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On the Stability of the Solutions of Inverse Problems for Elliptic Equations

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Abstract. The inverse problems on finding the unknown lower coefficient in linear and nonlinear second-order elliptic equations with integral overdetermination conditions are considered. The conditions of overdetermination are given on the boundary of the domain. The continuous dependence of the strong solution on the input data of the inverse problem for the linear equation is proved in the case of the mixed boundary condition. As to the nonlinear equation, the continuous dependence of the strong solution on the overdetermination data is established for the inverse problem with the Dirichlet boundary condition.

Keywords: inverse problem, elliptic equation, integral overdetermination, continuous dependence on input data.

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Introduction

In this paper the stability of the solutions of two inverse problems for second order elliptic equations are considered.

Problem 1. For given functions $f(x), \beta(x), h(x), \alpha(x)$ and a constant μ find the pair of functions u and constant k, satisfying the equation

$$-\operatorname{div}(\mathcal{M}(x)\nabla u) + m(x)u + ku = f, \tag{1}$$

the boundary condition

$$\left(\frac{\partial u}{\partial \overline{N}} + \alpha(x)u\right)\Big|_{\partial\Omega} = \beta(x), \tag{2}$$

and the condition of overdetermination

$$\int_{\partial\Omega} uh(x)ds = \mu. \tag{3}$$

Problem 2. For given functions $f(x), \beta(x), h(x)$ and a constant μ find the pair of funtcions u and constant k satisfying the equation

$$-\operatorname{div}(\mathcal{M}(x)\nabla u) + m(x)u + kr(u) = f, \tag{4}$$

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the boundary condition

$$u|_{\partial\Omega} = \beta(x), \tag{5}$$

and the condition of overdetermination

$$\int_{\partial\Omega} \frac{\partial u}{\partial N} h(x) ds = \mu. \tag{6}$$

Here $\Omega \cap \mathbf{R}^n$ is a bounded domain with a boundary $\partial\Omega \in C^2$, $\mathcal{M}(x) = m_{ij}(x)$ is a matrix of functions $m_{ij}, i, j = 1, 2, \ldots, n$, $m(x)$ is a scalar funtcion, $\dfrac{\partial}{\partial N} = (\mathcal{M}(x)\nabla, \mathbf{n}), \mathbf{n}$ is the unit vector of the outward normal to the boundary $\partial\Omega$.

A main goal of this paper is to establish stability (in the sense of continuous dependence on the source data) of strong generalized solutions of Problems 1 and 2. The conditions of the solvability and uniqueness of solutions to Problems 1 and 2 were established in [1,2]. The proof of the existence and uniqueness of the solutions follows the method developed by A. Sh. Lyubanova and A. Tani in [3,4] where inverse problems with integral overdetermination conditions were also considered. The method is based on the idea of reducing the inverse problem to an operator equation of the second kind for the unknown coefficient [5].

Practical interest in such inverse problems is due to many applications in the theory of diffusion and filtration [6] as well as the fact that filtration processes tend to stabilize over time [7]. The steady fluid flow in a fissured medium is described by a stationary equation in which the pressure u, coefficients and the right-hand side are independent of t. In general, the stationary equation of the compressible fluid filtration has the form

$$-\operatorname{div}(\mathbf{k}(x, u)\nabla\psi(u)) + \gamma(x, u) = f, \qquad x \in \Omega, \tag{7}$$

where $\mathbf{k}(x, u)$ is a matrix of functions, $\psi(u)$ и $\gamma(x, u)$ are scalar functions, $\Omega \subset \mathbb{R}^n$ is a bounded domain with the boundary $\partial\Omega$. An example of a diffusion model is the problem of finding the concentration of a pollutant in the environment [8]

$$-\lambda\Delta u + \mathbf{v}\nabla u + ku = f, \qquad u|_{\partial\Omega} = \beta,$$

where k is a value characterizing the breakdown of a pollutant due to chemical reactions, λ is the diffusion coefficient, f is the bulk source density, $\mathbf{v}$ is the velocity vector.

