Journal of Siberian Federal University. Mathematics & Physics 2024, 17(4), 519–527

EDN: EZQZAV
УДК 517.55+517.51

Maximal Functions and the Dirichlet Problem in the Class of m-convex Functions

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Received 16.01.2024, received in revised form 23.02.2024, accepted 14.04.2024

Abstract. In this work, we introduce the concept of maximal m-convex $(m - cv)$ functions and we solve the Dirichlet Problem with a given continuous boundary function for strictly m-convex domains $D \subset \mathbb{R}^n$. We prove that for the solution of the Dirichlet problem in the class $m - cv$ of functions, its Hessian $H_\omega^{n-m+1} = 0$ in the domain D.

Keywords: subharmonic functions, convex functions, m-convex functions, Borel measures, Hessians.

Citation: A. Sadullaev, R. Sharipov, Maximal Functions and the Dirichlet Problem in the Class of m-convex Functions, J. Sib. Fed. Univ. Math. Phys., 2024, 17(4), 519–527.
EDN: EZQZAV.

Introduction

In this work, we introduce the concept of maximal functions and for strictly m-convex domains $D \subset \mathbb{R}^n$ we solve the Dirichlet Problem with a given continuous boundary function. We prove that that for the solution of the Dirichlet problem in the class $m - cv$ of functions, its Hessian $H_\omega^{n-m+1} = 0$ in the domain D.

If the potential theory in the class of strongly m-subharmonic functions is based on differential forms and currents $(dd^c u)^k \wedge \beta^{n-k} \geqslant 0$, $k = 1, 2 \ldots, n - m + 1$, where $\beta = dd^c\|z\|^2$ the standard volume form in $\mathbb{C}^n$, then the theory of potential in the class of $m - cv$ functions, in particular, maximal $m - cv$ functions and the Dirichlet problem are related to Hessians $H^k(u) \geqslant 0$, $k = 1, 2, \ldots, n - m + 1$. The main method for studying maximal $m - cv$ functions, which in general are not smooth, is to connect $m - cv$ functions with strongly m-subharmonic (sh_m) functions. Theory of sh_m functions is well studied and currently the subject of study by many mathematicians (see Z. Błocki [6], S. Dinew and S. Kolodzej [7], S. Li [8], H. C. Lu [9, 10], A. Sadullaev, B. Abdullaev [11, 12] etc.)

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1. Strongly m-subharmonic and m-convex functions

Twice smooth function $u(z) \in C^2(D)$, $D \subset \mathbb{C}^n$, is called strongly m-subharmonic $u \in sh_m(D)$, if at each point of the domain D hold

$$sh_m(D) = \left\{u \in C^2 : (dd^c u)^k \wedge \beta^{n-k} \geqslant 0, \ \ k = 1, 2, \ldots, n - m + 1\right\} =$$

$$= \left\{u \in C^2 : dd^c u \wedge \beta^{n-1} \geqslant 0, (dd^c u)^2 \wedge \beta^{n-2} \geqslant 0, \ldots, (dd^c u)^{n-m+1} \wedge \beta^{m-1} \geqslant 0\right\}, \qquad (1)$$

where $\beta = dd^c\|z\|^2$ the standard volume form in $\mathbb{C}^n$.

Operators $(dd^c u)^k \wedge \beta^{n-k}$ closely related to the Hessians. For a twice smooth function, $u \in C^2(D)$ the second-order differential $dd^c u = \frac{i}{2}\sum\limits_{j,k} \frac{\partial^2 u}{\partial z_j \partial \bar{z}_k} dz_j \wedge d\bar{z}_k$ (at the fixed point $o \in D$) is a Hermitian quadratic form. After approaching unitary transformation of coordinate, it is reduced to diagonal form $dd^c u = \frac{i}{2}\left[\lambda_1 dz_1 \wedge d\,\bar{z}_1 + \ldots + \lambda_n dz_n \wedge d\,\bar{z}_n\right]$, where $\lambda_1, \ldots, \lambda_n$ the eigenvalues of the Hermitian matrix $\left(\frac{\partial^2 u}{\partial z_j \partial\,\bar{z}_k}\right)$, which are real: $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$. Note that the unitary transformation does not change the differential form $\beta = dd^c\|z\|^2$. It is easy to see that

$$(dd^c u)^k \wedge \beta^{n-k} = k!(n-k)!H^k(u)\beta^n, \qquad (2)$$

where $H^k(u) = \sum\limits_{1 \leqslant j_1 < \cdots < j_k \leqslant n} \lambda_{j_1} \ldots \lambda_{j_k}$ is the Hessian of dimension k of the vector $\lambda = \lambda(u) \in \mathbb{R}^n$.

