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A Classical Limit for the Dirac Equation in the Context of Magueijo-Smolin Model of the Doubly Special Relativity Using the Ehrenfest's Theorem

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Abstract. In this article, in the context of the Magueijo–Smolin model and employing Ehrenfest's theorem, we investigate the classical limit of the Dirac equation within doubly special relativity. This leads to obtaining deformed classical equations. Here, we assess the effectiveness of Ehrenfest's theorem in deriving the classical limit in the presence of Magueijo–Smolin model. Besides, we explore the deformed classical equations under the discrete, $\mathcal{CPT}$ and Lorentz symmetries.

Keywords: Dirac equation, doubly special relativity, Magueijo-Smolin model, Ehrenfest's theorem, classical limit.

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Introduction

The Dirac equation is a relativistic quantum mechanical equation that specifically describes massive particles with spin-1/2, such as electrons. It is a fundamental equation in quantum mechanics, providing a framework for understanding the behavior of these particles within the realm of relativistic effects. The classical limit (CL) of the Dirac equation can be investigated by neglecting the influences of quantum mechanics. In doing so, we can describe the system's behavior using classical physics, providing insights into the classical aspects of the system. In the CL, phenomena inherent to quantum mechanics, such as interference, superposition and entanglement, are expected to diminish at the macroscopic scale, however, this demise is not easy to explain. In this scenario, the quantum system adheres to the classical laws of physic. The CL is commonly defined in terms of the limit of a vanishing Planck's constant, i.e., $\hbar \to 0$ as scaled with the system's action. In this context, Hamilton's principle adopts its classical expression, and all operators commute. In the following, we present some scenarios and approaches that help explain the exploration of the CL of the Dirac equation. So, one can initiate the exploration by examining the solutions of the equation under conditions of large distances and durations, or under the conditions of large energies and momenta. Within these limits, the effects of quantum mechanics become negligible [1]. Put differently, the CL emerges if the system possesses a big quantum number, undergoes significant interactions with its surroundings, or if its de Broglie wavelength becomes significantly smaller compared to other relevant length measurements. A frequent example illustrating the CL of a quantum system is the Bohr correspondence principle [2], which asserts that in the limit of large quantum numbers, a quantum system exhibits behavior

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similar to the corresponding classical system. Also, the Ehrenfest's theorem is considerably used when exploring the CL of quantum mechanical systems [3]. This theorem establishes a connection between the evolution of expected values of observables and classical equations of motion. It serves as an effective tool for understanding the conduct of such systems. Through its application, we observe the way quantum mechanical influences dissipate, giving way to classical dynamics [4]. In the context of the Dirac equation, this theorem remains used to explore its CL, there, the quantum influences will be very small, leading simplify the Dirac equation to its classical counterpart.

In this work, we aim to investigate whether it can be asserted that Ehrenfest's theorem is applicable to the CL of the Dirac equation within a deformed framework, subject to specific conditions. Extensive research in the literature [5–14] has delved into the alignment between quantum and classical aspects. We also emphasize that other concepts may overlap with the concept of the CL, such as the semiclassical and non-relativistic limits. Note that the semiclassical limit of a quantum mechanical system, can be attained if external potentials vary slowly, like in the case of the electrostatic potential [15]. On the other hand, the non-relativistic limit of a relativistic quantum mechanical system as the Dirac equation [16, 17], is the limit where the speed of the particle is much less than the speed of light, i.e., $v \ll c$ or low energy in front of the rest energy, consequently, this limit permits to neglect the relativistic influences. However, the non-relativistic and classical limits are related but distinct concepts, they address different aspects of the system's behavior. It is important to highlight that in many physical situations, the CL and the non-relativistic limit can align, leading to similar descriptions of the system's behavior.

On the other side, recently, there has been a rising interest in the advancement of doubly special relativity (DSR) theories. This type of special relativity emerges at energy scales close to the domain of quantum gravity, specifically near the Planck energy $\kappa = \sqrt{\frac{\mathrm{c}^5\hbar}{\mathrm{G}}} = 10^{19}\mathrm{GeV}$, there special relativity may undergo deformation. This deformation entails κ transforming into an observer-independent constant, analogous to the speed of light c. Amelino-Camelia [18–20], in conjunction with Magueijo and Smolin [21, 22], advanced the concept of DSR, which requires adopted the parameter κ alongside the speed of light c. This incorporation implies a noncommutative structure in space-time, resulting in the formation of the κ-Minkowski space-time. The second parameter κ is assumed to be of the order of Planck energy $\kappa = E_P$, or in the form of an energy scale $\kappa = 1/l_p$. The models based on this assumption are referred to as DSR. However, as $\kappa \to \infty$, special relativity is regained. Many studies and research in this regard have been carried out in the literature. However, we will not delve into the historical background, as it is thoroughly covered in Amelino-Camelia's recent and comprehensive paper [23]. One of the latest extensions of the DSR is the Deformed General Relativity introduced in [24], which associates the geometric structure of an internal De Sitter space with a noncommutative curved space. It is also worth highlighting the significant role of noncommutative geometry in modern physics today. Its integration with several branches of physics greatly facililates understanding and overcoming many complexities, especially those related to quantum field theory, string theory, cosmology, black holes, and high energy. (For an overview, check Refs. [25–37]).