The study of the inverse problems for the elliptic equations goes back to fundamental works of M.M. Lavrentiev [9–11]. Various issues related to coefficient inverse problems for the linear and nonlinear equations (7) were discussed in [11–22]. Problems of finding highest coefficients of (7) from additional boundary data on $\partial\Omega$ or on some part of $\partial\Omega$ are of particular interest. In [15, 16, 20] this problem is considered in the case of $\psi(u) = u$, $\gamma(x, u) \equiv 0$, $\mathbf{k}(x, u) = k\mathbf{E}$, $\mathbf{E}$ is the identity matrix, and function k is unknown. It is assumed that $k = k(x)$ [15, 16], or $k = k(u)$ [20]. The overdetermination condition is $k\dfrac{\partial u}{\partial \mathbf{n}}\Big|_{\partial\Omega} = \nu(x)$ for the Dirichlet boundary problem and $u|_{\partial\Omega} = \nu(x)$ for the Neumann boundary problem. The pioneering work in this line is Calderon's one [16] where the inverse problem of finding the unknown $k(x, u) = k(x)$ with such overdetermination condition was first discussed and an approximate representation was suggested for the unknown coefficient close to a constant.

Problems of recovering unknown lowest coefficients in elliptic equations have been considered by many authors. The works of [8, 12, 21–23] should be noted here. In these works, unknown coefficients are recovered from information on the values of some integral operator over the whole domain or the solution trace on some surface inside the domain in which the problem is solved. Integral conditions on the boundary were not considered in such problems.

1. The preliminaries

We use the following notations: $\|\cdot\|_R$, $(\cdot,\cdot)_R$ — the norm and the inner product in $\mathbb{R}^n$; $\|\cdot\|$, $(\cdot,\cdot)$ — the norm and the inner product in $L^2(\Omega)$; $\|\cdot\|_j$, $\langle\cdot,\cdot\rangle_1$ — the norm in $W_2^j(\Omega)$, $j=1,2$, and the duality relation between $\overset{\circ}{W}{}_2^1(\Omega)$ and $W_2^{-1}(\Omega)$, respectively, $\|\cdot\|_{j+1/2}$ — the norm in $W_2^{s+1/2}(\partial\Omega)$, $s=0,1$.

Let us introduce the linear operator $M: W_2^1(\Omega)\to(W_2^1(\Omega))^*$ of the form

$$M=-\operatorname{div}(\mathcal{M}(x)\nabla)+m(x)I,$$

where I is the identity operator. We use the notation

$$\langle Mv_1,v_2\rangle_M=\int_\Omega((\mathcal{M}(x)\nabla v_1,\nabla v_2)_R+m(x)v_1v_2)dx$$

for any $v_1,v_2\in W_2^1(\Omega)$, and reason that the following assumptions of the operator M are fulfilled.

I. $m_{ij}(x)$, $\partial m_{ij}/\partial x_l$, $i,j,l=1,2,\ldots,n$, and $m(x)$ are bounded in Ω. Operator M is strongly elliptic, that is, there exist positive constants m_0 and m_1 such that for all $v\in W_2^1(\Omega)$

$$m_0\|v\|_1^2\leqslant\langle Mv,v\rangle_M\leqslant m_1\|v\|_1^2.$$

II. M is self-adjoint, that is $m_{ij}(x)=m_{ji}(x)$ for $i,j=1,\ldots,n$.

We also impose restrictions on function $r(\rho)$.

III. The function $r(\rho)$ is continuous and strictly monotone $(-\infty,+\infty)$, that is for all $\rho_1,\rho_2\in(-\infty,+\infty)$, $\rho_1\neq\rho_2$,

$$(r(\rho_1)-r(\rho_2))(\rho_1-\rho_2)>0,$$

and $r(0)=0$.

