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Some Classes of Sets Sufficient for Holomorphic Continuation of Integrable Functions

Bakhodir A. Shoimkulov*
National University of Uzbekistan
Tashkent, Uzbekistan
Baymurat J. Kutlimuratov†
Karakalpak State University
Nukus, Uzbekistan

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Abstract. In the present work we consider integrable functions defined on a boundary of a bounded domain D in $\mathbb{C}^n$, $n > 1$, and possessing a generalized Morera boundary property. We show that such functions possess a holomorphic continuation into the domain D for some sufficient sets Γ of complex lines.

Keywords: holomorphic continuation, Morera boundary condition, Bochner-Martinelli kernel.

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Introduction

This article contains some results related to the holomorphic extension of functions integrable on the boundary of a bounded domain into this domain. We consider functions that satisfy the multidimensional Morera boundary condition. It consists in the equality to zero of the integrals of a given function over the intersection of the boundary of the domain with complex lines. E. Grinberg [1] studied functions with the Morera property in a ball (in fact, this result was contained in the article by M. L. Agranovsky and R. E. Valsky [2]). I. Globevnik and E. L. Stout [3] obtained Morera's boundary theorem for an arbitrary bounded domain with a twice smooth boundary. A local version of Morera's theorem was considered by I. Globevnik [4], D. Govekar-Leban [5]. In the work of S. G. Myslivets [6] she considered functions with the Morera property along complex curves. In the works [7–11] and [12] there have been given some families of complex lines sufficient for holomorphic continuation functions. The monographs [13] and [14] present some results related to this problem.

This article considers integrable functions defined on the boundary of a bounded domain D in $\mathbb{C}^n$, $n > 1$, and possessing the generalized Morera boundary property.

*shoimkba@rambler.ru
†bayko-2020@mail.ru https://orcid.org/0000-0001-6683-5199

1. Basic notations and definitions

We consider the set of complex lines intersecting the germ of a smooth manifold of real dimension $(2n-2)$. Let $D \subset \mathbb{C}^n$ $(n > 1)$ be a bounded domain with a connected boundary of class C^1 of the form

$$D = \{z \in \mathbb{C}^n : \ \rho(z) < 0\},$$

where $\rho(z)$ is a smooth function of class C^1 being real in a neighbourhood of the set $\overline{D}$ such that $d\rho\,|_{\partial D} \neq 0$. We identify $\mathbb{C}^n$ with $\mathbb{R}^{2n}$ as follows: $z = (z_1, \ldots, z_n)$, where $z_j = x_j + iy_j$, x_j, $y_j \in \mathbb{R}$, $j = 1, \ldots, n$.

We consider complex lines $l_{z,b}$ of the form

$$l_{z,b} = \{\, \zeta \in \mathbb{C}^n : \zeta_j = z_j + b_j t, \ j = 1, 2, \ldots, n, \ t \in \mathbb{C}\} \tag{1}$$

passing through the point $z \in \mathbb{C}^n$ along the vector $b = \{b_1, \ldots, b_n\} \in \mathbb{C}P^{n-1}$ (the direction b is defined up to a multiplication by a complex number $\lambda \neq 0$).

Definition 1. *An integrable function f on ∂D $(f \in L^p(\partial D),\ p \geqslant 1)$ satisfies the Morera property along complex planes l of dimension k, $(1 \leqslant k \leqslant n-1)$, if*

$$\int\limits_{\partial D \cap l} f(\zeta)\, \beta(\zeta) = 0$$

for each differential form β of the type $(k,\, k-1)$ with constant coefficients.

The plane l is assumed to intersect the boundary of the domain D transversally.

If $l_{z,b}$ is a complex line intersecting ∂D transversally, then the Morera property along the planes $l_{z,b}$ becomes

$$\int\limits_{\partial D \cap l_{z,b}} f(z+bt)dt = \int\limits_{\partial D \cap l_{z,b}} f(z_1 + b_1 t, \ldots, z_n + b_n t)dt = 0 \tag{2}$$

for a given parameterization $\zeta = z + bt$ along a complex line $l_{z,b}$.

For complex lines we consider a more general condition. Let m be a fixed nonnegative integer, then the condition

$$\int\limits_{\partial D \cap l_{z,b}} f(z+bt)t^m dt = \int\limits_{\partial D \cap l_{z,b}} f(z_1 + b_1 t, \ldots, z_n + b_n t)t^m dt = 0 \tag{3}$$

we will call *the generalized Morera property.* For $m = 0$ the conditions (3) become the Morera boundary condition (2) (see [3]).

