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On a New Identity for Double Sum Related to Bernoulli Numbers

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Abstract. Let m, n and l be integers with $0 \leqslant l \leqslant m+n$. It is the main purpose of this paper to give an identity for the sum:

$$\sum_{\substack{a=0 \\ a+b \geqslant m+n-l}}^{m} \sum_{b=0}^{n} B_{m-a} B_{n-b} \frac{\binom{m}{a}\binom{n}{b}}{a+b+1} \binom{a+b+1}{m+n-l},$$

where B_m $(m = 0, 1, 2, \dots)$ is the Bernoulli number. As corollary we prove that the above sum equal to $\frac{1}{2}$ when $l = 0$.

Keywords: Bernoulli polynomial, Bernoulli number, generating function.

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1. Introduction and main results

As the years have gone by, Bernoulli polynomials and numbers have consistently affirmed their significance as crucial mathematical entities. Since their introduction in the 17th century, they have continuously piqued the curiosity of numerous mathematicians and have found applications across a multitude of mathematical disciplines. Bernoulli polynomials $B_m(x)$ $(m = 0, 1, 2, \dots)$ are defined by using the generating function (see e.g., [2–4]):

$$\frac{ze^{xz}}{e^z - 1} = \sum_{m=0}^{\infty} B_m(x) \frac{z^m}{m!}, \quad |z| < 2\pi.$$

The Bernoulli numbers B_m $(m = 0, 1, 2, \dots)$ are the values of the Bernoulli polynomials $B_m(x)$ at $x = 0$ or, equivalently, they are the coefficients in the power series expansion (see e.g., [2, 4]):

$$\frac{z}{e^z - 1} = \sum_{m=0}^{\infty} B_m \frac{z^m}{m!}, \quad |z| < 2\pi.$$

There are numerous properties associated with Bernoulli numbers and polynomials, which readers interested in this topic can explore, for instance, in the following references [3, 4]. In the forthcoming discussion, we will confine ourselves to enumerating the properties upon which we will rely for the demonstration of our results.

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The expression of the Bernoulli polynomials in terms of the Bernoulli numbers is given by (see e.g., [2–4]):

$$B_m(x) = \sum_{j=0}^{m} \binom{m}{j} B_{m-j} x^j. \tag{1}$$

The Bernoulli polynomials satisfy the well-known relation (see e.g., [2–4]):

$$\frac{d}{dx} B_m(x) = m B_{m-1}(x) \quad (n \geqslant 1). \tag{2}$$

The Bernoulli polynomials satisfy the difirence equation (see e.g., [4]):

$$B_m(x+1) - B_m(x) = m x^{n-1} \quad (n \geqslant 1),$$

from which

$$B_m(0) = B_m(1) \quad (n \geqslant 2), \tag{3}$$

Many mathematicians, over the course of time, has been deeply intrigued by the pursuit of identifying and rigorously establishing mathematical identities related to Bernoulli numbers. For example, in the work by Vassilev and Missana [4], an interesting identity was established for all positive integers m and n:

$$(-1)^m \sum_{a=0}^{m-1} \binom{m}{a} B_{m+a} = (-1)^n \sum_{a=0}^{n-1} \binom{n}{a} B_{n+a}.$$

In another research, Agoh and Dilcher [1, Lemma 1], for all $m, n \geqslant 0$, proved the following identity:

$$\sum_{a=0}^{m} (-1)^a \binom{m+n+1}{m-a} B_{m-a} B_{n+a+1} - \sum_{a=0}^{n} (-1)^a \binom{n+m+1}{n-a} B_{n-a} B_{m+a+1} = $$
$$= (-1)^n (m+n) B_{m+n+1}.$$

One can also find several identities in [4, Corollary 19.1.18].

The aim of this paper is to establish an identity for the sum associated with the Bernoulli numbers, which is presented as follows:

Let m, n and l be integers with $0 \leqslant l \leqslant m+n$. Set

$$S(m,n,l) := \sum_{\substack{a=0 \\ a+b \geqslant m+n-l}}^{m} \sum_{b=0}^{n} B_{m-a} B_{n-b} \frac{\binom{m}{a}\binom{n}{b}}{a+b+1} \binom{a+b+1}{m+n-l}.$$

Our main identity is the following:

Theorem 1.1. *Let $n < m$ be non-negative integers such that $m+n \geqslant 3$. If $0 \leqslant l \leqslant m+n-3$, then*

$$S(m,n,l) = \sum_{r=0}^{\lfloor \frac{m}{2} \rfloor} \left\{ n \binom{m}{2r} + m \binom{n}{2r} \right\} \frac{(m+n-2r-1)\cdots(l+2-2r)}{(m+n-l)!} B_{2r} B_{l+1-2r}, \tag{4}$$

where it understood that the sum is extended over those r such that $l+1-2r \geqslant 0$.

