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On the Boundedness of Maximal Operators Associated with Singular Surfaces

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Abstract. The paper is devoted to investigate maximal operators associated with singular surfaces. It is proved the boundedness of these operators in the space L^p, when singular surfaces are given by parametric equations in $\mathbb{R}^3$.

Keywords: maximal operator, averaging operator, fractional power series, nonsingular point, critical exponent.

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1. Introduction and preliminaries

We investigate maximal operators defined by the following formula:

$$\mathcal{M}f(y) := \sup_{t>0} | \mathcal{A}_t f(y) |, \tag{1}$$

where

$$\mathcal{A}_t f(y) := \int_S f(y - tx)\psi(x)dS(x) \tag{2}$$

is an averaging operator, $S \in \mathbb{R}^{n+1}$ is a hyper-surface, ψ is a fixed non-negative smooth function with compact support, i.e. $0 \leqslant \psi \in C_0^\infty(\mathbb{R}^{n+1})$ and $f \in C_0^\infty(\mathbb{R}^{n+1})$.

The maximal operator of the form (1) is said to be bounded in $L^p := L^p(\mathbb{R}^{n+1})$ if there exists a positive number C, such that for any function $f \in C_0^\infty(\mathbb{R}^{n+1})$ the following inequality

$$\|\mathcal{M}f\|_{L^p} \leqslant C\, \|f\|_{L^p}$$

holds, where $\|\cdot\|_{L^p}$ is the natural norm of the space L^p.

Denote by $p'(S)$ a minimal number such that for all p, satisfying $p > p'(S)$, the maximal operator (1) is bounded in L^p. A number $p'(S)$ is said a critical (boundedness) exponent of the maximal operator (1).

Firstly, the boundedness of the maximal operators (1) in $L^p(\mathbb{R}^n)$, when S is an unit sphere centered at the origin, was proved by I. M. Stein with $p'(S) = \dfrac{n}{n-1}$, for $n \geqslant 3$ [1]. Later these operators were investigated in the works of J. Bourgain [2], A. Greenleaf [3], K.D. Sogge [4, 5], A. Iosevich, E. Sawyer and A. Seeger [6, 7].

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Also, the boundedness problem for the maximal operators (1) were studied in the papers of I. A. Ikromov, M. Kempe and D. Müller [8, 9]. In these papers it is considered homogeneous and smooth hypersurfaces of a finite type and proved the boundedness of maximal operators in the space $L^p(\mathbb{R}^3)$, when $p > 2$.

In [10], it was investigated maximal operators (1) associated with smooth hypersurfaces and defined a boundedness exponent of these operators in the space $L^p(\mathbb{R}^{n+1})$.

The papers [11–14] were devoted to the study of the boundedness of maximal operators associated with singular surfaces.

2. Statement of the problem

The concept of fractional power series is defined using the following definition.

Definition. *Let $V \subseteq \mathbb{R}^n_+$ be an open connected set such that $0 \in \bar{V}$, f is called a fractional power series in the set V if there is an open set $W \subseteq \mathbb{R}^n$, containing $\bar{V}$, a natural number N and a real analytic function g in $\Phi_N^{-1}(W)$ such that the identity $f = g \circ \Phi_{1/N}$ holds in the set V, where $\Phi_N : \mathbb{R}^n \to \mathbb{R}^n$ is a map, given by the formula $\Phi_N(x) = (x_1^N, x_2^N, \ldots, x_n^N)$ [15].*

In the present work we consider singular surfaces in the space $\mathbb{R}^3$ given by the following parametric equations

$$
\begin{gathered}
x_1(u_1, u_2) = r_1 + u_1^{a_1} u_2^{a_2} g_1(u_1, u_2), \quad x_2(u_1, u_2) = r_2 + u_1^{b_1} u_2^{b_2} g_2(u_1, u_2), \\
x_3(u_1, u_2) = r_3 + u_1^{c_1} u_2^{c_2} g_3(u_1, u_2),
\end{gathered} \tag{3}
$$

where r_1, r_2, r_3 are arbitrary real numbers and $a_1, a_2, b_1, b_2, c_1, c_2$ are non-negative rational numbers, $u_1 \geqslant 0, u_2 \geqslant 0$, $\{g_k(u_1, u_2)\}_{k=1}^3$ are fractional power series.