IV. For all $\rho\in(-\infty,+\infty)$

$$|r(\rho)|\leqslant C_r|\rho|^{p-1}. \tag{8}$$

Here $C_r>0$, p – constants, $p>0$ when $n\leqslant 2$ and $0<p\leqslant n/(n-2)$ when $n>2$.

V. For any number $R>0$ and funtcions $v_1,v_2\in W_2^1(\Omega)$ such that $\|v_i\|_{L^{2(p-1)}(\Omega)}\leqslant R$, $i=1,2$, the inequality

$$\|r(v_1)-r(v_2)\|\leqslant c(R)\|v_1-v_2\|_1$$

is valid where constant $c(R)>0$ depends on R.

By the solution of Problem 1 is meant the pair, consisting of a function $u\in W_2^2(\Omega)$ and a constant $k>0$ which satisfies the equation (1), the boundary condition (2) and the overdetermination condition (3). By the solution of Problem 2 is meant the pair involving a function $u\in W_2^2(\Omega)$ and a constant $k>0$ which satisfies the equation (4), the boundary condition (5) and the overdetermination condition (6).

We define the auxiliary funtcions a,a^σ,b,d,d^τ and g as a solutions of the problems

$$Ma=f(x),\qquad\left(\frac{\partial a}{\partial\overline{N}}+\alpha(x)a\right)\Big|_{\partial\Omega}=\beta(x); \tag{9}$$

$$Ma^\sigma + \sigma a^\sigma = f, \qquad \left(\frac{\partial a^\sigma}{\partial \overline{N}} + \alpha(x) a^\sigma\right)\Big|_{\partial\Omega} = \beta(x); \tag{10}$$

$$Mb = 0, \qquad \left(\frac{\partial b}{\partial \overline{N}} + \alpha(x) b\right)\Big|_{\partial\Omega} = h(x); \tag{11}$$

$$Md = f(x), \qquad d|_{\partial\Omega} = \beta(x); \tag{12}$$

$$Md^\tau + \tau r(a^\tau) = f(x), \qquad d^\tau|_{\partial\Omega} = \beta(x); \tag{13}$$

$$Mg = 0, \qquad g|_{\partial\Omega} = h(x). \tag{14}$$

Here $\sigma > 0$, $\tau > 0$ — real numbers.

Existence and uniqueness theorems for strong solutions of the inverse Problems 1 and 2 were proven earlier in [1,2]. For the sake of convenience, we give their formulations.

Theorem 1 ([1]). *Let $\partial\Omega \in C^2$ and assumptions I and II be fulfilled. Suppose also that*

(i) $f(x) \in L^2(\Omega)$, $\beta(x), h(x) \in W_2^{3/2}(\partial\Omega)$, $\alpha(x) \in C(\partial\Omega)$;

(ii) $f(x) \geqslant 0$ *almost everywhere in* Ω; $\beta(x) \geqslant 0$, $\alpha(x) \geqslant 0$, $h(x) \geqslant 0$ *for almost all* $x \in \partial\Omega$ *and there is a smooth piece* Γ *of the boundary* $\partial\Omega$ *and a constant* $\delta > 0$ *such that* $\beta \geqslant \delta$ *and* $h \geqslant \delta$ *almost everywhere in* Γ.

Then Problem 1 has a solution $\{u, k\}$, *if*

$$0 \leqslant \mu - \Phi \leqslant \frac{m_0 (a, b)^2}{\|a\| \|b\|},$$

where $\Phi = \int\limits_{\partial\Omega} ah\, ds$, *and the estimates*

$$a^\sigma \leqslant u \leqslant a, \quad 0 \leqslant k \leqslant \sigma, \quad \|u\|_2 \leqslant C(\sigma + 1)\|a\| + \|a\|_2, \tag{15}$$

holds for some $\sigma > 0$, *constant* C *depends on* $mes\Omega, \sigma, m_0$ *and* m_1. *Moreover, if*

$$0 \leqslant \mu - \Phi < \frac{m_0 (a, b)^2}{\|a\| \|b\|} \tag{16}$$

then the solution of Problem 1 is unique.

