overline{\mathfrak{A}}_r) \mid r \in \Phi > .$$

By Lemma 2, to prove Theorem 1 it is enough to establish the following statement.

B) For any $t \in \overline{\mathfrak{A}}_r, u \in \overline{\mathfrak{A}}_s$ each factor from the right side of the commutator formula for elements $x_r(t)$ and $x_s(u)$ lies in M.

It is clear that we will be interested only in those cases for which there are two or more factors on the right side of the commutator formula (3).

Let Φ be of type B_2. In this case, there are two types of commutator formula (3) with more than one factor on the right side, these are the following formulas

$$[x_a(t), x_b(u)] = x_{a+b}(\varepsilon_1 tu)x_{2a+b}(\varepsilon_2 t^2 u), \tag{5}$$

$$[x_b(u), x_a(t)] = x_{a+b}(\varepsilon_3 tu)x_{2a+b}(\varepsilon_4 t^2 u), \tag{6}$$

where $\varepsilon_i = \pm 1$, $i = 1, 2, 3, 4$. The right sides of these two formulas differ only in sign, so it is enough to consider only one of them, for example, the first.

Let $\chi(a) = \chi(b) = -1$. According to Lemma 3, $h(\chi)$ normalizes M and by Lemma 1

$$h(\chi)[x_a(t), x_b(u)]h^{-1}(\chi) = x_{a+b}(\varepsilon_1 tu)x_{2a+b}(-\varepsilon_2 t^2 u). \tag{7}$$

Multiplying the right sides (5) and (7), we obtain the inclusion $x_{a+b}(\varepsilon_1 2tu) \in M$. Since the characteristic is odd, then $x_{a+b}(\pm tu) \in M$. Multiplying it to the (5), we get $x_{2a+b}(\varepsilon_2 t^2 u)$. Thus, statement B) is established.

Let Φ be of type G_2. Chevalley commutator formulas having more than one factor on the right side are represented by four cases

$$[x_a(t), x_b(u)] = x_{a+b}(\varepsilon_1 tu)x_{2a+b}(\varepsilon_2 t^2 u)x_{3a+b}(\varepsilon_3 t^3 u)x_{3a+2b}(\varepsilon_4 t^3 u^2), \tag{8}$$

$$[x_b(u), x_a(t)] = x_{a+b}(\varepsilon_1 tu)x_{2a+b}(\varepsilon_2 t^2 u)x_{3a+b}(\varepsilon_3 t^3 u)x_{3a+2b}(\varepsilon_4 2t^3 u^2), \tag{9}$$

$$[x_a(t), x_{a+b}(u)] = x_{2a+b}(\varepsilon_1 2tu)x_{3a+2b}(\varepsilon_2 3tu^2)x_{3a+b}(\varepsilon_3 3t^2 u), \tag{10}$$

$$[x_{a+b}(u), x_a(t)] = x_{2a+b}(\varepsilon_1 2tu)x_{3a+2b}(\varepsilon_2 3tu^2)x_{3a+b}(\varepsilon_3 3t^2 u), \tag{11}$$

where $\varepsilon_i = \pm 1$. Formulas (8) and (9) have different factors on the right side. The right sides of (10), (11) differ only in sign, so it is enough to consider only (10).

Let it begin with the formula (8). Let $\chi(a) = -1, \chi(b) = 1$. By Lemma 1

$$h(\chi)[x_a(t), x_b(u)]h^{-1}(\chi) = x_{a+b}(-\varepsilon_1 tu)x_{2a+b}(\varepsilon_2 t^2 u)x_{3a+b}(-\varepsilon_3 t^3 u)x_{3a+2b}(-\varepsilon_4 t^3 u^2). \tag{12}$$

Multiplying the right sides (8) and (12), we have $x_{2a+b}(\varepsilon_2 2t^2u) \in M$. Since the characteristic is odd, then $x_{2a+b}(-\varepsilon_2 t^2u) \in M$. Multiplying the right side (12) by it, we obtain the product

$$x_{a+b}(-\varepsilon_1 tu)x_{3a+b}(-\varepsilon_3 t^3u)x_{3a+2b}(-\varepsilon_4 t^3u^2). \tag{13}$$

Let $\chi(a) = -1, \chi(b) = -1$. Then

$$h(\chi)[x_{a+b}(\varepsilon_1 tu)x_{3a+b}(\varepsilon_3 t^3u)x_{3a+2b}(\varepsilon_4 t^3u^2)]h^{-1}(\chi) = x_{a+b}(\varepsilon_1 tu)x_{3a+b}(\varepsilon_3 t^3u)x_{3a+2b}(-\varepsilon_4 t^3u^2). \tag{14}$$