We use the following necessary denotations:

$$
B_1 = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}, \quad B_2 = \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}, \quad B_3 = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}.
$$

Remark 1. *If at least one of the numbers B_1, B_2, B_3 is nonzero, then the points of the surface* (3) *lie in a sufficiently small neighborhood of the origin of the coordinate system $Or_1r_2r_3$ and outside the coordinate planes are nonsingular points. The points of the surface* (3) *lie in a small neighborhood of zero and on the coordinate planes of the coordinate system $Or_1r_2r_3$ may be singular points (see lemma in [11]).*

In the paper we study the following averaging operator defined by the relations (2) and (3)

$$
\begin{gathered}
\mathcal{A}_t^{\phi} f(y) = \int_{\mathbb{R}^2_+} f\Big(y_1 - t\big(r_1 + u_1^{a_1} u_2^{a_2} g_1(u_1, u_2)\big), y_2 - t\big(r_2 + u_1^{b_1} u_2^{b_2} g_2(u_1, u_2)\big), \\
y_3 - t(r_3 + u_1^{c_1} u_2^{c_2} g_3(u_1, u_2))\Big) \psi_1(u_1, u_2) \sqrt{\phi(u_1, u_2)} du_1 du_2,
\end{gathered} \tag{4}
$$

here $\phi(u_1, u_2) = EG - F^2$ is fractional power series, as usual, E, G, F are the coefficients of the first quadratic form of the surface (3) and $f \in C_0^{\infty}(\mathbb{R}^3)$. Maximal operator, which corresponds to the operator $\mathcal{A}_t^{\phi} f$, is defined by the correlation

$$
\mathcal{M}^{\phi} f(y) := \sup_{t>0} \mid \mathcal{A}_t^{\phi} f(y) \mid, \; y \in \mathbb{R}^3.
$$

In this paper we investigate the maximal operators (1) associated with singular surfaces (3). More precisely, we study the maximal operator $\mathcal{M}^{\phi} f$ in a sufficiently small neighborhood of the point (r_1, r_2, r_3) of the surface (3) and prove that these operators are bounded in the space $L^p(\mathbb{R}^3)$ for some $p > 2$.

3. On the boundedness of the maximal operator $\mathcal{M}^{\phi}f$.

We use the following denotation:

$$p''(S) = \max\left\{\frac{a_1}{b_1+c_1}, \frac{a_2}{b_2+c_2}, \frac{b_1}{a_1+c_1}, \frac{b_2}{a_2+c_2}, \frac{c_1}{a_1+b_1}, \frac{c_2}{a_2+b_2}\right\}.$$

The main result of the present paper is the following

Theorem 3.1. *Let $\{g_k(u_1,u_2)\}_{k=1}^3$ be fractional power series at the origin in $\mathbb{R}^2$ such that $g_k(0,0)\neq 0$ and $B_1B_2B_3\neq 0$. If at least one of the numbers r_1, r_2, r_3 is non-zero, then there exists a neighborhood U of the point (r_1,r_2,r_3) such that for any function $\psi\in C_0^\infty(U)$, the maximal operator $\mathcal{M}^{\phi}f$ is bounded in $L^p(\mathbb{R}^3)$ for $p>\max\{p''(S),2\}$. Moreover, if $\psi_1(0,0)=\psi(r_1,r_2,r_3)>0$ and $p''(S)>2$, then the maximal operator $\mathcal{M}^{\phi}f$ is not bounded in $L^p(\mathbb{R}^3)$ when $2<p\leqslant p''(S)$.*