Let Γ be the germ of a C^1 manifold of real dimension $(2n-2)$. We assume that $0 \in \Gamma$ and in some neighbourhood of the origin the manifold Γ is of the form

$$\Gamma = \{\zeta \in \mathbb{C}^n : \Phi(\zeta) + i\psi(\zeta) = 0\},$$

where Φ, ψ are C^1 smooth, real-valued functions in the neighbourhood of the point zero. Here $\zeta = (\zeta_1, \ldots, \zeta_n)$ and $\zeta_j = \xi_j + i\eta_j$, $\xi_j, \eta_j \in \mathbb{R}$, $j = 1, \ldots, n$. The smoothness condition of the manifold Γ is that

$$rang\, A = rang \begin{pmatrix} \dfrac{\partial \Phi}{\partial \xi_1} \cdots \dfrac{\partial \Phi}{\partial \xi_n} & \dfrac{\partial \Phi}{\partial \eta_1} \cdots \dfrac{\partial \Phi}{\partial \eta_n} \\ \dfrac{\partial \psi}{\partial \xi_1} \cdots \dfrac{\partial \psi}{\partial \xi_n} & \dfrac{\partial \psi}{\partial \eta_1} \cdots \dfrac{\partial \psi}{\partial \eta_n} \end{pmatrix} = 2$$

at each point $\zeta \in \Gamma$.

We consider complex lines of form (1) and recall the following lemmas.

Lemma 1. *Let a vector $b^0 = (b_1^0, \dots, b_n^0) \in \mathbb{C}P^{n-1}$ be such that $D \cap l_{0,b^0} \neq \varnothing$. Then there exists $\varepsilon > 0$ such that for all z such that $|z| < \varepsilon$ and for all b such that $\left|b - b^0\right| < \varepsilon$, the following intersections are non-empty: $D \cap l_{z,b} \neq \varnothing$ and $\Gamma \cap l_{z,b} \neq \varnothing$.*

Lemma 2. *Let for some z and for all ζ, b such that $\Gamma \cap l_{z,b} \neq \varnothing$ for $\zeta \in \partial D \cap l_{z,b}$, the function ρ defining the domain D satisfies the conditions*

$$\sum_{j=1}^{n} \frac{\partial \rho}{\partial \zeta_j} b_j \neq 0, \tag{4}$$

then the curves $\partial D \cap l_{z,b}$ are smooth.

For instance, the assumptions of Lemma 2 are satisfied by domains in $\mathbb{C}^n$ that are strongly star-shaped with respect to a point $z \in D$, strongly convex, and strongly linear convex.

Lemma 3. *If $f \in L^p(\partial D)$, $(p \geqslant 1)$ then a function $f \in L^p(\partial D \cap l_{z,b})$ for almost all $b \in \mathbb{C}P^{n-1}$.*

Proof. Let $f \in L^p(\partial D)$, $(p \geqslant 1)$ and $l_{z,b}$ be one-dimensional complex lines of the form (1) passing through the point z. Consider an open set $W \subset \mathbb{C}P^{n-1}$ such that $\partial D \cap l_{z,b}$ be smooth curves for $b \in W$. We denote the open set by $S = \bigcup\limits_{b \in W} \partial D \cap l_{z,b}$. Then by Fubini's theorem we have

$$\int\limits_{S} |f(\zeta)|^p d\sigma(\zeta) = \int\limits_{W} d\sigma(b) \int\limits_{\partial D \cap l_{z,b}} |f((z+bt)|^p \left|\frac{d\sigma(\zeta)}{d\sigma(b)}\right| dt,$$

where $d\sigma(\zeta)$, $\sigma(b)$ and dt are Lebesgue measures, respectively, on S, W and $\partial D \cap l_{z,b}$, and $\left|\frac{d\sigma(\zeta)}{d\sigma(b)}\right|$ be module of the Jacobian arising when passing to the iterated integral. The inner integral converges for almost all $b \in W$. And since

$$0 < c_1 \leqslant \left|\frac{\sigma(\zeta)}{\sigma(b)}\right| \leqslant c_2 < \infty,$$

the following estimates hold

$$0 < c_1 \int\limits_{\partial D \cap l_{z,b}} |f|^p dt \leqslant \int\limits_{\partial D \cap l_{z,b}} |f|^p \left|\frac{\sigma(\zeta)}{\sigma(b)}\right| dt \leqslant c_2 \int\limits_{\partial D \cap l_{z,b}} |f|^p dt < \infty$$

The lemma is proven. □

2. Main result

Theorem 1. *Let a domain $D \subset \mathbb{C}^n$ satisfy conditions (4) for the points z lying in a neighbourhood of the manifold Γ such that $\partial D \cap \Gamma = \varnothing$. Let a function $f \in L^p(\partial D)$, $(p \geqslant 2)$ satisfy the generalized Morera conditions (3), that is,*

$$\int\limits_{\partial D \cap l_{z,b}} f(z_1 + b_1 t, \dots, z_n + b_n t) t^m dt = 0$$

for each $z \in \Gamma$, $b \in \mathbb{C}P^{n-1}$ and a fixed nonnegative integer number m. Then the function f has a holomorphic continuation into the domain D.

