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On Property $M(4)$ of the Graph $K_2^n + O_m$

Le Xuan Hung*
Hanoi University of Natural Resources and Environment
Hanoi, Vietnam

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Abstract. Given a list $L(v)$ for each vertex v, we say that the graph G is L-colorable if there is a proper vertex coloring of G where each vertex v takes its color from $L(v)$. The graph is uniquely k-list colorable if there is a list assignment L such that $|L(v)| = k$ for every vertex v and the graph has exactly one L-coloring with these lists. If a graph G is not uniquely k-list colorable, we also say that G has property $M(k)$. The least integer k such that G has the property $M(k)$ is called the m-number of G, denoted by $m(G)$. In this paper, we characterize uniquely list colorability of the graph $G = K_2^n + O_r$. We shall prove that $m(K_2^2 + O_r) = 4$ if and only if $r \geqslant 9$, $m(K_2^3 + O_r) = 4$ for every $1 \leqslant r \leqslant 5$ and $m(K_2^4 + O_1) = 4$.

Keywords: vertex coloring (coloring), list coloring, uniquely list colorable graph, complete r-partite graph.

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1. Introduction and preliminaries

All graphs considered in this paper are finite undirected graphs without loops or multiple edges. If G is a graph, then $V(G)$ and $E(G)$ (or V and E in short) will denote its vertex-set and its edge-set, respectively. The set of all neighbours of a subset $S \subseteq V(G)$ is denoted by $N_G(S)$ (or $N(S)$ in short). The subgraph of G induced by $W \subseteq V(G)$ is denoted by $G[W]$. The empty and complete graphs of order n are denoted by O_n and K_n, respectively. Unless otherwise indicated, our graph-theoretic terminology will follow [2].

A graph $G = (V, E)$ is called *r-partite graph* if V admits a partition indaguiActato r classes $V = V_1 \cup V_2 \cup \ldots \cup V_r$ such that the subgraphs of G induced by V_i, $i = 1, \ldots, r$, is empty. An r-partite graph in which every two vertices from different partition classes are adjacent is called complete r-partite graph and is denoted by $K_{|V_1|,|V_2|,\ldots,|V_r|}$. The complete r-partite graph $K_{|V_1|,|V_2|,\ldots,|V_r|}$ with $|V_1| = |V_2| = \ldots = |V_r| = s$ is denoted by K_s^r.

Let $G_1 = (V_1, E_1)$, $G_2 = (V_2, E_2)$ be two graphs such that $V_1 \cap V_2 = \emptyset$. Their *union* $G = G_1 \cup G_2$ has, as expected, $V(G) = V_1 \cup V_2$ and $E(G) = E_1 \cup E_2$. Their *join* defined is denoted $G_1 + G_2$ and consists of $G_1 \cup G_2$ and all edges joining V_1 with V_2.

Let $G = (V, E)$ be a graph and λ is a positive integer.

A λ-*coloring* of G is a mapping $f : V(G) \to \{1, 2, \ldots, \lambda\}$ such that $f(u) \neq f(v)$ for any adjacent vertices $u, v \in V(G)$. The smallest positive integer λ such that G has a λ-coloring is called the *chromatic number* of G and is denoted by $\chi(G)$. We say that a graph G is n-*chromatic* if $n = \chi(G)$.

Let $(L(v))_{v \in V}$ be a family of sets. We call a coloring f of G with $f(v) \in L(v)$ for all $v \in V$ is a *list coloring from the lists* $L(v)$. We will refer to such a coloring as an L-coloring. The graph G is called λ-*list-colorable*, or λ-*choosable*, if for every family $(L(v))_{v \in V}$ with $|L(v)| = \lambda$ for all v, there is a coloring of G from the lists $L(v)$. The smallest positive integer λ such that G has a

*lxhung@hunre.edu.vn

λ-choosable is called the *list-chromatic number*, or *choice number* of G and is denoted by $ch(G)$. The idea of list colorings of graphs is due independently to V. G. Vizing [19] and to P. Erdös, A. L. Rubin, and H. Taylor [7].