Theorem 2 ([2]). *Let assumptions I–V be fulfilled. Suppose also that*

(i) $f(x) \in L^2(\Omega)$, $\beta(x), h(x) \in W_2^{3/2}(\partial\Omega)$;

(ii) $f(x) \geqslant 0$ *almost everywhere in* $\Omega; \beta(x) \geqslant 0, h(x) \geqslant 0$ *for almost all* $x \in \partial\Omega$ *and there is a smooth piece* Γ *of the boundary* $\partial\Omega$ *and a constant* $\delta > 0$ *such that* $\beta \geqslant \delta$ *and* $h(x) \geqslant \delta$ *almost everywhere in* Γ.

If

$$0 \leqslant Q \equiv (f, g) - \langle Md, g\rangle_1 + \mu \leqslant \frac{m_1\,(r(d), g)^2}{4c_0^p\, C_r^{p/(p-1)}\, \Psi},$$

where $\Psi = c(\|d\|_{L^{2p-2}(\Omega)})\|d\|_1\|g\|_1$, c_0 — *embedding constant* $W_2^1(\Omega)$ *in* $L^p(\Omega)$, *then the problem (4)–(6) has a solution* $\{u, k\}$, *and estimates*

$$0 \leqslant k \leqslant \tau, \qquad d^\tau \leqslant u \leqslant d, \qquad \|u\|_2 \leqslant C_M(\tau\, C_r\, \|d\|_1^{p-1} + \|d\|) + \|d\|_2. \tag{17}$$

holds for some $\tau > 0$, *with a constant* C_M, *depends on* m_0, τ *and* $mes\Omega$. *Moreover, if*

$$0 \leqslant Q \equiv (f, g) - \langle Md, g\rangle_1 + \mu < \frac{m_1\,(r(d), g)^2}{4c_0^p\, C_r^{p/(p-1)}\, \Psi},$$

then the solution is unique.

2. Stability of the solutions of inverse problems

The main results of the work are theorems on the continuous dependence of strong solutions on the input data of the above inverse problems.

Let us consider Problem 1.

Theorem 3. *Let assumptions of Theorem 1 be fulfilled and a pair $\{u_j, k_j\}$ be the unique solution of Problem 1, where $f = f_j, \beta = \beta_j, h = h_j$, and $\mu = \mu_j, j = 1, 2$. Then the estimate*

$$\|u_1 - u_2\|_2 + |k_1 - k_2| \leqslant K(|\mu_1 - \mu_2| + \|f_1 - f_2\| + \|\beta_1 - \beta_2\|_{3/2} + \|h_1 - h_2\|_{1/2}) \tag{18}$$

holds with a constant $K > 0$.

Proof. Let a_j, a^σ, b_j are solutions of problems (9), (10), (11), where $f = f_j, \beta = \beta_j, h = h_j$, $j = 1, 2$. It was shown in [1] that k_j is the solution of the operator equation $k_j = A_j k_j$, where $A_j k_j$ is determined as

$$A_j k_j = \frac{\mu_j - \Phi_j}{(u_j, b_j)}, \tag{19}$$

where $\Phi_j = \int\limits_{\partial\Omega} a_j h_j ds$, and σ_j is given by the relation

$$\sigma_j = \frac{\sqrt{m_0}((a_j, b_j) - \sqrt{D_j})}{2\|a_j\|\|b_j\|}, \tag{20}$$

with

$$D_j \equiv (a_j, b_j)^2 - \frac{4(\mu_j - \Phi_j)\|a_j\|\|b_j\|}{\sqrt{m_0}} \geqslant 0,$$