Our objective in this study is to investigate the CL of the Dirac equation within the framework of the Magueijo–Smolin (MS) approach of the DSR by using the Ehrenfest's theorem. In the same context of the used framework, B. Hamil. et al. [38] have studied relativistic oscillators in the context of MS noncommutative model. Likewise, S. Mignemi and A. Samsarov [39] addressed the vacuum energy from withing noncommutative framework in several models including MS model from DSR. In addition, M. Coraddu and S. Mignemi [40], studied the non-relativistic limit of the motion of a classical particle from Klein-Gordon and Dirac equations in the MS model. Moreover, they found that the rest masses of particles and antiparticles differ and violating the $\mathcal{CPT}$ invariance. They claimed that this effect is close to observational limits and future experiments may give indications on its effective existence, etc. This work came as a continuation of some works on the CL we did before. For instance, in [41], we studied the CL and Ehrenfest's theorem of noncommutative Dirac equation in the context of Minimal Uncertainty in Momentum. Also, we explored the comparison between the CL and the non-relativistic limit of the noncommutative Dirac equation in presence of minimal length. Furthermore, in [42], we have investigated Ehrenfest's theorem from the Dirac equation in a noncommutative phase-space.

The rest of the paper is organized as follows. Section 1 provides a concise review of the MS modeld. In Section 2, the CL of the Dirac equation in the context of MS model using the Ehrenfest's theorem is explored, where in Sub-section A, a κ-deformed Dirac equation in presence of an electromagnetic field is derived. In Sub-section B, based on the Ehrenfest's theorem, κ-deformed classical equations are obtained, subsequently, these obtained classical equations are examined under the discrete, $\mathcal{CPT}$ and Lorentz symmetries in Sub-section C. Section 3 is devoted to the conclusion and remarks.

1. Review of Magueijo–Smolin model

The model we employ belongs to the κ-Poincarй class and is referred to as the MS model [21]. However, there exist alternative models that could be beneficial for our calculations. For example, we mention the Snyder model [43] and the Majid–Ruegg model [44]; the latter model belongs to the κ-Poincarй class as well. We opt to employ the MS approach primarily due to its profound and non-trivial results. The MS model considerations, similar to those outlined in [45–47], indicate that the Euclidean theory can be defined following the prescription $\mathrm{p}_0 \to \mathrm{i}\mathrm{p}_0$, $\kappa \to \mathrm{i}\kappa$. Now, the MS model is defined by the following transformation between noncommutative variables X_μ, P_μ and a canonical momentum variable p_μ [39]:

$$X_\mu = i\left(1+\frac{p_0}{\kappa}\right)\frac{\partial}{\partial p_\mu}, \quad P_\mu = \frac{p_\mu}{\left(1+\frac{p_0}{\kappa}\right)}, \tag{1}$$

with $p_0 = E$, where $-\infty < p_i < +\infty$, and $0 < p_0 < +\infty$. Note that the operators X_μ and P_μ are Hermitian and symmetric, i.e. $\langle\chi| X_\mu |\psi\rangle = \langle\psi| X_\mu |\chi\rangle$, $\langle\chi| P_\mu |\psi\rangle = \langle\psi| P_\mu |\chi\rangle$, with respsect to the scalar product $\langle\chi \mid \psi\rangle = \displaystyle\int \frac{dp}{\left(1+\frac{p_0}{\kappa}\right)}\chi^\times(p)\,\psi(p)$. The MS algebra, derived from equation (1) (Heisenberg relations) in the Granik basis [48], can be expressed as follows:

$$\begin{aligned} [X_i, X_0] &= \frac{i}{\kappa}X_i, \qquad [P_i, X_0] = -\frac{i}{\kappa}P_i, \\ [X_i, P_j] &= i\delta_{ij}, \quad [X_0, P_0] = -i\left(1-\frac{P_0}{\kappa}\right), \end{aligned} \tag{2}$$

where κ (with $\kappa > P_0$) is the Planck momentum, which implies an upper bound for the allowed particle energy in MS model. For this deformed Poincarй algebra, the Casimir operator takes the form of:

$$M^2 = \frac{P_0^2 - P_i^2}{\left(1-\frac{P_0}{\kappa}\right)^2}, \tag{3}$$

where M is the physical mass.