Consequently, a twice smooth function $u(z) \in C^2(D)$, $D \subset \mathbb{C}^n$, is strongly m-subharmonic if at each point $o \in D$ the next inequalities hold

$$H^k(u) = H_o^k(u) \geqslant 0, \quad k = 1, 2, \ldots, n - m + 1. \qquad (3)$$

Note that the concept of a strongly m-subharmonic function is defined, in general, in the distribution sense

Definition 1. *A function $u \in L^1_{loc}(D)$ is called sh_m in the domain $D \subset \mathbb{C}^n$, if it is upper semicontinuous and for any twice smooth sh_m functions $v_1, \ldots, v_{n-m}$ the current $dd^c u \wedge dd^c v_1 \wedge \cdots \wedge dd^c v_{n-m} \wedge \beta^{m-1}$ defined as*

$$\left[dd^c u \wedge dd^c v_1 \wedge \ldots \wedge dd^c v_{n-m} \wedge \beta^{m-1}\right](\omega) =$$

$$= \int u\, dd^c v_1 \wedge \cdots \wedge dd^c v_{n-m} \wedge \beta^{m-1} \wedge dd^c \omega, \qquad \omega \in F^{0,0} \qquad (4)$$

is positive.

In the work of Błocki [6], it was proven that this definition is correct, that for functions $u \in C^2(D)$ this definition coincides with the original definition of sh_m functions. Moreover, the class of bounded sh_m functions define the operators $(dd^c u)^k \wedge \beta^{n-k} \geqslant 0$, $k = 1, 2, \ldots, n - m + 1$ as Borel measures in the domain D (see [6, 11]).

Now let $D \subset \mathbb{R}^n$ and $u(x) \in C^2(D)$. Similar to (2) we want to define m-convex functions in the domain $D \subset \mathbb{R}^n$. The matrix $\left(\frac{\partial^2 u}{\partial x_j \partial\, x_k}\right)$ is orthogonal, $\frac{\partial^2 u}{\partial x_j \partial\, x_k} = \frac{\partial^2 u}{\partial x_k \partial\, x_j}$. Therefore, after a suitable orthonormal transformation, it is transformed into a diagonal form

$$\left(\frac{\partial^2 u}{\partial x_j \partial\, x_k}\right) \to \begin{pmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & \lambda_n \end{pmatrix},$$

where $\lambda_j = \lambda_j(x) \in \mathbb{R}$ the eigenvalues of the matrix $\left(\dfrac{\partial^2 u}{\partial x_j \partial x_k}\right)$. Let $H^k(u) = H^k(\lambda) = \sum_{1 \leqslant j_1 < \dots < j_k \leqslant n} \lambda_{j_1} \dots \lambda_{j_k}$ the Hessian of the dimension k of the eigenvalue vector $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$.

Definition 2. *A twice smooth function $u \in C^2(D)$ is called m-convex in $D \subset \mathbb{R}^n$, $u \in m-cv(D)$, if its eigenvalue vector $\lambda = \lambda(x) = (\lambda_1(x), \lambda_2(x), \dots, \lambda_n(x))$ satisfies the conditions*

$$m - cv \cap C^2(D) = \left\{ H^k(u) = H^k(\lambda(x)) \geqslant 0,\ \forall x \in D,\ \ k = 1, \dots, n-m+1 \right\}.$$

Theory $m-cv$ functions is a poorly-studied and new direction in the theory of real geometry. However, when $m = n$ the class $n - cv \cap C^2(D) = \{\lambda_1 + \dots + \lambda_n \geqslant 0\}$ coincides with the class of subharmonic functions, and when $m = 1$ this class $1 - cv \cap C^2(D) = \left\{H^1(\lambda) \geqslant 0\right\} = \{\lambda_1 \geqslant 0, \dots, \lambda_n \geqslant 0\}$ coincides with functions that are convex functions in $\mathbb{R}^n$. The class of convex functions is well studied (A. Alexandrov, I. Bakelman, A. Pogorelov, see [1–5]). This $m > 1$ class was studied in a series of works by N. Ivochkina, N. Trudinger, H. Wang, etc. (see. [16–22]).