Proof. Assume first that $r_3\neq 0$. We investigate the maximal operator $\mathcal{M}^{\phi}f$ at nonsingular points of the surface (3). After direct calculations for the function $\phi(u_1,u_2)$ in (4) we have

$$\phi(u_1,u_2) := u_1^{m_1}u_2^{m_2}h_1^2(u_1,u_2) + u_1^{n_1}u_2^{n_2}h_2^2(u_1,u_2) + u_1^{l_1}u_2^{l_2}h_3^2(u_1,u_2), \tag{5}$$

where

$$m_1 = 2(a_1+b_1-1),\ m_2 = 2(a_2+b_2-1),\ n_1 = 2(a_1+c_1-1),$$

$$n_2 = 2(a_2+c_2-1),\quad l_1 = 2(b_1+c_1-1),\quad l_2 = 2(b_2+c_2-1)$$

and

$$\begin{aligned}
h_1(u_1,u_2) =& \Big(a_1g_1(u_1,u_2) + u_1\frac{\partial g_1(u_1,u_2)}{\partial u_1}\Big)\Big(b_2g_2(u_1,u_2) + u_2\frac{\partial g_2(u_1,u_2)}{\partial u_2}\Big) - \\
&- \Big(a_2g_1(u_1,u_2) + u_2\frac{\partial g_1(u_1,u_2)}{\partial u_2}\Big)\Big(b_1g_2(u_1,u_2) + u_1\frac{\partial g_2(u_1,u_2)}{\partial u_1}\Big),\\
h_2(u_1,u_2) =& \Big(a_1g_1(u_1,u_2) + u_1\frac{\partial g_1(u_1,u_2)}{\partial u_1}\Big)\Big(c_2g_3(u_1,u_2) + u_2\frac{\partial g_3(u_1,u_2)}{\partial u_2}\Big) - \\
&- \Big(a_2g_1(u_1,u_2) + u_2\frac{\partial g_1(u_1,u_2)}{\partial u_2}\Big)\Big(c_1g_3(u_1,u_2) + u_1\frac{\partial g_3(u_1,u_2)}{\partial u_1}\Big),\\
h_3(u_1,u_2) =& \Big(b_1g_2(u_1,u_2) + u_1\frac{\partial g_2(u_1,u_2)}{\partial u_1}\Big)\Big(c_2g_3(u_1,u_2) + u_2\frac{\partial g_3(u_1,u_2)}{\partial u_2}\Big) - \\
&- \Big(b_2g_2(u_1,u_2) + u_2\frac{\partial g_2(u_1,u_2)}{\partial u_2}\Big)\Big(c_1g_3(u_1,u_2) + u_1\frac{\partial g_3(u_1,u_2)}{\partial u_1}\Big)
\end{aligned}$$

are fractional power series.

From the conditions $B_1B_2B_3\neq 0$, $g_i(0,0)\neq 0$ follow that $h_i(0,0)\neq 0$.

We need to consider the following cases.

Case 1. Suppose that either $\min\{m_1,n_1,l_1\}=m_1$, $\min\{m_2,n_2,l_2\}=m_2$, or $\min\{m_1,n_1,l_1\}=n_1$, $\min\{m_2,n_2,l_2\}=n_2$, or $\min\{m_1,n_1,l_1\}=l_1$, $\min\{m_2,n_2,l_2\}=l_2$. For these cases, we can find easily that by formulas (4), (5) and by Theorem 3.1 in [13] the critical exponent of the maximal operator $\mathcal{M}^{\phi}f$ is equal to

$$p_1(S) = \max\left\{\frac{c_1}{0,5m_1+1}, \frac{c_2}{0,5m_2+1}\right\},$$

i.e., the maximal operator $\mathcal{M}^{\phi} f$ is bounded in $L^p(\mathbb{R}^3)$ for $p > \max\{p_1(S), 2\}$, and if $\psi_1(0,0) > 0$, $p_1(S) > 2$, then this operator is unbounded when $2 < p \leqslant p_1(S)$.