2. Classical limit of the κ-deformed Dirac equation

In this section, we obtain the Dirac equation in the context of MS model and then use it to investigate its CL through the Ehrenfest's theorem. Additionally, the resulting classical equations will be examined under the $\mathcal{CPT}$ symmetry.

A. κ-Deformed Dirac equation

In a commutative phase-space, the time-independent Dirac equation in interaction with an electromagnetic four-potential $A^{\mu}\left(\overrightarrow{A},\Phi\right)$ is

$$\left\{c\overrightarrow{\alpha}\cdot\left(\overrightarrow{p}-e\overrightarrow{A}\left(\overrightarrow{x}\right)\right)+e\Phi\left(\overrightarrow{x}\right)+\beta mc^{2}\right\}\psi\left(\overrightarrow{x}\right)=E\psi\left(\overrightarrow{x}\right), \tag{4}$$

where $\psi\left(\overrightarrow{r}\right)=\begin{pmatrix}\phi(\overrightarrow{x}) & \chi(\overrightarrow{x})\end{pmatrix}^{T}$ is the bispinor in the Dirac representation. The momentum $\overrightarrow{p}$ is given by $\overrightarrow{p}=-i\hbar\overrightarrow{\nabla}$ and α_i and β are the Dirac matrices, which satisfy the following anticommutation relations:

$$\{\alpha_i,\alpha_j\}=2\delta_{ij}\ ,\ \{\alpha_i,\beta\}=0\ ,\ \alpha_i^2=\beta^2=1. \tag{5}$$

Additionally, there is a clarification to make regarding equation (4). In our previous works [17, 41], where we employed $\frac{e}{c}\overrightarrow{A}$ rather than $e\overrightarrow{A}$. Here, suppose to be no $\frac{1}{c}$ factor in SI units; instead, it appears in Gaussian units (old notation).

Now, the Dirac equation resulting from the DSR based on the MS model in the representation of the noncommutative operators X_μ, P_μ is given as

$$\left\{c\overrightarrow{\alpha}\cdot\left(\overrightarrow{P}-e\overrightarrow{A}\left(\overrightarrow{X}\right)\right)+e\Phi\left(\overrightarrow{X}\right)+\beta mc^{2}\right\}\psi\left(\overrightarrow{X}\right)=P_0\psi\left(\overrightarrow{X}\right), \tag{6}$$

then, by applying the definition of the position and momentum operators reported in equation (1), we obtain the following deformed Dirac equation

$$\left\{c\overrightarrow{\alpha}\cdot\left(\frac{\overrightarrow{p}}{(1+\frac{p_0}{\kappa})}-e\tilde{\overrightarrow{A}}\left(\overrightarrow{x}\right)\right)+e\tilde{\Phi}\left(\overrightarrow{x}\right)+\beta mc^{2}\right\}\psi^{MS}=\frac{p_0}{(1+\frac{p_0}{\kappa})}\psi^{MS}. \tag{7}$$

Note that $\tilde{\overrightarrow{A}}\left(\overrightarrow{x}\right)=\overrightarrow{A}\left(\overrightarrow{X}\right)$ and $\tilde{\Phi}\left(\overrightarrow{x}\right)=\Phi\left(\overrightarrow{X}\right)$ with

$$\overrightarrow{X}=i\left(1+\frac{p_0}{\kappa}\right)\frac{\partial}{\partial\overrightarrow{p}}, \tag{8}$$

then in more elegant simple form, we have

$$\left\{c\overrightarrow{\alpha}\cdot\tilde{\overrightarrow{\Pi}}+\left(1+\frac{p_0}{\kappa}\right)e\tilde{\Phi}+\left(1+\frac{p_0}{\kappa}\right)\beta mc^{2}\right\}\psi^{MS}=E\psi^{MS}, \tag{9}$$

where the minimal substitution $\overrightarrow{p}-e\left(1+\frac{p_0}{\kappa}\right)\tilde{\overrightarrow{A}}=\tilde{\overrightarrow{\Pi}}$. Here ψ^{MS} is the wave function in the DSR framework. Next, we move to employ the obtained deformed Dirac equation (9) to explore the CL through Ehrenfest's theorem.