Estimating the right side of the difference

$$k_1 - k_2 = A_1 k_1 - A_2 k_2 = \frac{\Phi_2 - \Phi_1 + \mu_1 - \mu_2}{(u_1, b_1)} + k_2 \Big[\frac{(u_2 - u_1, b_1) - (u_1, b_1 - b_2)}{(u_1, b_1)}\Big].$$

in absolute value with (15) and the relation [1]

$$(u_1, b_1) \geqslant (a_1^\sigma, b_1) = (a_1, b_1) - (a_1 - a_1^\sigma, b_1) \geqslant (a_1, b_1) - \frac{\sigma_1}{\sqrt{m_0}}\|a_1\|\|b_1\| \geqslant 0,$$

we come to the inequality

$$|k_1 - k_2| \leqslant K_1(|\mu_1 - \mu_2| + |\Phi_1 - \Phi_2| + \|b_1 - b_2\|_1) + \frac{k_2\sqrt{m_0}\|b_1\|\|u_1 - u_2\|}{\sqrt{m_0}(a_1, b_1) - \sigma_1\|a_1\|\|b_1\|}, \tag{21}$$

where positive constant K_1 depends on m_0, mesΩ, μ_j, Φ_j, $\|a_j\|_1$, $\|b_j\|_1$, $j = 1, 2$.

On the other hand, difference $u = u_1 - u_2$ satisfies the relations (1)–(2), where $k = k_1$, $f = (k_2 - k_1)u_2 + f_1 - f_2$ and $\beta = \beta_1 - \beta_2$. Using (15) and (19), for u_j, k_j when $a = a_j$ and $\sigma = \sigma_j$, $j = 1, 2$, and also estimate [24]

$$\|v\|_2 \leqslant C_M(\|Mv\| + \|v\|), \tag{22}$$

valid for all $v \in \overset{\circ}{W}{}_2^1(\Omega) \bigcap W_2^2(\Omega)$, we obtain

$$\|u_1 - u_2\|_1 \leqslant \frac{1}{m_0}(\sigma_1\|a_1 - a_2\| + |k_1 - k_2|\|a_1\|) + \|a_1 - a_2\|_1,$$

$$\|u_1 - u_2\|_2 \leqslant \frac{C_M(m_0+1)}{m_0}(\sigma_1\|a_1 - a_2\| + |k_1 - k_2|\|a_1\|) + \|a_1 - a_2\|_2. \tag{23}$$

Without loss of geberality, it may be suggested that $k_1 \geqslant k_2$. Then (15), (16), (20) for $j = 1$ and (21) lead to inequality

$$|k_1 - k_2| \leqslant K_2\Big[|\mu_1 - \mu_2| + |\Phi_1 - \Phi_2| + \|b_1 - b_2\|_1\Big], \tag{24}$$

where K_2 depends on K_1, m_0, σ_1, $\|a_1\|$. For $a_1 - a_2$ and $b_1 - b_2$, we have [25, Chapter 2]

$$\|a_1 - a_2\|_j \leqslant C_2(\|f_1 - f_2\| + \|\beta_1 - \beta_2\|_{j-1/2}), \quad j = 1, 2, \tag{25}$$

$$\|b_1 - b_2\|_1 \leqslant C_1\|h_1 - h_2\|_{1/2}, \tag{26}$$

where constants $C_i > 0,\ i = 1, 2$, depend on n, m_0, m_1 and $mes\Omega$. Taking into account definition of $\Phi_j,\ j = 1, 2$, and relations (23)–(26), we come to the estimate (18). Theorem is proved. □

Let us turn our attention to the theorem on stability of the strong solution of Problem 2.

Theorem 4. *Let the assumptions of Theorem 2 be fulfilled and a pair $\{u_j, k_j\}$ be a solution of Problem 2 where $\mu = \mu_j, j = 1, 2$. Then the estimate*

$$\|u_1 - u_2\|_2 + |k_1 - k_2| \leqslant H|\mu_1 - \mu_2| \tag{27}$$

holds with a constant $H > 0$.