Case 2. Assume that $\min\{m_1, n_1, l_1\} = m_1$, $\min\{m_2, n_2, l_2\} = n_2$. Then the function $\phi(u_1, u_2)$ in (5) can be written in the form

$$\phi(u_1, u_2) = u_1^{m_1} u_2^{n_2} \eta(u_1, u_2), \tag{6}$$

where

$$\eta(u_1, u_2) = u_2^{m_2 - n_2} h_1^2(u_1, u_2) + u_1^{n_1 - m_1} h_2^2(u_1, u_2) + u_1^{l_1 - m_1} u_2^{l_2 - n_2} h_3^2(u_1, u_2) \tag{7}$$

is fractional power series.

Suppose that $m_2 - n_2$, $n_1 - m_1$, $l_1 - m_1$, $m_2 - l_2$, $n_1 - l_1$, $l_2 - n_2$ are positive rational numbers. In this case $\eta(0,0) = 0$ and consider the following two cases.

Case 2.1. Assume that the Newton diagram (see [10, 16]) of the function $\eta(u_1, u_2)$ consists of segments γ_1 and γ_2 connecting points $(l_1 - m_1, l_2 - n_2)$, $(0, m_2 - n_2)$ and $(l_1 - m_1, l_2 - n_2)$, $(n_1 - m_1, 0)$. In this case the point $(l_1 - m_1, l_2 - n_2)$ lies below the line connecting the points $(n_1 - m_1, 0)$, $(0, m_2 - n_2)$ and we have

$$\frac{l_1 - m_1}{n_1 - m_1} + \frac{l_2 - n_2}{m_2 - n_2} < 1. \tag{8}$$

Consider an open small square $E = \{(u_1, u_2) \in \mathbb{R}^2 : 0 < u_1, u_2 < \varepsilon\}$, where ε is a sufficiently small positive number. Now following Section 2 of [16], we can divide E into the regions

$$V_1 = \big\{(u_1, u_2) \in E : M_1 u_1^{s_1} \leqslant u_2 \leqslant \delta_1 u_1^{s_1}\big\},$$

$$V_2 = \big\{(u_1, u_2) \in E : M_2 u_1^{s_2} \leqslant u_2 \leqslant \delta_2 u_1^{s_2}\big\},$$

which correspond to the edges γ_1, γ_2 and

$$V_3 = \big\{(u_1, u_2) \in E : \delta_2 u_1^{s_2} < u_2 < M_1 u_1^{s_1}\big\}$$
$$V_4 = \big\{(u_1, u_2) \in E : u_2 < M_2 u_1^{s_2}\big\}$$
$$V_5 = \big\{(u_1, u_2) \in E : u_2 > \delta_1 u_1^{s_1}\big\}$$

corresponding to the vertices $(l_1 - m_1, l_2 - n_2)$, $(n_1 - m_1, 0)$, $(0, m_2 - n_2)$, respectively. Here $M_1, M_2, \delta_1, \delta_2$ are some positive numbers,

$$s_1 = \frac{l_1 - m_1}{m_2 - l_2} = \frac{c_1 - a_1}{a_2 - c_2}, \quad s_2 = \frac{n_1 - l_1}{l_2 - n_2} = \frac{a_1 - b_1}{b_2 - a_2}$$

and $s_1 < s_2$, $-\dfrac{1}{s_1}$, $-\dfrac{1}{s_2}$ are slopes of the edges γ_1, γ_2, respectively.

Following Lemma 2.2 in [16], we make the power transformation

$$u_1 = v_1, \ u_2 = v_1^{s_1} v_2 \tag{9}$$

in V_1. Then from the relations (4), (6) and (9) follows

$$\mathcal{A}_t^{\phi_1} f(y) = \int_{\mathbb{R}_+^2} f\Big(y_1 - t v_1^{a_1 + s_1 a_2} v_2^{a_2} \widetilde{g}_1(v_1, v_2), \ y_2 - t v_1^{b_1 + s_1 b_2} v_2^{b_2} \widetilde{g}_2(v_1, v_2),$$
$$y_3 - t(1 + v_1^{c_1 + s_1 c_2} v_2^{c_2} \widetilde{g}_3(v_1, v_2)\Big) \widetilde{\psi}_1(v_1, v_2) \times v_1^{0,5m_1 + (0,5m_2 + 1)s_1} v_2^{0,5n_2} \sqrt{\tilde{\eta}_1(v_1, v_2)} dv_1 dv_2,$$