B. Ehrenfest's theorem in the context of MS model

Ehrenfest's theorem, which originates from the Dirac equation, establishes that the time evolution of expected values of observables in quantum mechanics aligns with classical equations of motion. Essentially, it suggests that the average behavior of a quantum system corresponds to classical physics. Additionally, it is noteworthy that this theorem applies to all quantum systems. However, in the present context, we are computing the time derivatives of position and kinetic momentum operators for Dirac particles interacting with an electromagnetic field in the context of MS model of DSR. Consequently, the equation of motion for an arbitrary operator $\hat{\mathcal{F}}$ is expressed as follows:

$$\frac{d\hat{\mathcal{F}}}{dt}=\frac{\partial\hat{\mathcal{F}}}{\partial t}+\frac{i}{\hbar}\left[\hat{\mathscr{H}},\hat{\mathcal{F}}\right], \tag{10}$$

where $\hat{\mathscr{H}}$ is the Hamiltonian operator. Now, let commence with the operator of position

$$\frac{d\hat{\vec{x}}}{dt}=\frac{\partial\hat{\vec{x}}}{\partial t}+\frac{i}{\hbar}\left[\hat{\mathscr{H}}_D^{MS},\hat{\vec{x}}\right]=\frac{i}{\hbar}\left[\hat{\mathscr{H}}_D^{MS},\hat{\vec{x}}\right], \tag{11}$$

and the Hamiltonian operator from equation (9) is given as:

$$\mathscr{H}_D^{MS}=c\vec{\alpha}\cdot\tilde{\vec{\Pi}}+\left(1+\frac{p_0}{\kappa}\right)e\tilde{\Phi}+\left(1+\frac{p_0}{\kappa}\right)\beta mc^2, \tag{12}$$

subsequently, the commutator expressed in equation (11) is as follows:

$$\begin{aligned}\left[\hat{\mathscr{H}}_D^{MS},\hat{\vec{x}}\right]=c\left[\hat{\vec{\alpha}}\cdot\hat{\vec{p}},\hat{\vec{x}}\right]-ec\left(1+\frac{p_0}{\kappa}\right)\left[\hat{\vec{\alpha}}\cdot\tilde{\vec{A}},\hat{\vec{x}}\right]+e\left(1+\frac{p_0}{\kappa}\right)\left[\tilde{\Phi},\hat{\vec{x}}\right]\\+\left(1+\frac{p_0}{\kappa}\right)mc^2\left[\hat{\beta},\hat{\vec{x}}\right],\end{aligned} \tag{13}$$

The position operator $\hat{x}$ is diagonal concerning the spinor indices, i.e., $\hat{\vec{x}}\psi=\vec{x}\psi$ and contains no differentiation, thus $\left[\hat{\beta},\hat{\vec{x}}\right]=\left[\hat{\vec{\alpha}},\hat{\vec{x}}\right]=0$, then for three arbitrary vectors $\vec{A}_1$, $\vec{A}_2$ and $\vec{A}_3$ we use the identity

$$\left[\vec{A}_1\vec{A}_2,\vec{A}_3\right]=\left[\vec{A}_1,\vec{A}_3\right]\vec{A}_2+\vec{A}_1\left[\vec{A}_2,\vec{A}_3\right]. \tag{14}$$

Then we have

$$\left[\hat{\vec{\alpha}}\cdot\hat{\vec{p}},\hat{\vec{x}}\right]=-i\hbar\hat{\vec{\alpha}}, \tag{15}$$

also

$$\left[\tilde{\vec{A}},\hat{\vec{x}}\right]=\left[\tilde{\Phi},\hat{\vec{x}}\right]=0, \tag{16}$$

because both $\tilde{\vec{A}}$, $\tilde{\Phi}$ are functions of $\frac{\partial}{\partial\vec{p}}$. Consequently, we obtain

$$\frac{d\hat{\vec{x}}}{dt}=c\hat{\vec{\alpha}}. \tag{17}$$

Let us subsequently see how the operator (17) acts on the Dirac spinor. By considering single components ψ, then we obtain

$$\frac{d\hat{x}}{dt}\psi=\pm c\psi, \tag{18}$$

where the eigenvalues of $\hat{\vec{\alpha}}$ are $\pm$ 1. This result has no classical analogy because despite the considered effects, the Dirac particle is still moving at the speed of light.