Let G be a graph with n vertices and suppose that for each vertex v in G, there exists a list of k colors $L(v)$, such that there exists a unique L-coloring for G, then G is called a *uniquely k-list colorable graph* or a UkLC graph for short. If a graph G is not uniquely k-list colorable, we also say that G has property $M(k)$. So G has the property $M(k)$ if and only if for any collection of lists assigned to its vertices, each of size k, either there is no list coloring for G or there exist at least two list colorings. The least integer k such that G has the property $M(k)$ is called the *m-number* of G, denoted by $m(G)$. The idea of uniquely colorable graph was introduced independently by Dinitz and Martin [6] and by Mahmoodian and Mahdian [14].

For example, one can easily see that the graph $K_{1,1,2}$ has the property M(3) and it is U2LC, so $m(K_{1,1,2}) = 3$.

The list coloring model can be used in the channel assignment. The fixed channel allocation scheme leads to low channel utilization across the whole channel. It requires a more effective channel assignment and management policy, which allows unused parts of channel to become available temporarily for other usages so that the scarcity of the channel can be largely mitigated [20]. It is a discrete optimization problem. A model for channel availability observed by the secondary users is introduced in [20]. The research of list coloring consists of two parts: the choosability and the unique list colorability. In [10], we characterized list-chromatic number of the graph $G = K_2^n + O_r$. In [11] and [12], we characterized uniquely list colorability of the graph $G = K_2^n + K_r$. In [13], we characterized uniquely list colorability of complete tripartite graphs. In this paper, we characterize uniquely list colorability of the graph $G = K_2^n + O_r$. We shall prove that $m(K_2^2 + O_r) = 4$ if and only if $r \geqslant 9$, $m(K_2^3 + O_r) = 4$ for every $1 \leqslant r \leqslant 5$ and $m(K_2^4 + O_1) = 4$.

2. Preliminaries

We need the following Lemmas 1–20 to prove our results.

Lemma 1 ([14])**.** *Each UkLC graph is also a U$(k-1)$LC graph.*

Lemma 2 ([14])**.** *The graph G is UkLC if and only if $k < m(G)$.*

Lemma 3 ([14])**.** *A connected graph G has the property $M(2)$ if and only if every block of G is either a cycle, a complete graph, or a complete bipartite graph.*

Lemma 4 ([14])**.** *For every graph G we have $m(G) \leqslant |E(\overline{G})| + 2$.*

Lemma 5 ([14])**.** *Every UkLC graph has at least $3k - 2$ vertices.*

If $n = 1$ then $G = K_2^n + O_r$ is a complete bipartite graph, by Lemma 3, G has the property $M(2)$.

Lemma 6. *With $G = K_2^n + O_r$, we have $m(G) \leqslant \dfrac{r^2 - r + 2n + 4}{2}$.*

Proof. It is clear that $|E(\overline{G})| = \dfrac{r^2 - r + 2n}{2}$. By Lemma 4,

$$m(G) \leqslant \frac{r^2 - r + 2n + 4}{2}.$$

□

Lemma 7. *If $G = K_2^n + O_r$ is UkLC then $k \leqslant 2n + 1$.*

Proof Suppose that G is UkLC. Then there exists a list of k colors $L(v)$, such that there exists a unique L-coloring f for G. Let $V(G) = V_1 \cup V_2 \cup \ldots \cup V_{n+1}$ is a partition of $V(G)$ such that $|V_1| = |V_2| = \ldots = |V_n| = 2$, $|V_{n+1}| = r$ and for every $i = 1, 2, \ldots, n+1$ the subgraphs of G induced by V_i, is empty graph. Set $V_{n+1} = \{v_1, v_2, \ldots, v_r\}$.