where $\widetilde{\psi}_1(v_1, v_2) = \psi_1(v_1, v_1^{s_1} v_2)$, $\widetilde{g}_i(v_1, v_2) = g_i(v_1, v_1^{s_1} v_2)$, $i = 1, 2, 3$,

$$\tilde{\eta}_1(v_1, v_2) = v_2^{m_2-n_2} \tilde{h}_1^2(v_1, v_2) + v_1^{n_1-m_1-s_1(m_2-n_2)} \tilde{h}_2^2(v_1, v_2) + v_2^{l_2-n_2} \tilde{h}_3^2(v_1, v_2),$$

$\tilde{h}_i(v_1, v_2) = h_i\big(v_1, v_1^{s_1} v_2\big)$, $0 < v_1 < \varepsilon$, $M_1 \leqslant v_2 \leqslant \delta_1$.

By (8) we have $n_1 - m_1 - s_1(m_2 - n_2) > 0$ and $\tilde{\eta}_1(0, v_2) > 0$.

It is easy to see that the maximal operator $\mathcal{M}^{\phi_1} f$, which corresponds to the averaging operator $\mathcal{A}_t^{\phi_1} f$, satisfies assumptions of Theorem 3.1 in [13]. Therefore, according to this theorem, maximal operator $\mathcal{M}^{\phi_1} f$ is bounded in $L^p(\mathbb{R}^3)$ for

$$p > p_2(S) = \max\left\{ \frac{c_1 + c_2 s_1}{0,5m_1 + 1 + (0,5m_2 + 1)s_1}, \frac{c_2}{0,5n_2 + 1} \right\}$$

and is not bounded for $2 < p \leqslant p_2(S)$, while $p_2(S) > 2$, $\psi_1(0,0) > 0$.

Similarly, one can show that the critical exponent of the maximal operator $\mathcal{M}^{\phi} f$ is equal to

$$p_3(S) = \max\left\{ \frac{c_1 + c_2 s_2}{0,5m_1 + 1 + (0,5m_2 + 1)s_2}, \frac{c_2}{0,5n_2 + 1} \right\}$$

in V_2.

Next, to prove the boundedness of the maximal operator $\mathcal{M}^{\phi} f$ in V_3 we apply Lemma 2.1 in [16]. Let us write $\eta(u_1, u_2)$ in (7) in the form $\eta(u_1, u_2) = \alpha(u_1, u_2) + \beta(u_1, u_2)$, where

$$\alpha(u_1, u_2) = u_2^{m_2-n_2} h_1^2(u_1, u_2) + 0,5 u_1^{l_1-m_1} u_2^{l_2-n_2} h_3^2(u_1, u_2),$$

$$\beta(u_1, u_2) = u_1^{n_1-m_1} h_2^2(u_1, u_2) + 0,5 u_1^{l_1-m_1} u_2^{l_2-n_2} h_3^2(u_1, u_2).$$

Using the change of variables

$$u_1 = w_1,\ u_2 = w_1^{s_1} w_2,$$

in V_3 the function $\alpha(u_1, u_2)$ is represented as

$$\alpha_1(w_1, w_2) = w_1^{l_1-m_1+s_1(l_2-n_2)} \times w_2^{l_2-n_2} \Big(w_2^{m_2-l_2} \hat{h}_1^2(w_1, w_2) + 0,5 \hat{h}_3^2(w_1, w_2) \Big), \tag{10}$$

where $0 < w_1 < \varepsilon$, $\delta_2 w_1^{s_2-s_1} < w_2 < M_1$, $\hat{h}_1(w_1, w_2) = h_1(w_1, w_1^{s_1} w_2)$, $\hat{h}_3(w_1, w_2) = h_3(w_1, w_1^{s_1} w_2)$. Assume that M_1 is a sufficiently small positive number.