Now, the equation of motion for the kinetic momentum operator $\hat{\vec{\Pi}}=\hat{\vec{p}}-\frac{e}{c}\vec{A}$ is

$$\frac{d\hat{\vec{\Pi}}}{dt}=\frac{\partial\hat{\vec{\Pi}}}{\partial t}+\frac{i}{\hbar}\left[\mathscr{H}_D^{MS},\hat{\vec{\Pi}}\right]=-e\frac{\partial\vec{A}}{\partial t}+\frac{i}{\hbar}\left[\hat{\mathscr{H}}_D^{MS},\hat{\vec{\Pi}}\right], \tag{19}$$

consequently, the commutator is given by

$$\left[\hat{\mathscr{H}}_D^{MS},\hat{\vec{\Pi}}\right]=\left[\hat{\mathscr{H}}_D^{MS},\hat{\vec{p}}\right]-e\left[\hat{\mathscr{H}}_D^{MS},\vec{A}\right]. \tag{20}$$

At first, we calculate the first commutator in equation (20)

$$\begin{aligned}\left[\mathscr{H}_D^{MS},\hat{\overrightarrow{p}}\right]=c\left[\hat{\overrightarrow{\alpha}}\cdot\hat{\overrightarrow{p}},\hat{\overrightarrow{p}}\right]-ce\left(1+\frac{p_0}{\kappa}\right)\left[\hat{\overrightarrow{\alpha}}\cdot\tilde{\overrightarrow{A}},\hat{\overrightarrow{p}}\right]+\left(1+\frac{p_0}{\kappa}\right)e\left[\tilde{\Phi},\hat{\overrightarrow{p}}\right]\\+\left(1+\frac{p_0}{\kappa}\right)mc^2\left[\hat{\beta},\hat{\overrightarrow{p}}\right],\end{aligned}\tag{21}$$

with $\left[\hat{\beta},\hat{\overrightarrow{p}}\right]=\left[\hat{\overrightarrow{\alpha}},\hat{\overrightarrow{p}}\right]=0$ because $\hat{\beta}$ and $\hat{\overrightarrow{\alpha}}$ are independent of space coordinates. Furthermore, we have

$$\left[\tilde{\Phi},\hat{\overrightarrow{p}}\right]=i\hbar\left[\overrightarrow{\nabla},\tilde{\Phi}\right]=i\hbar\left(\overrightarrow{\nabla}\tilde{\Phi}-\tilde{\Phi}\overrightarrow{\nabla}\right),\tag{22}$$

then through equation (22), we have

$$\left[\tilde{\Phi},\hat{\overrightarrow{p}}\right]\psi=i\hbar\left(\overrightarrow{\nabla}\tilde{\Phi}-\tilde{\Phi}\overrightarrow{\nabla}\right)\psi=i\hbar\left(\overrightarrow{\nabla}\tilde{\Phi}\right)\psi,\tag{23}$$

and

$$\left[\hat{\overrightarrow{\alpha}}\cdot\hat{\overrightarrow{p}},\hat{\overrightarrow{p}}\right]=0.\tag{24}$$

Also

$$\left[\hat{\overrightarrow{\alpha}}\cdot\tilde{\overrightarrow{A}},\hat{\overrightarrow{p}}\right]=-i\hbar\sum_{i,j}\hat{\alpha}_i\left[\tilde{A}_i,\nabla_j\right]e_j,\tag{25}$$

then by considering the effect of equation (25) on ψ, we obtain:

$$\left[\hat{\overrightarrow{\alpha}}\cdot\tilde{\overrightarrow{A}},\hat{\overrightarrow{p}}\right]\psi=i\hbar\sum_{i,j}\hat{\alpha}_i\left(\nabla_j\tilde{A}_i\psi-\tilde{A}_i\nabla_j\psi\right)e_j=i\hbar\sum_{i,j}\hat{\alpha}_i\left(\nabla_j\tilde{A}_i\right)e_j\psi.\tag{26}$$

Now, we pass to the second commutator in equation (20), so, we have

$$\begin{aligned}\left[\mathscr{H}_D^{MS},\overrightarrow{A}\right]=c\left[\hat{\overrightarrow{\alpha}}\cdot\hat{\overrightarrow{p}},\overrightarrow{A}\right]-ce\left(1+\frac{p_0}{\kappa}\right)\left[\hat{\overrightarrow{\alpha}}\cdot\tilde{\overrightarrow{A}},\overrightarrow{A}\right]+\left(1+\frac{p_0}{\kappa}\right)e\left[\tilde{\Phi},\overrightarrow{A}\right]\\+\left(1+\frac{p_0}{\kappa}\right)mc^2\left[\hat{\beta},\overrightarrow{A}\right].\end{aligned}\tag{27}$$