Principal difficulties in the theory of $m-cv$ functions are the introduction of class $m-cv \cap L^1_{loc}$, i.e. definition $m - cv(D)$ functions in the class of upper semicontinuous, locally integrable or bounded functions. So, for $m = n$ (the case of subharmonic functions) in the class of upper semicontinuous, locally integrable functions $u(x) \in n - cv(D)$ is defined as a distribution, where the Laplace operator Δu is a Borel measure.

The key point to study $m - cv \cap L^1_{loc}$ functions is the following relationship $m - cv$ and sh_m functions (see. [14]). We embed $\mathbb{R}^n_x$ into $\mathbb{C}^n_z$, by $\mathbb{R}^n_x \subset \mathbb{C}^n_z = \mathbb{R}^n_x + i\mathbb{R}^n_y (z = x + iy)$, as a real n-dimensional subspace of the complex space $\mathbb{C}^n_z$.

Theorem 1. *A twice smooth function $u(x) \in C^2(D)$, $D \subset \mathbb{R}^n_x$, is $m - cv$ in D if and only if a function $u^c(z) = u^c(x + iy) = u(x)$ that does not depend on variables $y \in \mathbb{R}^n_y$, is sh_m in the domain $D \times \mathbb{R}^n_y$.*

Definition 3. *An upper semicontinuous function $u(x)$ in a domain $D \subset \mathbb{R}^n_x$ is called m-convex D, if the function $u^c(z)$ is strongly m-subharmonic, $u^c(z) \in sh_m\left(D \times \mathbb{R}^n_y\right)$.*

If a function $u(x)$ is locally bounded and m-convex in the domain $D \subset \mathbb{R}^n_x$, then $u^c(z)$ will be also locally bounded, strongly m-subharmonic function in the domain $D \times \mathbb{R}^n_y \subset \mathbb{C}^n_z$, $u^c(z) \in sh_m \cap L^\infty_{loc}\left(D \times \mathbb{R}^n_y\right)$. Therefore, the operators are correctly defined

$$(dd^c u^c)^k \wedge \beta^{n-k},\ \ k = 1, 2, \dots, n-m+1$$

as Borel measures in the domain $D \times \mathbb{R}^n_y \subset \mathbb{C}^n_z$, $\mu_k = (dd^c u^c)^k \wedge \beta^{n-k}$.

Since for a twice smooth function $(dd^c u^c)^k \wedge \beta^{n-k} = k!(n-k)! H^k(u^c)\beta^n$, then for a locally bounded, strongly m-subharmonic function in the domain, $D \times \mathbb{R}^n_y \subset \mathbb{C}^n_z$ it is natural to define its Hessians, equating them to the measure

$$H^k(u^c) = \frac{\mu_k}{k!(n-k)!} = \frac{1}{k!(n-k)!}(dd^c u^c)^k \wedge \beta^{n-k}. \tag{5}$$

By using (5) we can now define Hessians H^k, $k = 1, 2, \dots, n-m+1$, in the class of locally bounded, m-convex functions in the domain $D \subset \mathbb{R}^n_x$. Let $u(x)$ be locally bounded, m-convex function in the domain $D \subset \mathbb{R}^n_x$. Let us define Borel measures in the domain $D \times \mathbb{R}^n_y \subset \mathbb{C}^n_z$

$$\mu_k = (dd^c u^c)^k \wedge \beta^{n-k},\ \ k = 1, 2, \dots, n-m+1.$$

Since $u^c \in sh_m\left(D \times \mathbb{R}_y^n\right)$ does not depend on $y \in \mathbb{R}_y^n$, then for any Borel sets, $E_x \subset D$, $E_y \subset \mathbb{R}_y^n$ the measures $\frac{1}{mesE_y}\mu_k\left(E_x \times E_y\right)$ does not depend on the set $E_y \subset \mathbb{R}_y^n$, i.e. $\frac{1}{mesE_y}\mu_k\left(E_x \times E_y\right) = \nu_k\left(E_x\right)$. Borel measures