It is clear that $|f(V_1 \cup V_2 \cup \ldots \cup V_n)| \leqslant 2n$ and $|L(v_1) \setminus \{f(v_1)\}| = k - 1$. If $k > 2n+1$ then $|L(v_1) \setminus \{f(v_1)\}| > 2n$. So there exists $c \in L(v_1) \setminus \{f(v_1)\}$ such that $c \notin f(V_1 \cup V_2 \cup \ldots \cup V_n)$. It follows that there exists a unique L-coloring f' for G: $f(v) = f'(v)$ if $v \in V(G) \setminus \{v_1\}$ and $f(v_1) = c$. It is not difficult to see that $f \neq f'$, a contradiction. Thus, $k \leqslant 2n+1$. □

Lemma 8. *$m(K_2^2 + O_r) = 3$ for every $1 \leqslant r \leqslant 2$.*

Proof. By Lemma 3, $G = K_2^2 + O_r$ is U2LC. Suppose that G is U3LC. By Lemma 5, $|V(G)| \geqslant 7$, a contradiction. So $m(G) = 3$. □

Lemma 9 ([14]). *$m(K_2^2 + O_3) = 3$.*

Lemma 10 ([23]). *$m(K_2^2 + O_r) = 3$ for every $4 \leqslant r \leqslant 8$.*

Lemma 11. *$m(K_2^2 + O_r) = 3$ for every $1 \leqslant r \leqslant 8$.*

Proof. It follows from Lemma 8, Lemma 9 and Lemma 10. □

Lemma 12 ([23]). *$G = K_2^2 + O_9$ is U3LC.*

The join of O_r and K_n, $O_r + K_n = S(r, n)$, is called a complete split graph. It is clear that $S(1, n)$ is a complete graph for every $n \geqslant 1$, by Lemma 3, $m(S(1, n)) = 2$ for every $n \geqslant 1$.

Lemma 13. *(i) $m(S(1, n)) = 2$ for every $n \geqslant 1$;*
(ii) $m(S(r, 1)) = 2$ for every $r \geqslant 1$;
(iii) $m(S(2, n)) = 3$ for every $n \geqslant 2$.

Proof. (i) It is clear that $S(1, n)$ is a complete graph for every $n \geqslant 1$, by Lemma 3, $m(S(1, n)) = 2$ for every $n \geqslant 1$.

(ii) It is clear that $S(r, 1)$ is a complete bipartite graph for every $r \geqslant 1$, by Lemma 3, $m(S(r, 1)) = 2$ for every $r \geqslant 1$.

(iii) By Lemma 3, $G = S(2, n)$ is U2LC for every $n \geqslant 2$.

It is not difficult to see that $|E(\overline{G})| = 1$. By Lemma 4, $m(S(2, n)) \leqslant 3$ for every $n \geqslant 2$.

Thus, $m(S(2, n)) = 3$ for every $n \geqslant 2$. □

Lemma 14 ([8]). *$m(S(3, n)) = 3$ for every $n \geqslant 2$;*

Lemma 15 ([8]). *For every $r \geqslant 2$, $m(S(r, 3)) = 3$.*

Lemma 16 ([9]). *The graph $S(4, 4)$ has the property $M(3)$.*

Lemma 17 ([16]). *The graph $S(4, 5)$ has the property $M(3)$.*

Lemma 18. *(i) $G = S(4, n)$ has the property $M(4)$ for every $n \geqslant 2$;*
(ii) $S(4, n)$ is U3LC for every $n \geqslant 6$;
(iii) $m(S(4, n)) = 4$ for every $n \geqslant 6$.

Proof. Let $G = S(4, n)$ is a complete split graph with $V(G) = I \cup K$, $G[I] = O_4$, $G[K] = K_n$, $n \geqslant 2$. Set

$$I = \{u_1, u_2, u_3, u_4\},\ K = \{v_1, v_2, \ldots, v_n\}.$$

(i) For suppose on the contrary that graph $G = S(4, n)$ is U4LC. So there exists a list of 4 colors $L(v)$ for each vertex $v \in V(G)$, such that there exists a unique L-coloring f for G. We consider separately four cases.

Case 1: $|f(I)| = 1$.

In this case, let $f(u_i) = a$ for every $i = 1, 2, 3, 4$. Set graph $H = G - I$, it is not difficult to see that H is a complete graph K_n. We assign the following lists $L'(v)$ for the vertices v of H:

(a) If $a \in L(v)$ then $L'(v) = L(v) \setminus \{a\}$.

(b) If $a \notin L(v)$ then $L'(v) = L(v) \setminus \{b\}$, where $b \in L(v)$ and $b \neq f(v)$.