Thereafter, we move to calculate each commutator in equation (27), thus we start with

$$\left[\hat{\overrightarrow{\alpha}}\cdot\hat{\overrightarrow{p}},\overrightarrow{A}\right]=-i\hbar\sum_{i,j}\hat{\alpha}_i\left[\nabla_i,A_j\right]e_j,\tag{28}$$

and its act on ψ yields

$$\left[\hat{\overrightarrow{\alpha}}\cdot\hat{\overrightarrow{p}},\overrightarrow{A}\right]\psi=-i\hbar\sum_{i,j}\hat{\alpha}_i\left(\nabla_iA_j\right)e_j\psi.\tag{29}$$

Note that in equations (26, 29), the gradient acts on $\overrightarrow{A}$ only. Then, we continue with

$$\left[\hat{\beta},\overrightarrow{A}\right]=\left[\hat{\overrightarrow{\alpha}},\overrightarrow{A}\right]=0,\tag{30}$$

and

$$\left[\tilde{\Phi},\overrightarrow{A}\right]=\left[\tilde{\overrightarrow{A}},\overrightarrow{A}\right]=0.\tag{31}$$

Now in total, we have

$$\frac{d\hat{\overrightarrow{\Pi}}}{dt}=-e\left\{\frac{\partial\overrightarrow{A}}{\partial t}+\left(1+\frac{p_0}{\kappa}\right)\left(\overrightarrow{\nabla}\tilde{\Phi}\right)\right\}+e\sum_{i,j}\left(c\hat{\alpha}_i\right)\left\{\left(1+\frac{p_0}{\kappa}\right)\nabla_j\tilde{A}_i-\nabla_iA_j\right\}e_j.\tag{32}$$

By using equation (17), and after some simplifications we get

$$\frac{d\hat{\overrightarrow{\Pi}}}{dt}=-e\left(1+\frac{p_0}{\kappa}\right)\left\{\frac{\partial\frac{\overrightarrow{A}}{\left(1+\frac{p_0}{\kappa}\right)}}{\partial t}+\left(\overrightarrow{\nabla}\tilde{\Phi}\right)\right\}+e\left(1+\frac{p_0}{\kappa}\right)\sum_{i,j}v_i\left(\nabla_j\tilde{A}_i-\nabla_i\frac{A_j}{\left(1+\frac{p_0}{\kappa}\right)}\right)e_j. \quad (33)$$

But if $\tilde{\overrightarrow{A}}=\dfrac{\overrightarrow{A}}{\left(1+\frac{p_0}{\kappa}\right)}$, one has

$$\tilde{\overrightarrow{E}}=-\frac{1}{c}\frac{\partial\tilde{\overrightarrow{A}}}{\partial t}-\overrightarrow{\nabla}\tilde{\Phi}, \quad (34)$$

and

$$\sum_{i,j}v_i\left(\nabla_j\tilde{A}_i-\nabla_i\tilde{A}_j\right)e_j=\overrightarrow{v}\times\mathrm{curl}\tilde{\overrightarrow{A}}. \quad (35)$$

Then, we have

$$\frac{d\hat{\overrightarrow{\Pi}}}{dt}=e\left(1+\frac{p_0}{\kappa}\right)\left\{\tilde{\overrightarrow{E}}+\overrightarrow{v}\times\tilde{\overrightarrow{B}}\right\}, \quad (36)$$

where $\mathrm{curl}\tilde{\overrightarrow{A}}=\tilde{\overrightarrow{B}}$.

As can be seen, equation (36) is a κ-deformed Lorentz force in the classical case. It is a force exerted by the electromagnetic field on an electron having an electric charge e. Unlike the case of velocity in equation (17), here the effect of MS model of DSR on the Lorentz force appears widely in equation (36) through the parameter κ. In the limit of $\kappa\to\infty$, we have

$$\frac{d\hat{\overrightarrow{\Pi}}}{dt}=e\left\{\overrightarrow{E}+\overrightarrow{v}\times\overrightarrow{B}\right\},\text{with }\tilde{\Phi}\to\Phi, \quad (37)$$

which is the Lorentz force in the classical case. Now, let us discuss the findings:

It is observed that $\hat{\overrightarrow{x}}$ does not comply with classical equations of motion. Nevertheless, a classical equation of motion can be established for the operator $\hat{\overrightarrow{\Pi}}$. Interestingly, equation (36) appears to formally align with the corresponding classical equation; however, it is crucial to bear in mind that any expectation values derived from (37) lack utility due to Zitterbewegung, with a reduction in velocity. At best, the projection of the even contributions from (37) yields result pertinent to a classical single-particle description. Shifting focus to equation (36), it illustrates the impact of DSR on the Lorentz force, which undergoes deformation based on these considerations. Conversely, equation (17) indicates that DSR does not exert an influence on velocity. Notably, the applied considerations of the MS model are found to impact Ehrenfest's theorem.