If we exchange the roles of the u_1 and u_2 axes, then we have

$$V_3' = \big\{ (u_1, u_2) \in E : M_1^{-\frac{1}{s_1}} u_2^{\frac{1}{s_1}} < u_1 < \delta_2^{-\frac{1}{s_2}} u_2^{\frac{1}{s_2}} \big\}.$$

After changing variables

$$u_1 = \nu_1 \nu_2^{\frac{1}{s_2}}, \quad u_2 = \nu_2 \tag{11}$$

in V_3' the function $\beta(u_1, u_2)$ takes the form

$$\beta_1(\nu_1, \nu_2) = \nu_1^{l_1-m_1} \nu_2^{\frac{1}{s_2}(n_1-m_1)} \Big(\nu_1^{n_1-l_1} \bar{h}_2^2(\nu_1, \nu_2) + 0,5 \bar{h}_3^2(\nu_1, \nu_2) \Big), \tag{12}$$

where $M_1^{-\frac{1}{s_1}} \nu_2^{\frac{1}{s_1} - \frac{1}{s_2}} < \nu_1 < \delta_2^{-\frac{1}{s_2}}$, $0 < \nu_2 < \varepsilon$. We assume that δ_2 is a sufficiently large number.

Consequently, by (6), (10) and (6), (12) we have

$$\phi_2(w_1, w_2) = w_1^{l_1+s_1 l_2} w_2^{l_2} \tilde{\eta}_2(w_1, w_2), \tag{13}$$

$$\overline{\phi}_2(\nu_1,\nu_2)=\nu_1^{l_1}\nu_2^{\frac{1}{s_2}n_1+n_2}\overline{\eta}_2(\nu_1,\nu_2), \tag{14}$$

where

$$\tilde{\eta}_2(w_1,w_2)=w_2^{m_2-l_2}\hat{h}_1^2(w_1,w_2)+0,5\hat{h}_3^2(w_1,w_2),$$
$$\overline{\eta}_2(\nu_1,\nu_2)=\nu_1^{n_1-l_1}\bar{h}_2^2(\nu_1,\nu_2)+0,5\bar{h}_3^2(\nu_1,\nu_2)$$

and $\tilde{\eta}_2(0,0)>0$, $\overline{\eta}_2(0,0)>0$.

Thus, from the formulas (4), (13) and (14), we get

$$\mathcal{A}_t^{\phi_2}f(y)=\int_{\mathbb{R}^2_+}f\Big(y_1-tw_1^{a_1+s_1a_2}w_2^{a_2}\hat{g}_1(w_1,w_2),\ y_2-tw_1^{b_1+s_1b_2}w_2^{b_2}\hat{g}_2(w_1,w_2),$$
$$y_3-t(1+w_1^{c_1+s_1c_2}w_2^{c_2}\hat{g}_3(w_1,w_2)\Big)\hat{\psi}_1(w_1,w_2)w_1^{0,5l_1+(0,5l_2+1)s_1}w_2^{0,5l_2}\times\sqrt{\tilde{\eta}_2(w_1,w_2)}dw_1dw_2,$$
$$\mathcal{A}_t^{\overline{\phi}_2}f(y)=\int\limits_{\mathbb{R}^2_+}f\Big(y_1-t\nu_1^{a_1}\nu_2^{\frac{a_1}{s_2}+a_2}\bar{g}_1(\nu_1,\nu_2),y_2-t\nu_1^{b_1}\nu_2^{\frac{b_1}{s_2}+b_2}\bar{g}_2(\nu_1,\nu_2),y_3-$$
$$-t(1+\nu_1^{c_1}\nu_2^{\frac{c_1}{s_2}+c_2}\bar{g}_3(\nu_1,\nu_2)\Big)\bar{\psi}_1(\nu_1,\nu_2)\nu_1^{0,5l_1}\nu_2^{(0,5n_1+1)\frac{1}{s_2}+0,5n_2}\times\sqrt{\overline{\eta}_2(\nu_1,\nu_2)}d\nu_1d\nu_2,$$