Proof. We consider the Bochner–Martinelli kernel

$$U(\zeta,z)=\frac{(n-1)!}{(2\pi i)^n}\sum_{k=1}^{n}(-1)^{k-1}\frac{\overline{\zeta}_k-\overline{z}_k}{|\zeta-z|^{2n}}d\overline{\zeta}[k]\wedge d\zeta.$$

As it is known, in coordinates b and t (see [13], Lemma 3.2.1), the kernel $U(\zeta,z)$ reads as:

$$U(\zeta,z)=\lambda(b)\wedge\frac{dt}{t}, \tag{5}$$

where $\lambda(b)$ is a differential form of type $(n-1,\,n-1)$ in $\mathbb{C}P^{n-1}$, independent of t, while $z\notin\partial D$.

We consider the integral

$$M_\alpha f(z)=\int\limits_{\partial D_\zeta}(\zeta-z)^\alpha f(\zeta)U(\zeta,z),$$

where $\alpha=(\alpha_1,\dots,\alpha_n)$ is an arbitrary multi-index such that

$$\|\alpha\|=\alpha_1+\dots+\alpha_n=m+1$$

and

$$(\zeta-z)^\alpha=(\zeta_1-z_1)^{\alpha_1}\dots(\zeta_n-z_n)^{\alpha_n}.$$

By the Fubini theorem and the form of the kernel (5) we obtain:

$$M_\alpha f(z)=\int\limits_{\mathbb{C}\mathbf{P}^{n-1}}b^\alpha\lambda(b)\int\limits_{\partial D\cap l_{z,b}}f(z_1+b_1t,\dots,z_n+b_nt)t^m dt.$$

By the conditions of Theorem 1 and Lemma 1, the identities

$$\int\limits_{\partial D\cap l_{z,b}}f(z_1+b_1t,\dots,z_n+b_nt)t^m dt=0$$

hold for all sufficiently small z, nearby to b^0, and any b. Then

$$M_\alpha f(z)=\int\limits_{\partial D_\zeta}(\zeta-z)^\alpha f(\zeta)U(\zeta,z)\equiv 0 \tag{6}$$

for all z such that $|z|<\varepsilon$.

We rewrite a function $M_\alpha f(z)$ in another form. We consider differential forms $U_s(\zeta,z)$:

$$\begin{aligned}U_s(\zeta,z\;=\;&\frac{(-1)^s(n-2)!}{(2\pi i)^n}\left(\sum_{j=1}^{s-1}(-1)^j\frac{\overline{\zeta}_j-\overline{z}_j}{|\zeta-z|^{2n-2}}d\overline{\zeta}[j,s]+\right.\\&\left.+\sum_{j=s+1}^{n}(-1)^{j-1}\frac{\overline{\zeta}_j-\overline{z}_j}{|\zeta-z|^{2n-2}}d\overline{\zeta}[s,j]\right)\wedge d\zeta.\end{aligned}$$

Here

$$\begin{aligned}\overline{\partial}\left(\tfrac{1}{\zeta_s-z_s}U_s(\zeta,z)\right)&=\frac{\partial}{\partial\overline{z}_s}\left(\frac{1}{\zeta_s-z_s}U_s(\zeta,z)\right)=\\&=\frac{(n-1)!}{(2\pi i)^n}\sum_{j=1}^{n}(-1)^{j-1}\frac{\overline{\zeta}_j-\overline{z}_j}{|\zeta-z|^{2n}}d\overline{\zeta}[j]\wedge d\zeta=U(\zeta,z)\end{aligned}$$

at $\zeta_s \neq z_s,\ s = 1,\dots,n$. Then condition (6) can be rewritten as

$$\int\limits_{\partial D_\zeta} f(\zeta)\overline{\partial}\Big((\zeta - z)^\beta U_s(\zeta, z)\Big) \equiv 0, \tag{7}$$

for z such that $|z| < \varepsilon$ and for all monomials $(\zeta - z)^\beta$ with $\|\beta\| = m$.