It is clear that $|L'(v)| = 3$ for every $v \in V(H)$. By Lemma 3, H has the property $M(2)$, so H has the property $M(3)$. It follows that with lists $L'(v)$, there exist at least two list colorings for the vertices v of H. So it is not difficult to see that with lists $L(v)$, there exist at least two list colorings for the vertices v of G, a contradiction.

Case 2: $|f(I)| = 2$.

In this case, let $f(I) = \{a, b\}$. Set graph $H = G - I$, it is not difficult to see that H is a complete graph K_n. We assign the following lists $L'(v)$ for the vertices v of H:

(a) If $a, b \in L(v)$ then $L'(v) = L(v) \setminus \{a, b\}$.

(b) If $a \in L(v), b \notin L(v)$ then $L'(v) = L(v) \setminus \{a, c\}$, where $c \in L(v)$ and $c \neq f(v)$.

(c) If $a \notin L(v), b \in L(v)$ then $L'(v) = L(v) \setminus \{b, c\}$, where $c \in L(v)$ and $c \neq f(v)$.

(d) If $a, b \notin L(v)$ then $L'(v) = L(v) \setminus \{c, d\}$, where $c, d \in L(v)$, $c \neq d$ and $c, d \neq f(v)$.

It is clear that $|L'(v)| = 2$ for every $v \in V(H)$. By Lemma 3, H has the property $M(2)$. It follows that with lists $L'(v)$, there exist at least two list colorings for the vertices v of H. So it is not difficult to see that with lists $L(v)$, there exist at least two list colorings for the vertices v of G, a contradiction.

Case 3: $|f(I)| = 3$.

In this case, let $f(I) = \{a, b, c\}$. Without loss of generality, we may assume that $f(u_1) = f(u_2) = a, f(u_3) = b, f(u_4) = c$. Set graph $G' = (V', E')$, with

$$V' = I \cup K, \quad E' = (E(G) \cup \{u_1u_3, u_1u_4, u_2u_3, u_2u_4\}).$$

It is clear that G' is complete split graph $S(2, n+2)$ with $V(G') = I' \cup K'$, where

$$I' = \{u_1, u_2\}, \quad K' = \{u_3, u_4, v_1, v_2, \ldots, v_n\}$$

By (iii) of Lemma 13, with lists $L(v)$, there exist at least two list colorings for the vertices v of G'. So it is not difficult to see that with lists $L(v)$, there exist at least two list colorings for the vertices v of G, a contradiction.

Case 4: $|f(I)| = 4$.

In this case, $f(u_i) \neq f(u_j)$ for every $i, j \in \{1, 2, 3, 4\}$, $i \neq j$. Set graph $G'' = (V'', E'')$, with

$$V'' = I \cup K, \quad E'' = E(G) \cup \{u_iu_j | i, j = 1, 2, 3, 4; i \neq j\}.$$

It is clear that G'' is a complete graph K_{n+4}. By Lemma 3, G'' has the property $M(2)$, so with lists L_v, there exist at least two list colorings for the vertices v of G''. Since $V(G) = V(G'')$, it is not difficult to see that with lists L_v, there exist at least two list colorings for the vertices v of G, a contradiction.

(ii) We assign the following lists for the vertices of G:

$L(u_1) = \{1, 3, 4\}$, $L(u_2) = \{1, 7, 8\}$, $L(u_3) = \{2, 5, 6\}$, $L(u_4) = \{2, 7, 8\}$;

$L(v_1) = \{1, 2, 3\}$, $L(v_2) = \{1, 2, 4\}$, $L(v_3) = \{1, 2, 5\}$, $L(v_4) = \{1, 2, 6\}$, $L(v_5) = \{1, 2, 7\}$, $L(v_6) = L(v_7) = \ldots = L(v_n) = \{1, 2, 8\}$.

A unique coloring f of G exists from the assigned lists:

$f(u_1) = 1, f(u_2) = 1, f(u_3) = 2, f(u_4) = 2$;

$f(v_1) = 3, f(v_2) = 4, f(v_3) = 5, f(v_4) = 6, f(v_5) = 7, f(v_6) = f(v_7) = \ldots = f(v_n) = 8$.