Proof. Let d_j^τ be the solution of (13) with $\tau = \tau_j$, $j = 1, 2$, where

$$\tau_j = \frac{((r(d), g) - \sqrt{G_j})m_0}{2Q_j\ C_r^{p/(p-1)}\ c_0^p\ \Psi}, \qquad G_j = (r(d), g)^2 - 4Q_j\frac{C_r^{p/(p-1)}c_0^p\ \Psi}{m_0},$$

and

$$Q_j = (f, g) - \langle Md, g\rangle_1 + \mu_j.$$

As was shown in [2], k_j is a solution of the operator equation

$$k_j = B_jk_j = \frac{Q_j}{(r(u_j), g)}. \tag{28}$$

For the sake of convenience, we denote by $\{\bar{u}, \bar{k}\}$ the difference of solutions $\{u_1, k_1\}$ and $\{u_2, k_2\}$. $\bar{k}$ is a solution of the equation

$$\bar{k} = B_1k_1 - B_2k_2 = \frac{Q_1}{(r(u_1), g)} - \frac{Q_2}{(r(u_2), g)} = \frac{(Q_1 - Q_2)(r(u_1), g) - Q_1(r(u_1) - r(u_2), g)}{(r(u_1), g)(r(u_2), g)},$$

or, by the defenition of Q_j and (28),

$$\bar{k} = \frac{Q_1 - Q_2}{(r(u_2), g)} - \frac{k_1(r(u_1) - r(u_2), g)}{(r(u_2), g)} = \frac{\mu_1 - \mu_2}{(r(u_2), g)} - \frac{k_1(r(u_1) - r(u_2), g)}{(r(u_2), g)}. \tag{29}$$

Let us estimate the last term of the right side of the resulting relation by absolute value, taking into account (8), (17), (28), assumption V and the inequality [2]

$$(r(u_2), g) \geqslant (r(d), g) + (r(d_2^\tau) - r(d), g) \geqslant (r(d), g) - \tau_2\frac{C_r^{p/(p-1)}c_0^p}{m_0}c(\|d\|_{L^{2p-2}})\|d\|_1^{p-1}\|g\|_1.$$

Without loss of generality one may suggest that $k_1 \leqslant k_2$. We have

$$\Big|\frac{k_1(r(u_1)-r(u_2),g)}{(r(u_2),g)}\Big| = Q_2 \cdot \frac{|(r(u_1)-r(u_2),g)|}{(r(u_2),g)^2} \leqslant$$

$$\leqslant \frac{m_0(r(d),g)^2}{4C_r^{p/(p-1)}\, c_0^p\Psi} \cdot \frac{c(\|d\|_{L^{2p-2}})\|u_1-u_2\|_1\|g\|}{((r(d),g)-\tau_2 C_r^{p/(p-1)} c_0^p m_1{}^{-1} c(\|d\|_{L^{2p-2}}(\Omega))\|d\|_1^{p-1}\|g\|_1)^2} =$$

$$= \frac{m_0(r(d),g)^2}{C_r^{p/(p-1)} c_0^p \|d\|_1^{p-1}((r(d),g)+\sqrt{G_2})^2} \cdot \|u_1-u_2\|_1. \tag{30}$$

On the other hand, difference $\{\bar{u},\bar{k}\}$ satisfies the equation

$$M\bar{u} + k_1(r(u_1)-r(u_2)) = (k_2-k_1)r(u_2) \tag{31}$$

and the boundary condition

$$\bar{u}|_{\partial\Omega} = 0. \tag{32}$$

Then multiplying (31) by $\bar{u}$ in terms of the inner product in $L^2(\Omega)$ and integrating by parts in the first term with regard to (32) give

$$\langle M\bar{u},\bar{u}\rangle_1 + k_1(r(u_1)-r(u_2),\bar{u}) = (k_2-k_1)(r(u_2),\bar{u}). \tag{33}$$