C. $\mathcal{CPT}$ and Lorentz symmetries of κ-deformed Lorentz force

Both $\mathcal{CPT}$ and Lorentz symmetries hold a crucial role in modern quantum field theory, including the standard model of particle physics, and its potential violation could have profound implications for our understanding of fundamental physics and the nature of spacetime. Ongoing experimental efforts aim to test the $\mathcal{CPT}$ symmetry with increasing precision, providing valuable insights into the symmetrical underpinnings of the universe. However, the $\mathcal{CPT}$ symmetry combines charge conjugation $\mathcal{C}$, parity inversion $\mathcal{P}$ and time reversal $\mathcal{T}$ into a more encompassing symmetry. This combined operation must be an exact symmetry. It ensures that the physics laws remain unchanged when particles are replaced by their antiparticles, space is inverted and time flows backward simultaneously. The $\mathcal{CPT}$ symmetry is a powerful concept that underlies our understanding of the fundamental symmetries of the universe. On the other hand, Lorentz symmetry, ensures that laws of physics are the same for all observers in inertial reference frames.

Now, by applying the transformation rules from Tab. 1 to equation (33), we examine the deformed Lorentz force under the discrete symmetries $\mathcal{C}$, $\mathcal{P}$, and $\mathcal{T}$, as well as the $\mathcal{CPT}$ symmetry. Consequently,

$$\begin{aligned} \mathcal{C}\left\{\frac{d\tilde{\vec{\Pi}}}{dt}\right\} &\neq \left\{\frac{d\tilde{\vec{\Pi}}}{dt}\right\}, \\ \mathcal{P}\left\{\frac{d\tilde{\vec{\Pi}}}{dt}\right\} &\neq \left\{\frac{d\tilde{\vec{\Pi}}}{dt}\right\}, \\ \mathcal{T}\left\{\frac{d\tilde{\vec{\Pi}}}{dt}\right\} &\neq \left\{\frac{d\tilde{\vec{\Pi}}}{dt}\right\}, \end{aligned} \tag{38}$$

where $\vec{A}$, Φ depend on x, but $\tilde{\vec{A}}$, $\tilde{\Phi}$ depend on p. Then

$$\mathcal{CPT}\left\{\frac{d\tilde{\vec{\Pi}}}{dt}\right\} \neq \left\{\frac{d\tilde{\vec{\Pi}}}{dt}\right\}, \tag{39}$$

this clearly means that the Lorentz force operator in the MS model of DSR violates the $\mathcal{CPT}$ symmetry, this in turn violates the Lorentz symmetry. Moreover, under the discrete symmetries $\mathcal{C}$, $\mathcal{P}$ and $\mathcal{T}$, the κ-deformed Lorentz force is not invariant. Note that the discrete symmetries of $\vec{X}$ (defined in equation (8)), $\tilde{\vec{A}} = \vec{A}\left(\vec{X}, B\right)$ and $\tilde{\Phi} = \Phi\left(\vec{X}, e\right)$ are successively given as follows:

$$\begin{aligned} \vec{X} &\xrightarrow{\mathcal{C}} -\vec{X}, \\ \vec{X} &\xrightarrow{\mathcal{P}} -\vec{X}, \\ \vec{X} &\xrightarrow{\mathcal{T}} -\vec{X}, \end{aligned} \tag{40}$$

then

$$\begin{aligned} \tilde{\vec{A}} &\xrightarrow{\mathcal{C}} \tilde{\vec{A}}, \\ \tilde{\vec{A}} &\xrightarrow{\mathcal{P}} -\tilde{\vec{A}}, \\ \tilde{\vec{A}} &\xrightarrow{\mathcal{T}} \tilde{\vec{A}}, \end{aligned} \tag{41}$$

and

$$\begin{aligned} \tilde{\Phi} &\xrightarrow{\mathcal{C}} \tilde{\Phi}, \\ \tilde{\Phi} &\xrightarrow{\mathcal{P}} -\tilde{\Phi}, \\ \tilde{\Phi} &\xrightarrow{\mathcal{T}} -\tilde{\Phi}. \end{aligned} \tag{42}$$

However, based on the equation (40), one can see that the noncommutative variable $\vec{X}$ undergoes changes under discrete $\mathcal{C}$, $\mathcal{P}$, $\mathcal{T}$ and $\mathcal{CPT}$ transformations. Consequently, other related physical aspects may also exhibit alterations.