(iii) It follows from (i) and (ii). □

Lemma 19 ([21]). *(i) For every $n \geqslant 2$, $S(5, n)$ has the property $M(4)$;*

(ii) If $n \geqslant 5$ then $m(S(5, n)) = 4$.

Lemma 20. *$m(S(r,n)) \leqslant 4$ for every $1 \leqslant r \leqslant 5$ and $n \geqslant 6$.*

Proof It follows from Lemma 13 to Lemma 19. □

3. On property $M(4)$ of the graph $K_2^n + O_m$

Set the graph $G = K_2^n + O_r$. Let $V(G) = V_1 \cup V_2 \cup \ldots \cup V_{n+1}$ is a partition of $V(G)$ such that $|V_1| = |V_2| = \ldots = |V_n| = 2$, $|V_{n+1}| = r$ and for every $i = 1, 2, \ldots, n+1$ the subgraphs of G induced by V_i, is empty graph. Set $V_i = \{u_{i1}, u_{i2}\}$ for every $i = 1, 2, \ldots, n$ and $V_{n+1} = \{v_1, v_2, \ldots, v_r\}$.

Theorem 21. *$m(K_2^n + O_r) \leqslant m(K_2^{n-1} + O_r) + 2$ for every $r \geqslant 1, n \geqslant 2$.*

Proof. Put $m(K_2^{n-1} + O_r) = t$. For suppose on the contrary that graph $G = K_2^n + O_r$ satisfies $m(G) = k > t + 2$. So there exists a list of $k-1$ colors $L(v)$ for each vertex $v \in V(G)$, such that there exists a unique L-coloring f for G. Set graph $H = G - V_1$, it is not difficult to see that H is a graph $K_2^{n-1} + O_r$. We consider separately two cases.

Case 1: $|f(V_1)| = 1$.

In this case, $f(u_{11}) = f(u_{12}) = a$. We assign the following lists $L'(v)$ for the vertices v of H:

(a) If $a \in L(v)$ then $L'(v) = L(v) \setminus \{a\}$.

(b) If $a \notin L(v)$ then $L'(v) = L(v) \setminus \{b\}$, where $b \in L(v)$ and $b \neq f(v)$.

It is clear that $|L'(v)| = k - 2 \geqslant t + 1$ for every $v \in V(H)$. Since H has the property $M(t)$, by Lemma 1, H has the property $M(t+1)$, so H has the property $M(k-2)$. It follows that with lists $L'(v)$, there exist at least two list colorings for the vertices v of H. So it is not difficult to see that with lists $L(v)$, there exist at least two list colorings for the vertices v of G, a contradiction.

Case 2: $|f(V_1)| = 2$.

In this case, $f(u_{11}) = a, f(u_{12}) = b, a \neq b$. We assign the following lists $L'(v)$ for the vertices v of H:

(a) If $a, b \in L(v)$ then $L'(v) = L(v) \setminus \{a, b\}$.

(b) If $a \in L(v), b \notin L(v)$ then $L'(v) = L(v) \setminus \{a, c\}$, where $c \in L(v)$ and $c \neq f(v)$.

(c) If $a \notin L(v), b \in L(v)$ then $L'(v) = L(v) \setminus \{b, c\}$, where $c \in L(v)$ and $c \neq f(v)$,

(d) If $a, b \notin L(v)$ then $L'(v) = L(v) \setminus \{c, d\}$, where $c, d \in L(v)$, $c \neq d$ and $c, d \neq f(v)$.

It is clear that $|L'(v)| = k - 3 \geqslant t$ for every $v \in V(H)$. Since H has the property $M(t)$, by Lemma 1, H has the property $M(k-3)$. It follows that with lists $L'(v)$, there exist at least two list colorings for the vertices v of H. So it is not difficult to see that with lists $L(v)$, there exist at least two list colorings for the vertices v of G, a contradiction.