$$\nu_k: \ \nu_k\left(E_x\right) = \frac{1}{mesE_y}\mu_k\left(E_x \times E_y\right), \ \ k = 1, 2, \ldots, n-m+1, \tag{6}$$

we call by Hessians $H^k = H^k\left(E_x\right)$, $k = 1, 2, \ldots, n-m+1$, for a locally bounded, m-convex $u(x) \in m - cv(D)$ function in the domain $D \subset \mathbb{R}_x^n$. For a twice smooth function, $u(x) \in m - cv(D) \cap C^2(D)$ the Hessians are ordinary functions, however, for a non-twice smooth, but bounded upper semicontinuous function, $u(x) \in m - cv(D) \cap L^\infty(D)$, the Hessians H^k, $k = 1, 2, \ldots, n-m+1$, are positive Borel measures (see [13, 15]).

2. Maximal functions and the Dirichlet problem

Similar to the Monge-Ampere operator $(dd^c u)^{n-m+1} \wedge \beta^{m-1}$ in the class sh_m of functions, the Hessian measures H_u^{n-m+1} in the class $m - cv(D)$ also has the property of dominance: the function, with smaller its total mass, is closer to the maximal.

Theorem 2 (Comparison principle). *If $u, v \in m - cv(D) \cap C(D)$ and a set $F = \{x \in D : u(x) < v(x)\} \subset\subset D$, then*

$$H_u^{n-m+1}(F) \geqslant H_v^{n-m+1}(F). \tag{7}$$

Proof. The proof of the theorem is carried out in several stages.

1) If $D \subset \mathbb{R}^n$ a bounded domain with a smooth boundary ∂D and $u, v \in m - cv(D) \cap C^2(\bar{D})$: $u|_D < v|_D$, $u|_{\partial D} \equiv v|_{\partial D}$, then $H_u^{n-m+1}(D) \geqslant H_v^{n-m+1}(D)$.

Actually, let us put $\mathbb{R}_x^n$ in $\mathbb{C}_z^n$, $\mathbb{R}_x^n \subset \mathbb{C}_z^n = \mathbb{R}_x^n + i\mathbb{R}_y^n$ $(z = x + iy)$, and construct the functions $u^c(z) = u(x) \in sh_m(D \times \mathbb{R}_y^n)$, $v^c(z) = v(x) \in sh_m(D \times \mathbb{R}_y^n)$. We take the cylinder $\Omega = \left\{(x, y) \in D \times \mathbb{R}_y^n : x \in D, \|y\| < 1\right\}$. The boundary of the cylinder is $\partial\Omega = S_1 \cup S_2$, where $S_1 = D \times \{\|y\| = 1\}$, $S_2 = \partial D \times \{\|y\| < 1\}$.

According to the Stokes formula we have

$$\int \Omega \left[(dd^c u^c)^{n-m+1} \wedge \beta^{m-1} - (dd^c v^c)^{n-m+1} \wedge \beta^{m-1}\right] =$$

$$= \int_\Omega \left[(dd^c u^c) - (dd^c v^c)\right] \wedge$$

$$\left[(dd^c u^c) \wedge (dd^c v^c)^{n-m} + (dd^c u^c)^2 \wedge (dd^c v^c)^{n-m-1} + \cdots + (dd^c u^c)^{n-m} \wedge (dd^c v^c)\right] \wedge \beta^{m-1} =$$

$$= \int_{\partial\Omega} \left[(d^c u^c) - (d^c v^c)\right] \wedge$$

$$\left[(dd^c u^c) \wedge (dd^c v^c)^{n-m} + (dd^c u^c)^2 \wedge (dd^c v^c)^{n-m-1} + \cdots + (dd^c u^c)^{n-m} \wedge (dd^c v^c)\right] \wedge \beta^{m-1}.$$