Remark 1.2. *When relying on the right-hand side of Formula (4), we opt for $B_1 = \frac{1}{2}$ rather than $-\frac{1}{2}$, and this selection is quite common, as many researchers adopt it (see e.g., [3, Remark 1.2]).*

In the special case, when $l = 0$, the sum in Formula (4) becomes restricted to only one term (for $r = 0$), and then we have:

$$(m+n)\frac{(m+n-1)(m+n-2)\cdots(2)}{(m+n)!}B_0B_1 = B_1,$$

which proves the following corollary:

Corollary 1.3. *Let $n < m$ be non-negative integers such that $m + n \geqslant 3$. Then $S(m, n, 0) = \frac{1}{2}$.*

2. Proof of Theorem 1.1

The subsequent lemma will assume a pivotal role in establishing the proof for Theorem 1.1.

Lemma 2.1. *Let m and n be positive integers. Then*

$$B_m(x)B_n(x) = \sum_{r=0}^{M_{m,n}} \left\{ n\binom{m}{2r} + m\binom{n}{2r} \right\} \frac{B_{2r}B_{m+n-2r}(x)}{m+n-2r} + (-1)^{m+1}\frac{m!n!}{(m+n)!}B_{m+n},$$

where $M_{m,n} = \max\left\{\lfloor \frac{m}{2} \rfloor, \lfloor \frac{n}{2} \rfloor\right\}$

Proof. See e.g., [2, Ex. 19 p. 276]. □

Now, we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. Suppose that $n < m$ and l be non-negative integers such that $m + n \geqslant 3$ and $0 \leqslant l \leqslant m + n - 3$. Then according to Formula (1) we have

$$\sum_{a=0}^{m}\sum_{b=0}^{n}\binom{m}{a}\binom{n}{b}B_{m-a}B_{n-a}\ x^{a+b} = B_m(x)B_n(x). \tag{5}$$

Differentiating $(m + n - l - 1)$ times both sides of Formula (5) with respect to x, then dividing by $(m + n - l)!$ gives

$$\sum_{a=0}^{m}\sum_{b=0}^{n}\binom{m}{a}\binom{n}{b}B_{m-a}B_{n-a}\frac{1}{m+n-l}\binom{a+b}{m+n-l-1}\ x^{a+b-m-n+l+1} =$$

$$= \frac{1}{(m+n-l)!}\Big(B_m(x)B_n(x)\Big)^{(m+n-l-1)}. \tag{6}$$

By using the following elementary identity:

$$\frac{1}{m+n-l}\binom{a+b}{m+n-l-1} = \frac{1}{a+b+1}\binom{a+b+1}{m+n-l}$$

we can rewrite Formula (6) as:

$$\sum_{a=0}^{m}\sum_{b=0}^{n}B_{m-a}B_{n-a}\frac{\binom{m}{a}\binom{n}{b}}{a+b+1}\binom{a+b+1}{m+n-l}\ x^{a+b-m-n+l+1} =$$

$$= \frac{1}{(m+n-l)!}\Big(B_m(x)B_n(x)\Big)^{(m+n-l-1)}.$$

Taking $x = 1$ gives

$$S(m,n,l) = \frac{1}{(m+n-l)!}\Big(B_m(x)B_n(x)\Big)^{(m+n-l-1)}(1). \tag{7}$$

Now, taking into consideration Formulas (2) and (3), Lemma 2.1 allows us to get

$$\Big(B_m(x)B_n(x)\Big)^{(m+n-l-1)}(1) = \sum_{r=0}^{\lfloor \frac{m}{2} \rfloor} \left\{ n\binom{m}{2r} + m\binom{n}{2r} \right\} \times$$
$$\times (m+n-2r-1)(m+n-2r-2)\cdots(l+2-2r)B_{2r}B_{l+1-2r}. \tag{8}$$

Consequently, one can show that Formulas (7) and (8) imply Formula (4). This completes the proof. □

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О новом тождестве для двойной суммы, связанной с числами Бернулли

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Аннотация. Пусть m, n и l — целые числа с $0 \leqslant l \leqslant m+n$. Основной целью данной статьи является дать тождество для суммы:

$$\sum_{\substack{a=0 \\ a+b\geqslant m+n-l}}^{m}\sum_{b=0}^{n} B_{m-a}B_{n-b}\frac{\binom{m}{a}\binom{n}{b}}{a+b+1}\binom{a+b+1}{m+n-l},$$

где B_m $(m = 0,1,2,\dots)$ — число Бернулли. В качестве следствия мы доказываем, что указанная выше сумма равна $\frac{1}{2}$ при $l = 0$.

Ключевые слова: многочлен Бернулли, число Бернулли, производящая функция.