Thus, $m(K_2^n + O_r) \leqslant m(K_2^{n-1} + O_r) + 2$ for every $r \geqslant 1, n \geqslant 2$. □

Corollary 22. *The graph $G = K_2^2 + O_r$ has the property M(4) for every $r \geqslant 1$.*

Proof. It is clear that $K_2^1 + O_r$ is a complete bipartite graph $K_{2,r}$. By Lemma 3, $K_2^1 + O_r$ has the property $M(2)$. By Theorem 13, $G = K_2^2 + O_r$ has the property $M(4)$ for every $r \geqslant 1$. □

Theorem 23. *(i) $G = K_2^2 + O_r$ is U3LC if and only if $r \geqslant 9$;*

(ii) $m(K_2^2 + O_r) = 4$ if and only if $r \geqslant 9$.

Proof. (i) Firrst we prove the necessity. Suppose that $G = K_2^2 + O_r$ is U3LC. If $1 \leqslant r \leqslant 8$ then by Lemma 11, $m(G) = 3$, a contradiction. Therefore, $r \geqslant 9$.

Now we prove the sufficiency. We assign the following lists for the vertices of G: $L(u_{11}) = \{1, 2, 6\}$, $L(u_{12}) = \{3, 4, 5\}$; $L(u_{21}) = \{1, 3, 6\}$, $L(u_{22}) = \{2, 4, 6\}$;

$L(v_1) = \{1, 4, 5\}$, $L(v_2) = \{1, 3, 6\}$, $L(v_3) = \{1, 4, 6\}$, $L(v_4) = \{1, 5, 6\}$, $L(v_5) = \{2, 3, 4\}$, $L(v_6) = \{2, 3, 5\}$, $L(v_7) = \{2, 3, 6\}$, $L(v_8) = \{2, 4, 6\}$, $L(v_9) = L(v_{10}) = \ldots = L(v_r) = \{2, 5, 6\}$.

A unique coloring f of G exists from the assigned lists: $f(u_{11}) = 6$, $f(u_{12}) = 5$; $f(u_{21}) = 3$, $f(u_{22}) = 4$;

$f(v_1) = f(v_2) = f(v_3) = f(v_4) = 1$, $f(v_5) = f(v_6) = \ldots = f(v_r) = 2$.

(ii) It follows from (i) and Corollary 22. □

Theorem 24. *(i) $G = K_2^3 + O_r$ is U3LC for every $r \geqslant 1$;*
(ii) $m(K_2^3 + O_r) = 4$ for every $1 \leqslant r \leqslant 5$.

Proof. (i) We assign the following lists for the vertices of G: $L(u_{11}) = \{1, 4, 5\}$, $L(u_{12}) = \{2, 4, 5\}$, $L(u_{21}) = \{1, 2, 3\}$, $L(u_{22}) = \{3, 4, 5\}$, $L(u_{31}) = \{1, 2, 4\}$, $L(u_{32}) = \{3, 4, 5\}$, $L(v_1) = L(v_2) = \ldots = L(v_r) = \{3, 4, 5\}$.

A unique coloring f of G exists from the assigned lists: $f(u_{11}) = 1$, $f(u_{12}) = 2$, $f(u_{21}) = 3$, $f(u_{22}) = 3$, $f(u_{31}) = 4$, $f(u_{32}) = 4$, $f(v_1) = f(v_2) = \ldots = f(v_r) = 5$.

(ii) By (i), $m(K_2^3 + O_r) \geqslant 4$ for every $r \geqslant 1$. For suppose on the contrary that $m(K_2^3 + O_r) = t \geqslant 5$ for every $1 \leqslant r \leqslant 5$. Then for each vertex v in $G = K_2^3 + O_r$, there exists a list of $t - 1$ colors $L(v)$, such that there exists a unique L-coloring for G. We consider separately two cases.

Case 1: There exists $i \in \{1, 2, 3\}$ such that $|f(V_i)| = 1$.

Without loss of generality, we may assume that $|f(V_1)| = 1$ and $f(u_{11}) = f(u_{12}) = a$. Set graph $H = G - V_1$, it is not difficult to see that H is graph $K_2^2 + O_r$. We assign the following lists $L'(v)$ for the vertices v of H:

(a) If $a \in L(v)$ then $L'(v) = L(v) \setminus \{a\}$.

(b) If $a \notin L(v)$ then $L'(v) = L(v) \setminus \{b\}$, where $b \in L(v)$ and $b \neq f(v)$.