Note that the differential form

$$\left[(dd^c u^c) \wedge (dd^c v^c)^{n-m} + (dd^c u^c)^2 \wedge (dd^c v^c)^{n-m-1} + \cdots + (dd^c u^c)^{n-m} \wedge (dd^c v^c)\right]$$

is positive and $[(d^c u^c) - (d^c v^c)] = d^c(u^c - v^c)$ represents the derivative by the internal normal $[(d^c u^c) - (d^c v^c)] = d^c(u^c - v^c) \sim \dfrac{\partial(u^c - v^c)}{\partial n}$. Since the function $u^c - v^c$ does not depend on y, $\left.\dfrac{\partial(u^c - v^c)}{\partial n}\right|_{\|y\|=1} = 0$. Therefore,

$$\int_{S_1} [(d^c u^c) - (d^c v^c)] \wedge$$

$$\Big[(dd^c u^c) \wedge (dd^c v^c)^{n-m} + (dd^c u^c)^2 \wedge (dd^c v^c)^{n-m-1} + \cdots + (dd^c u^c)^{n-m} \wedge (dd^c v^c)\Big] \wedge \beta^{m-1} = 0.$$

For the integral over S_2

$$\int_{S_2} [(d^c u^c) - (d^c v^c)] \wedge$$

$$\Big[(dd^c u^c) \wedge (dd^c v^c)^{n-m} + (dd^c u^c)^2 \wedge (dd^c v^c)^{n-m-1} + \cdots + (dd^c u^c)^{n-m} \wedge (dd^c v^c)\Big] \wedge \beta^{m-1} \geqslant 0,$$

since $u^c - v^c < 0$ inside D and $(u^c - v^c)|_{\partial D} = 0$. Therefore, $d^c(u^c - v^c)$ is positive on S_2.

That's why,

$$\int_{\Omega} \Big[(dd^c u^c)^{n-m+1} \wedge \beta^{m-1} - (dd^c v^c)^{n-m+1} \wedge \beta^{m-1}\Big] =$$

$$= \int\limits_{D\times\{\|y\|\leqslant 1\}} \Big[(dd^c u^c)^{n-m+1} \wedge \beta^{m-1} - (dd^c v^c)^{n-m+1} \wedge \beta^{m-1}\Big] \geqslant 0.$$

From here,

$$\int\limits_{D\times\{\|y\|\leqslant 1\}} (dd^c u^c)^{n-m+1} \wedge \beta^{m-1} \geqslant \int\limits_{D\times\{\|y\|\leqslant 1\}} (dd^c v^c)^{n-m+1} \wedge \beta^{m-1}$$

and according to (6) $H_u^{n-m+1}(D) \geqslant H_v^{n-m+1}(D)$.

2) If $u, v \in C^2(D)$ and the open set $F = \{u < v\} \subset\subset D$, then from 1) it easily follows that

$$H_u^{n-m+1}(F) \geqslant H_v^{n-m+1}(F).$$

3) In general: $u, v \in C(D)$. Then set

$$F = \{x \in D : \ u(x) < v(x)\}$$

will be an open set. Fixing domain G, $G' : F \subset\subset G \subset\subset G' \subset\subset D$, number $\delta > 0$ and open set $F_\delta = \{u(x) + \delta < v(x)\} \subset\subset F$. Let's construct sequences of approximations $u_j, v_j \in m{-}cv(G') \cap C^\infty(G')$, $j = 1, 2, \ldots :\ u_j \downarrow u,\ \ v_j \downarrow v$. Due to continuity u, v the convergence $u_j \downarrow u,\ \ v_j \downarrow v$ will be uniform in G and, therefore, $\exists j_0, k_0 :\ F_{3\delta} \subset F' = \{u_k + 2\delta < v_j\} \subset F_\delta,\ j \geqslant j_0,\ k \geqslant k_0$. According to 2) we have

$$H_{u_k}^{n-m+1}(F') \geqslant H_{v_j}^{n-m+1}(F'),\ \ k \geqslant k_0,\ j \geqslant j_0.$$

Hence for such k and j

$$H_{v_j}^{n-m+1}(F_{3\delta}) \leqslant H_{v_j}^{n-m+1}(F') \leqslant H_{u_k}^{n-m+1}(F') \leqslant H_{u_k}^{n-m+1}(\bar{F}_\delta).$$