3. Conclusion and remarks

In this study, using Ehrenfest's theorem, we have analytically explored the CL of the Dirac equation in interaction with electromagnetic potential and in the context of MS model of DSR. We successfully examined the effects of the MS model on the CL, which yields a κ-deformed classical equations. Our findings affirm the feasibility of obtaining CL within the framework of DSR, specifically, the MS model. Once again, Ehrenfest's theorem demonstrates its efficacy in deriving CL of the Dirac equation, regardless of the effects on the relativistic system. Consequently, we emphasize the significance of this type of theorems. In addition, it is shown that considering MS model of DSR in the CL of Dirac equation is not suitable for the invariance of the $\mathcal{CPT}$ and Lorentz symmetries. Clearly, our results can be considered a useful tool for exploring further related studies, encompassing non-relativistic and semiclassical limits, and other scenarios involving specific models within the framework of DSR such as Snyder and Majid–Ruegg models. Additionally, expanding the study to include more generalizations, such as particles with arbitrary higher spins, would be a promising avenue for future research. Knowing that, in the limit of $\kappa \to \infty$, the κ-deformed Dirac and the obtained classical equations reduce to those of ordinary quantum mechanics, confirms that our results are consistent with and reducible to those found and discussed in the literature.

Appendix A: $\mathcal{C}$, $\mathcal{P}$ and $\mathcal{T}$ discrete symmetries

The discrete symmetries play a fundamental role in modern theoretical physics and among these symmetries, $\mathcal{C}$, $\mathcal{P}$ and $\mathcal{T}$ are particularly significant and basic.

- The $\mathcal{C}$ symmetry, i.e., charge conjugation, involves exchanging particles with their corresponding antiparticles while reversing their charges, e.g., $e \to -e$ and $i \to -i$.
- The $\mathcal{P}$ symmetry, i.e., parity, reflects the spatial inversion of a physical system, interchanging left and right, e.g., $\overrightarrow{x} \to -\overrightarrow{x}$.
- The $\mathcal{T}$ symmetry, i.e., time reversal, entails reversing the direction of time in a process, e.g., $t \to -t$.

In classical mechanics, the definitions of physical quantities like momentum, angular momentum and energy etc., decide their transformation properties under $\mathcal{P}$ and $\mathcal{T}$ symmetries, but, $\mathcal{C}$ symmetry does not enter the classical field. However, one could define it as an operation which changes the charge of a particle, leaving other attributes the same. Consequently, classical electrodynamics is invariant under $\mathcal{C}$, provided the fields change sign under $\mathcal{C}$. On the other hand, while $\mathcal{C}$ has no place in non-relativistic quantum mechanics, it arises naturally in relativistic quantum mechanics, particularly, it represents a symmetry between matter and antimatter.

Tab. 1 shows some of the discrete $\mathcal{C}$, $\mathcal{P}$ and $\mathcal{T}$ symmetries operations known in the literature [41].

Table 1. Summary of some discrete symmetry operations

Quantity	Notation	$\mathcal{P}$	$\mathcal{C}$	$\mathcal{T}$
Electric charge	e	1	–1	1
Time derivative	$\frac{\partial}{\partial t}$	1	1	–1
Nabla vector	$\overrightarrow{\nabla}$	–1	1	1
Position	$\overrightarrow{x}$	–1	1	1
Velocity	$\overrightarrow{v}$	–1	1	–1
Momentum	$\overrightarrow{p}$	–1	1	–1
Electric field	$\overrightarrow{E}$	–1	–1	1
Magnetic field	$\overrightarrow{B}$	1	–1	–1
Scalar potential	Φ	1	–1	1
Electromagnetic vector	$\overrightarrow{A} \propto Bx$	–1	–1	–1

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Conflicts of Interest

The author declares no conflict of interest.

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Классический предел уравнения Дирака в контексте модели Магейхо-Смолина двойной специальной теории относительности с использованием теоремы Эренфеста

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Аннотация. В этой статье в контексте модели Магейхо–Смолина и с использованием теоремы Эренфеста мы исследуем классический предел уравнения Дирака в рамках двойной специальной теории относительности. Это приводит к получению деформированных классических уравнений. Здесь мы оцениваем эффективность теоремы Эренфеста при выводе классического предела в присутствии модели Магейхо–Смолина. Кроме того, мы исследуем деформированные классические уравнения относительно дискретной, $\mathcal{CPT}$ и симметрии Лоренца.

Ключевые слова: уравнение Дирака, дважды специальная теория относительности, модель Магейхо-Смолина, теорема Эренфеста, классический предел.