It is clear that $|L'(v)| = t - 2 \geqslant 3$ for every $v \in V(H)$. Since $1 \leqslant r \leqslant 5$, by Lemma 11, $m(H) = 3$. It follows that with lists $L'(v)$, there exist at least two list colorings for the vertices v of H. So it is not difficult to see that with lists $L(v)$, there exist at least two list colorings for the vertices v of G, a contradiction.

Case 2: $|f(V_i)| = 2$ for every $i \in \{1, 2, 3\}$.

Set graph $G' = (V', E')$, with

$$V' = V(G),\ E' = (E(G) \cup \{u_{11}u_{12}, u_{21}u_{22}, u_{31}u_{32}\}).$$

It is clear that G' is complete split graph $S(m, 6)$ with $V(G') = I' \cup K'$, where

$$I' = \{v_1, v_2, \ldots, v_r\},\ K' = \{u_{11}, u_{12}, u_{21}, u_{22}, u_{31}, u_{32}\}.$$

By Lemma 20, $m(G') \leqslant 4$. So with lists $L(v)$, there exist at least two list colorings for the vertices v of G'. So it is not difficult to see that with lists $L(v)$, there exist at least two list colorings for the vertices v of G, a contradiction. □

Theorem 25. *(i) If $n \geqslant 4$ and $r \geqslant 1$ then $G = K_2^n + O_r$ is UkLC with $k = \left\lfloor \frac{n}{2} \right\rfloor + 1$;*
(ii) $m(K_2^4 + O_1) = 4$.

Proof. (i) Put $t = \left\lfloor \frac{n}{2} \right\rfloor$. We assign the following lists for the vertices of G:

$L(u_{i1}) = \{1, 2, \ldots, t + 1\}$ for every $i = 1, 2, \ldots, t + 1$;

$L(u_{i2}) = \{t + 2, t + 3, \ldots, 2t + 1, i\}$ for every $i = 1, 2, \ldots, t + 1$;

$L(u_{(t+i)j}) = \{2, 3, \ldots, t + 1, t + 1 + i\}$ for every $i = 1, 2, \ldots, n - t,\ j = 1, 2$;

$L(v_1) = L(v_2) = \ldots = L(v_r) = \{2, 3, \ldots, t + 1, n + 2\}$.

A unique coloring f of G exists from the assigned lists:

$f(u_{i1}) = i$ for every $i = 1, 2, \ldots, t + 1$;

$f(u_{i2}) = i$ for every $i = 1, 2, \ldots, t + 1$;

$f(u_{(t+i)j}) = t + 1 + i$ for every $i = 1, 2, \ldots, n - t$, $j = 1, 2$;

$f(v_1) = f(v_2) = \ldots = f(v_r) = n + 2$. □

(ii) By (i), G is U3LC. If G is U4LC then by Lemma 5, $|V(G)| \geqslant 10$, a contradiction. So $m(G) = 4$.

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О свойстве $M(4)$ графа $K_2^n + O_m$

Ли Хуан Ханг
Ханойский университет природных ресурсов и окружающей среды
Ханой, Вьетнам

Аннотация. Учитывая список $L(v)$ для каждой вершины v, мы говорим, что граф G является L-раскрашиваемым, если существует правильная раскраска вершин графа G, при которой каждая вершина v принимает свой цвет из $L(v)$. Граф однозначно раскрашивается в k-список, если существует такое задание списка L, что $|L(v)| = k$ для каждой вершины v и граф имеет ровно одну L-раскраску этими списками. Если граф G не является однозначно раскрашиваемым в k-списке, мы также говорим, что G обладает свойством $M(k)$. Наименьшее целое число k такое, что G обладает свойством $M(k)$, называется m-числом G и обозначается $m(G)$. В этой статье мы однозначно характеризуем список раскрашиваемости графа $G = K_2^n + O_r$. Мы докажем, что $m(K_2^2 + O_r) = 4$ тогда и только тогда, когда $r \geqslant 9$, $m(K_2^3 + O_r) = 4$ для каждого $1 \leqslant r \leqslant 5$ и $m(K_2^4 + O_1) = 4$.

Ключевые слова: раскраска вершин (раскраска), раскраска списков, граф, однозначно раскрашиваемый списком, полный r-раздельный граф.