When, $j \to \infty$, $k \to \infty$ according to the properties of Borel measures, we have

$$H_v^{n-m+1}(F_{3\delta}) \leqslant H_u^{n-m+1}\left(\bar{F}_\delta\right).$$

Tending $\delta \to 0$ from here we get that $H_v^{n-m+1}(\{u < v\}) \leqslant H_u^{n-m+1}\left(\overline{\{u < v\}}\right)$. Applying this inequality to the functions $u + \varepsilon, v$ we have $H_v^{n-m+1}(\{u + \varepsilon < v\}) \leqslant H_u^{n-m+1}\left(\overline{\{u + \varepsilon < v\}}\right)$ and by tending $\varepsilon \to 0$ we obtain the proof of the theorem. □

Definition 4. *A function $u(x) \in m - cv(D)$ is called maximal in the domain $D \subset \mathbb{R}^n$ if for this function the maximum principle holds in the class of $m - cv(D)$, i.e. if $v \in m - cv(D)$: $\varliminf_{x \to \partial D} (u(x) - v(x)) \geqslant 0$, then $u(x) \geqslant v(x)$, $\forall x \in D$.*

Note that the following convenient maximality criterion is often used: a function $u(x) \in m - cv(D)$ is maximal in the domain $D \subset \mathbb{R}^n$ if and only if for any domain $G \subset\subset D$ the inequality $u(x) \geqslant v(x)$, $\forall x \in G$ holds for all functions $v \in m - cv(D)$: $u|_{\partial G} \geqslant v|_{\partial G}$.

Maximal functions are closely related to the Dirichlet problem.

Theorem 3. *Let $D = \{\rho(x) < 0\}$ strictly $m - cv$ convex domain in $\mathbb{R}^n$ and $\varphi(\xi)$ a continuous function defined on the boundary ∂D. Let's put*

$$\mathcal{U}(\varphi, D) = \left\{u \in m - cv(D) \cap C(\bar{D}) : \ u|_{\partial D} \leqslant \varphi\right\}$$

and

$$\omega(x) = \sup\left\{u(x) : \ u \in \mathcal{U}(\varphi, D)\right\}. \tag{8}$$

Then, $\omega(x) \in m - cv(D) \cap C(\bar{D})$, $\omega|_{\partial D} = \varphi$ and in addition, $\omega(x)$ is the maximal $m - cv$ function in D.

We remember, a domain $D = \{\rho(x) < 0\}$ is strictly $m - cv$ convex if the function $\rho(x)$ is strictly $m - cv$ in a neighborhood $D^+ \supset \bar{D}$, $\rho(x) \in m - cv(D^+)$, $\rho(x) - \delta|x|^2 \in m - cv(D^+)$ for some $\delta > 0$.

It is natural to call the function $\omega(x)$ as a solution to the Dirichlet problem: $\omega(x)$ maximal and $\omega|_{\partial D} = \varphi$. Moreover, it is easy to see that a function $u \in m - cv(D) \cap C(D)$ is maximal if and only if the function $u^c(z) \in sh_m(D \times \mathbb{R}^n_y) \cap C(D \times \mathbb{R}^n_y)$ is a maximal sh_m function. It follows that $(dd^c u^c)^{n-m+1} \wedge \beta^{m-1} = 0$ or $H^{n-m+1}(u^c) = 0$. This is equivalent to $H^{n-m+1}(u(x)) = 0$.

Proof of Theorem 3. Note that if in (8) instead of a class $m - cv(D)$ we consider a wider class of subharmonic functions $n - cv(D) = sh(D) \supset m - cv(D)$, then we would obtain a solution to the classical Dirichlet problem: $\nu(x) = \sup\left\{u \in sh(D) \cap C(\bar{D}) : \ u|_{\partial D} \leqslant \varphi\right\}$. In this case $\Delta\nu \equiv 0$, $\nu|_{\partial D} \equiv \varphi$. It is clear that $\omega(x) \leqslant \nu(x)$ and

$$\varlimsup_{x \to \xi} \omega(x) \leqslant \varphi(\xi), \quad \forall \xi \in \partial D. \tag{9}$$

On the other hand, any fixed boundary point $\xi^0 \in \partial D$ of a strictly $m-$convex domain $D = \{\rho(x) < 0\}$, $\rho(x)-$strictly $m - cv$ function in some neighborhood $D^+ \supset \bar{D}$, is a peak point: there exists $v \in m - cv(D) \cap C(\bar{D})$: $v(\xi^0) = 0$, $v|_{\bar{D} \setminus \{\xi^0\}} < 0$.