where $\hat{\psi}_1(w_1,w_2)=\psi_1(w_1,w_1^{s_1}w_2)$, $\hat{g}_i(w_1,w_2)=g_i(w_1,w_1^{s_1}w_2)$,
$\overline{\psi}_1(\nu_1,\nu_2)=\psi_1(\nu_1\nu_2^{\frac{1}{s_2}},\nu_2)$, $\overline{g}_i(\nu_1,\nu_2)=g_i(\nu_1\nu_2^{\frac{1}{s_2}},\nu_2)$, $i=1,2,3$.

Obviously, the maximal operators $\mathcal{M}^{\phi_2}f$ and $\mathcal{M}^{\overline{\phi}_2}f$, which correspond to the operators $\mathcal{A}_t^{\phi_2}f$ and $\mathcal{A}_t^{\overline{\phi}_2}f$, satisfy assumptions of Theorem 3.1 in [13]. Therefore, by means of this theorem the boundedness exponent of these maximal operators is equal to

$$p_4(S)=\max\left\{\frac{c_1+c_2s_1}{0,5l_1+1+(0,5l_2+1)s_1},\frac{c_1+c_2s_1}{0,5n_1+1+(0,5n_2+1)s_1},\frac{c_1}{0,5l_1+1},\frac{c_2}{0,5l_2+1}\right\}.$$

Analogously, it can be proved that using the power transformations (9) and (11) in the domains V_4 and V_5, respectively, we get the following critical exponent for the maximal operator $\mathcal{M}^{\phi}f$

$$p_5(S)=\max\left\{\frac{c_1+c_2s_2}{0,5n_1+1+(0,5n_2+1)s_2},\frac{c_1+c_2s_1}{0,5m_1+1+(0,5m_2+1)s_1},\frac{c_2}{0,5n_2+1},\frac{c_1}{0,5m_1+1}\right\}.$$

Case 2.2. Assume that the Newton diagram of the function $\eta(u_1,u_2)$ in (7) is a segment connecting the points $(n_1-m_1,0)$ and $(0,m_2-n_2)$.

Following section 2 of [16], we can divide the set E into the regions

$$D_1=\big\{(u_1,u_2)\in E: N_1u_1^{s_3}\leqslant u_2\leqslant\lambda_1u_1^{s_3}\big\},$$

which corresponds to the edge connecting the vertices $(n_1-m_1,0)$, $(0,m_2-n_2)$ and

$$D_2=\big\{(u_1,u_2)\in E: u_2<N_1u_1^{s_3}\big\},\quad D_3=\big\{(u_1,u_2)\in E: u_2>\lambda_1u_1^{s_3}\big\}$$

corresponding to the vertices $(n_1-m_1,0)$, $(0,m_2-n_2)$, respectively. Here N_1,λ_1 are positive numbers, $s_3=\dfrac{n_1-m_1}{m_2-n_2}=\dfrac{c_1-b_1}{b_2-c_2}$ and $-\dfrac{1}{s_3}$ is a slope of the edge, $n_1-m_1>0$, $m_2-n_2>0$.

Similarly, as in the case 2.1, we obtain the boundedness exponent $p_1(S)$ for the maximal operator $\mathcal{M}^{\phi}f$ in the regions D_1, D_2 and D_3 (see also Theorem 2, [11]).

It is easy to see that if at least one of the numbers n_1-m_1, m_2-n_2 is zero, i.e., $s_3=0$ or $s_3=+\infty$ or $n_1-m_1=0$, $m_2-n_2=0$, then the boundedness exponent of the maximal operator $\mathcal{M}^{\phi}f$ remains unchanged.