Multiplying the right sides (13) and (14), we obtain $x_{3a+2b}(-2t^3u^2) \in M$, and therefore

$$x_{a+b}(-\varepsilon_1 tu)x_{3a+b}(-\varepsilon_3 t^3u).$$

This product cannot be split using the sets $\chi(r) = \pm 1$. Let us choose other values of $\chi(r)$ from the multiplicative groups of the field. Since the characteristic $p > 3$ is odd, the number 2 is different from ± 1 and invertible in the field K. We use this fact to choose $\chi(r)$. Let $\chi(a) = 2, \chi(b) = -2$, then

$$h(\chi)[x_{a+b}(\varepsilon_1 tu), x_{3a+b}(\varepsilon_2 t^3u)]h^{-1}(\chi) = x_{a+b}(-4\varepsilon_1 tu)x_{3a+b}(-16\varepsilon_2 t^3u). \tag{15}$$

Let $k \leqslant p$ be the inverse element for -4 in the ring, then we raise $x_{a+b}(-4\varepsilon_1 tu)x_{3a+b}(-16\varepsilon_2 t^3u)$ to the power k and get $x_{a+b}(\varepsilon_1 tu)x_{3a+b}(4\varepsilon_3 t^3u)$. Adding this result to the product $x_{a+b}(-\varepsilon_1 tu)x_{3a+b}(-\varepsilon_3 t^3u)$, we obtain $x_{3a+b}(3\varepsilon_3 t^3u) \in M$. Since the characteristic of the ring K is not divisible by 3 , then $x_{3a+b}(\varepsilon_3 t^3u) \in M$. So, we managed to split the factors of the formula (8). Formula (9) differs in the multiplier coefficient $x_{3a+2b}(\varepsilon_4 2t^3u^2)$, which, as above, splits off from the product at the second step and does not play a role in the proof. Thus, statement B) holds for (8), (9).

Carry out a similar procedure for the formula (10). Let $\chi(a) = 1, \chi(b) = -1$. By Lemma 1

$$h(\chi)[x_a(t), x_{a+b}(u)]h^{-1}(\chi) = x_{2a+b}(-2\varepsilon_1 tu)x_{3a+2b}(3\varepsilon_2 t^2u)x_{3a+b}(-3\varepsilon_3 t^2u). \tag{16}$$

Let $\chi(a) = -1, \chi(b) = 1$. By Lemma 1

$$h(\chi)[x_a(t), x_{a+b}(u)]h^{-1}(\chi) = x_{2a+b}(2\varepsilon_1 tu)x_{3a+2b}(-3\varepsilon_2 t^2u)x_{3a+b}(-3\varepsilon_3 t^2u). \tag{17}$$

Multiplying the right sides (10) and (16), we have $x_{3a+2b}(6\varepsilon_2 t^2u)$. Multiplying the right sides (10) and (17), we have $x_{2a+b}(4\varepsilon_1 tu)$. Since the numbers 6 and 4 are coprime with the characteristic of the ring K, then the elements $x_{2a+b}(\varepsilon_1 tu), x_{3a+2b}(\varepsilon_2 t^2u)$ lie in M. Consequently, the factors of Chevalley's formula (10) are able to be split. The theorem has been proven.

The author expresses deep gratitude to his scientific supervisor Ya. Nuzhin for setting the problem, constant attention to the work and support at all stages of its implementation.

This work is supported by the Krasnoyarsk Mathematical Center and financed by the Ministry of Science and Higher Education of the Russian Federation (Agreement No. 075-02-2024-1429).

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К вопросу о замыкании ковра

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Аннотация. Для системы корней Φ набор $\mathfrak{A} = \{\mathfrak{A}_r \mid r \in \Phi\}$ аддитивных подгрупп $\mathfrak{A}_r$ коммутативного кольца K называется ковром типа Φ, если при коммутировании двух корневых элементов $x_r(t), t \in \mathfrak{A}_r$ и $x_s(u), u \in \mathfrak{A}_s$, каждый сомножитель из правой части коммутативной формулы Шевалле лежит в подгруппе $\Phi(\mathfrak{A})$, порожденной корневыми элементами $x_r(t), t \in \mathfrak{A}_r, r \in \Phi$. Подгруппа $\Phi(\mathfrak{A})$ называется ковровой подгруппой. Она определяет новый набор аддитивных подгрупп $\overline{\mathfrak{A}} = \{\overline{\mathfrak{A}}_r \mid r \in \Phi\}$, называемый замыканием ковра $\mathfrak{A}$, который задается равенствами $\overline{\mathfrak{A}}_r = \{t \in K \mid x_r(t) \in \Phi(\mathfrak{A})\}$. Я.Н. Нужин записал в Коуровской тетради следующий вопрос. *Является ли замыкание $\overline{\mathfrak{A}}$ ковра $\mathfrak{A}$ ковром? (вопрос 19.61).* В статье доказано, что замыкание ковра типа Φ над коммутативным кольцом нечетной характеристики p является ковром, если 3 не делит p, когда Φ типа G_2.

Ключевые слова: коммутативное кольцо, группа Шевалле, ковер аддитивных подгрупп, K-характер.