In fact, since $\rho(x)$ strictly $m - cv$ function in a certain neighborhood $D^+ \supset \bar{D}$, then for a sufficiently small positive number $\delta > 0$ the difference $\rho(x) - \delta\left\|x - \xi^0\right\|^2$ is m-convex in D^+. Considering instead $\rho(x)$ function

$$v(x) = \rho(x) - \delta\left\|x - \xi^0\right\|^2 \in m - cv(D) \cap C(\bar{D}) : \ v(\xi^0) = 0, \quad v|_{\bar{D} \setminus \{\xi^0\}} < 0$$

we'll make sure that the point $\xi^0 \in \partial D$ is peak point.

Hence, for any fixed number $\varepsilon > 0$ there is a large number $M > 0$ that $M \cdot v(x) + \varphi(\xi^0) - \varepsilon \in \mathcal{U}(\varphi, D)$. Therefore, $M \cdot v(x) + \varphi(\xi^0) - \varepsilon \leqslant \omega(x)$ and $\varliminf_{x \to \xi^0} \omega(x) \geqslant \varphi(\xi^0) - \varepsilon$. Since the number

$\varepsilon > 0$ and point $\xi^0 \in \partial D$ are arbitrary, then $\varliminf_{x\to\xi} \omega(x) \geqslant \varphi(\xi),\ \ \forall \xi \in \partial D$. Combining this with (9) we get $\lim_{x\to\xi} \omega(x) = \varphi(\xi),\ \ \forall \xi \in \partial D$.

For regularization ω^* which is $m-cv$ function in the domain D condition of continuity on the boundary is also satisfied: $\lim_{x\to\xi} \omega^*(x) = \varphi(\xi),\ \ \forall \xi \in \partial D$. From $\omega^*(x) \in m-cv(D),\ \ \lim_{x\to\partial D} \omega^* = \varphi$ follows that $\omega^*(x) \leqslant \omega(x)$, i.e. $\omega^*(x) \equiv \omega(x)$ and $\omega(x)$ is $m-cv$ function. Let us show that it is maximal.

Assume the contrary, let there be a domain $G \subset\subset D$ and a function $\phi(x) \in m-cv(D)$: $\phi|_{\partial G} \leqslant \omega|_{\partial G}$, but $\phi(x^0) > \omega(x^0)$ at some point x^0.

Function

$$w(x) = \begin{cases} \max\{\omega(x),\, \phi(x)\} & if\ \ x \in \bar{G} \\ \omega \quad if\ \ x \in D\backslash G \end{cases}$$

is m-convex, $w(x) \in m-cv(D),\ \ w|_{\partial D} = \omega|_{\partial D} = \varphi$. Therefore, $w(x) \leqslant \omega(x)$ and $\phi(x^0) \leqslant \omega(x^0)$. This is contradiction.

It remains to prove that the function ω will be continuous in the closure. Let's build an approximation

$$\omega_\delta(x) = \omega \circ K_\delta(x-y) \in m-cv(D_\delta) \cap C^\infty(D_\delta),\ \ D_\delta = \{x \in D:\ \rho(x) < \delta\},$$

$\omega_\delta(x) \downarrow \omega(x)$, as $\delta \downarrow 0$. For small enough $\delta > 0$ each interior normal $n_\xi,\ \xi \in \partial D$ intersects ∂D_δ at a single point $\eta(\xi) \in \partial D_\delta$, so that a homeomorphism n_δ is defined $n_\delta:\ \ \partial D \to \partial D_\delta$. Let us put $\varphi_\delta(\eta) = \varphi(n_\delta(\xi)),\ \eta \in \partial D_\delta,\ \xi \in D$. Since $\lim_{x\to\xi} \omega(x) = \varphi(\xi),\ \ \forall \xi \in \partial D$, then for any fixed $\varepsilon > 0$ there is a $\delta_0 > 0$ such that $|\omega(x) - \varphi_{\delta_0}(x)| < \varepsilon,\ \ \forall x \in \partial D_{\delta_0}$. For a fixed $\delta_0 > 0$ the domain $D_{\delta_0} \subset\subset D$ and the approximation $\omega_\delta(x) \downarrow \omega(x)$, for $\delta \downarrow 0$ covers the domain D_{δ_0}.