Analogously, one can investigate that if either $\min\{m_1, n_1, l_1\} = m_1$, $\min\{m_2, n_2, l_2\} = l_2$, or $\min\{m_1, n_1, l_1\} = n_1$, $\min\{m_2, n_2, l_2\} = m_2$, or $\min\{m_1, n_1, l_1\} = l_1$, $\min\{m_2, n_2, l_2\} = m_2$, or $\min\{m_1, n_1, l_1\} = n_1$, $\min\{m_2, n_2, l_2\} = l_2$, or $\min\{m_1, n_1, l_1\} = l_1$, $\min\{m_2, n_2, l_2\} = n_2$, then the boundedness exponent of the maximal operator is equal to $p_1(S)$.

Hence, we obtain $p'(S) = \max\{p_1(S), p_2(S), p_3(S), p_4(S), p_5(S)\} = \max\left\{\dfrac{c_1}{a_1 + b_1}, \dfrac{c_2}{a_2 + b_2}\right\}$.

Then making similar arguments for $r_1 \neq 0$ or $r_2 \neq 0$, we can get $p_6(S) = \max\left\{\dfrac{a_1}{b_1 + c_1}, \dfrac{a_2}{b_2 + c_2}\right\}$ or $p_7(S) = \max\left\{\dfrac{b_1}{a_1 + c_1}, \dfrac{b_2}{a_2 + c_2}\right\}$, respectively.

Thus, assuming $p''(S) = \max\{p'(S), p_6(S), p_7(S)\}$, we complete the proof of Theorem 3.1.

In the proof of the main result, we assumed that $l_1 - m_1$, $m_2 - l_2$, $n_1 - l_1$, $l_2 - n_2$ are positive rational numbers. It should be noted that if at least one of these numbers is equal to zero then the exponent of the boundedness of maximal operator remains unchanged. It is not difficult to see that the following remarks hold.

Remark 2. *By the conditions of Theorem 3.1 there are no cases when all numbers $l_1 - m_1$, $m_2 - l_2$, $n_1 - l_1$, $l_2 - n_2$ or any three of these numbers are zero. In other words, if either s_1 and s_2 does not exist, i.e., $\nexists s_1$, $\nexists s_2$, either $\nexists s_1$, $s_2 = +\infty$, or $s_1 = +\infty$, $\nexists s_2$, or $\nexists s_1$, $s_2 = 0$, or $s_1 = 0$, $\nexists s_2$, then they are contradictions to the conditions $B \neq 0$, $B_1 \neq 0, B_2 \neq 0$.*

Remark 3. *If either $\nexists s_1, s_2 > 0$, or $s_1 = +\infty, s_2 = 0$, or $s_1 > 0, \nexists s_2$, or $s_1 = +\infty, s_2 > 0$, or $s_1 > 0, s_2 = 0$, then they contradict the to inequality* (8).

Remark 4. *If either $s_1 = 0, s_2 = 0$, or $s_1 = 0, s_2 = +\infty$, or $s_1 = +\infty, s_2 = +\infty$, then the boundedness indicator of the maximal operator is equal to $p_1(S)$.*

Remark 5. *If either $s_1 = 0, s_2 > 0$ or $s_1 > 0, s_2 = +\infty$, then it is easy to show that the critical exponent of the maximal operator $\mathcal{M}^\phi f$ is equal to $p_1(S)$.*

Proposition 1. *Let $\{g_i(u_1, u_2)\}_{i=1}^3$, $\phi(u_1, u_2)$ be real analytic functions at the origin in $\mathbb{R}^2$. Then the statements of Theorem 3.1 are true.*

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Об ограниченности максимальных операторов, ассоциированных с сингулярными поверхностями

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Аннотация. Статья посвящена к исследованию максимальных операторов, ассоциированных с сингулярными поверхностями. Доказана ограниченность этих операторов в пространстве L^p, когда сингулярные поверхности задаются параметрическими уравнениями в $\mathbb{R}^3$.

Ключевые слова: максимальный оператор, оператор усреднения, дробно-степенной ряд, несингулярная точка, критический показатель.