We are going to show that the condition (7) holds for monomials $(\zeta - z)^\gamma$ with $\|\gamma\| = m$. Indeed, we consider a monomial $(\zeta - z)^\gamma$ with $\|\gamma\| = m - 1$. Then the condition (7) holds for the monomials of the form:

$$(\zeta - z)^\beta (\zeta_k - z_k), \quad k = 1,\dots,n,$$

since the degree of these monomials is equal to m.

The identity holds:

$$\begin{gathered}\frac{\partial}{\partial \zeta_k}\left((\zeta - z)^\gamma(\zeta_k - z_k)U_s(\zeta, z)\right) = (\gamma_k + 1)(\zeta - z)^\gamma U_s(\zeta, z) - \\ -(n-1)(\zeta - z)^\gamma \frac{(\zeta_k - z_k)(\overline{\zeta}_k - \overline{z}_k)}{|\zeta - z|^2} U_s(\zeta, z).\end{gathered} \tag{8}$$

Summing up identities (8) over k, we obtain:

$$\sum_{k=1}^{n} \frac{\partial}{\partial \zeta_k}\left((\zeta - z)^\gamma(\zeta_k - z_k)U_s(\zeta, z)\right) = (\|\gamma\| + 1)\,(\zeta - z)^\gamma U_s(\zeta, z). \tag{9}$$

Since the condition (7) can be differentiated in z as $|z| < \varepsilon$, and the derivatives of expressions (9) in z and ζ differ only by the sign, it follows from (9) that the degree of the monomial in (7) can be lessen by one. By successively decreasing this degree, we arrive at the following conditions

$$\int\limits_{\partial D_\zeta} f(\zeta)\overline{\partial}U_s(\zeta, z) \equiv 0$$

for $|z| < \varepsilon$ and $s = 1,\dots,n$, i.e.

$$\int\limits_{\partial D_\zeta} (\zeta_s - z_s) f(\zeta) U(\zeta - z) \equiv 0 \tag{10}$$

for $|z| < \varepsilon$ and $s = 1,\dots,n$.

Applying the Laplace operator with respect to z to the left side of the equality (10)

$$\Delta = \frac{\partial^2}{\partial z_1 \partial \overline{z}_1} + \dots + \frac{\partial^2}{\partial z_n \partial \overline{z}_n}$$

we obtain that

$$\frac{\partial}{\partial \overline{z}_s} \int\limits_{\partial D_\zeta} f(\zeta) U(\zeta, z) \equiv 0$$

for $|z| < \varepsilon$ and $s = 1,\dots,n$. Here we used the harmonicity of the kernel $U(\zeta, z)$ and the identity

$$\Delta(gh) = h\Delta g + g\Delta h + \sum_{j=1}^{n} \frac{\partial g}{\partial \overline{z}_j} \cdot \frac{\partial h}{\partial z_j} + \sum_{j=1}^{n} \frac{\partial g}{\partial z_j}\frac{\partial h}{\partial \overline{z}_j}\cdot$$

Hence, the Bochner–Martinelli integral of the function $f \in L^p(\partial D)$, namely,

$$Mf(z) = \int_{\partial D_\zeta} f(\zeta)U(\zeta, z)$$

is a function holomorphic in a neighbourhood of the origin.

If $\Gamma \subset \mathbb{C}^n \backslash \overline{D}$, then $Mf(z) \equiv 0$ outside $\overline{D}$ since the boundary is connected and then the function f is continued holomorphically in the domain D (see [13]).

If $\Gamma \subset D$, then the function Mf is holomorphic in D and the boundary values of Mf coincide with f. □

Corollary 1. *Let a domain D satisfy the conditions of Theorem 1 and the function $f \in L^p(\partial D)\,(p \geqslant 2)$ satisfies the condition (2) for all $z \in \Gamma$ and $b \in \mathbb{C}P^{n-1}$. Then the function f can be holomorphically continued into D.*

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О некоторых классах множеств, достаточных для голоморфного продолжения интегрируемых функций

Баходир А. Шоимкулов
Национальный университет Узбекистана
Ташкент, Узбекистан
Баймурат Ж. Кутлымуратов
Каракалпакский госуниверситет
Нукус, Узбекистан

Аннотация. В данной статье рассматриваются интегрируемые функции, заданные на границе ограниченной области D в $\mathbb{C}^n$, $n > 1$, и обладающие обобщенным граничным свойством Морера. Исследуется вопрос о существовании голоморфного продолжения таких функций в область D для некоторых достаточных множеств Γ комплексных прямых.

Ключевые слова: голоморфное продолжение, граничное условие Морера, ядро Бохнера-Мартинелли.