Now applying Hartogs' lemma to a compact set ∂D_{δ_0} and a function $\varphi_{\delta_0}(x) \in C(\partial D_{\delta_0})$ we find $0 < \delta' < \delta_0$ such that

$$\omega_\delta(x) < \omega_{\delta_0}(x) + 3\varepsilon,\ \ \forall x \in \partial D_{\delta_0},\ \ \delta < \delta'. \tag{10}$$

Since the solution to the Dirichlet problem $\omega(x)$ is maximal in D, from $\omega_\delta(x) < \varphi_{\delta_0}(x) + 3\varepsilon,\ \forall x \in \partial D_{\delta_0},\ \delta < \delta'$ follows that $\omega_\delta(x) < \omega(x) + 4\varepsilon,\ \forall x \in D_{\delta_0},\ \delta < \delta'$ because $\omega(x) > \varphi_{\delta_0}(x) - 3\varepsilon,\ \ \forall x \in \partial D_{\delta_0}$. From here, $\omega(x) < \omega_\delta(x) < \omega(x) + 4\varepsilon,\ \forall x \in \partial D_{\delta_0},\ \ \delta < \delta'$, i.e. $|\omega_\delta(x) - \omega(x)| < 4\varepsilon,\ \forall x \in D_{\delta_0},\ \ \delta < \delta'(\delta_0)$. Since $\varepsilon > 0$ arbitrary, then the convergence $\omega_\delta(x) \downarrow \omega(x)$ will be uniform inside D and $\omega(x) \in C(D)$, because $\omega_\delta(x) \in C^\infty(D_\delta)$. The theorem is proven □

Theorem 4. *A continuous $m-cv$ function $u(x) \in m-cv(D) \cap C(D)$ is maximal if and only if the Borel measure is $H_u^{n-m+1} = 0$.*

Proof. We proved above the equality $H_u^{n-m+1} = 0$ for the maximal function $u(x) \in m-cv(D) \cap C(D)$. Let now $H_u^{n-m+1} = 0$ and we will show that u maximal. Assume that u is not the maximal. Then for some domain $G \subset\subset D$ there is a function $v \in m-cv(D) : u|_{\partial G} \geqslant v|_{\partial G}$, but $v(x^0) - u(x^0) = \varepsilon > 0$ for some point $x^0 \in G$.

Approximating v by infinitely smooth $m-cv$ functions $v_j \downarrow v$, and then using Hartog's lemma, we find $j_0 \in \mathbb{N}$ such that $v_{j_0}|_{\partial G} < u|_{\partial G} + \frac{\varepsilon}{2}$. Let us compare the function $u(x)$ with the function $v_{j_0}(x) + \delta\,\|x\|^2$, where $\delta = \dfrac{\varepsilon}{3 \cdot \max\left\{\|x\|^2:\ x \in \bar{G}\right\}}$. For such $\delta > 0$ a set

$F = \left\{ u(x) + \frac{\varepsilon}{2} < v_{j_0}(x) + \delta \|x\|^2 \right\}$ is not empty and lies compactly in G. Then according to the comparison principle (Theorem 2)

$$\delta^n \int_F \left(dd^c \|x\|^2 \right)^n \leqslant \int_F \left(dd^c v + \delta dd^c \|x\|^2 \right)^n \leqslant \int_F (dd^c u)^n = 0,$$

which contradicts to $\int_F \left(dd^c \|x\|^2 \right)^n > 0$. The theorem is proven. □

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Максимальные функции и задача Дирихле в классе m-выпуклых функций

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Аннотация. В этой работе мы вводим понятие максимальных m-выпуклых $(m-cv)$ функций и для строго m-выпуклых областей $D \subset \mathbb{R}^n$ решаем Задачу Дирихле с заданной граничной непрерывной функцией. Докажем, что для решения задачи Дирихле в классе $m-cv$ функций его Гессиан $H_\omega^{n-m+1} = 0$ в области D.

Ключевые слова: субгармонические функции, выпуклые функции, m-выпуклые функции, Борелевские меры, Гессианы.