We estimate the right-hand side of (33) by the absolute value using the embedding theorem $W_2^1(\Omega)$ in $L^p(\Omega)$ and (8).

$$|(k_2-k_1)(r(u_2),\bar{u})| \leqslant C_r^{p/(p-1)}|k_2-k_1|\|u_2\|_{L^p(\Omega)}^{p-1}\|\bar{u}\|_{L^p(\Omega)} \leqslant$$

$$\leqslant C_r^{p/(p-1)} c_0^p\, |k_2-k_1|\|d\|_1^{p-1}\|\bar{u}\|_1 \leqslant \frac{C_r^{2p/(p-1)} c_0^{2p}}{2m_0}|k_2-k_1|^2\|d\|_1^{2p-2} + \frac{m_1}{2}\|\bar{u}\|_1^2.$$

By the assumptions I – V, the equalty (33) and the last relation lead us to the inequality

$$\|\bar{u}\|_1 \leqslant \frac{C_r^{p/(p-1)} c_0^p}{m_1}\|d\|_1^{p-1}|k_2-k_1|. \tag{34}$$

Combining (29), (30) and (33) we obtain the estimate

$$|\bar{k}| \leqslant \frac{|\mu_1-\mu_2|}{(r(d),g)+\sqrt{G_2}} + \frac{(r(d),g)^2}{((r(d),g)+\sqrt{G_2})^2}|\bar{k}|,$$

which implies that

$$|\bar{k}| \leqslant \frac{(r(d),g)+\sqrt{G_2}}{\sqrt{G_2}(2(r(d),g)+\sqrt{G_2})}|\mu_1-\mu_2|. \tag{35}$$

Inequalities (34) and (35) give us an estimates

$$\|\bar{u}\|_1 \leqslant \frac{C_r^{p/(p-1)} c_0^p}{m_1}\|d\|_1^{p-1}\frac{(r(d),g)+\sqrt{G_2}}{\sqrt{G_2}(2(r(d),g)+\sqrt{G_2})}|\mu_1-\mu_2|$$

and in view of IV

$$\|r(u_1)-r(u_2)\| \leqslant\ c(\|d\|_{L^{2p-2}(\Omega)})\|u_1-u_2\|\ \leqslant$$

$$\leqslant\ \frac{C_r^{p/(p-1)} c_0^p}{m_0}\|d\|_1^{p-1} c(\|d\|_{L^{2p-2}(\Omega)})\frac{(r(d),g)+\sqrt{G_2}}{\sqrt{G_2}(2(r(d),g)+\sqrt{G_2})}|\mu_1-\mu_2| \equiv C_3|\mu_1-\mu_2|.$$

We now multiply (31) by $M\bar{u}$ in terms of inner product in $L_2\Omega$.

$$\|M\bar{u}\|^2 = (k_2 - k_1)(r(u_2), M\bar{u}) - k_1(r(u_1) - r(u_2), M\bar{u}).$$

We estimate the right hand side of the last relation taking into account (35). This gives

$$|(k_2 - k_1)(r(u_2), M\bar{u}) - k_1(r(u_1) - r(u_2), M\bar{u})| \leqslant \frac{1}{2}(\tau_1 C_3|\mu_1 - \mu_2| + |\bar{k}|\|r(u_2)\|)^2 + \frac{1}{2}\|M\bar{u}\|^2 \leqslant$$

$$\leqslant \frac{1}{2}\left(\tau_1 C_3 + \frac{r(d), g) + \sqrt{G_2}}{\sqrt{G_2}(2(r(d), g) + \sqrt{G_2})}\right)^2 |\mu_1 - \mu_2|,$$

whence, due to the inequality (22), we obtain the estimate

$$\|\bar{u}\|_2 \leqslant C_M\left(\tau_1 C_3 + \frac{(r(d), g) + \sqrt{G_2}}{\sqrt{G_2}(2(r(d), g) + \sqrt{G_2})} + \frac{C_r^{p/(p-1)} c_0^p}{m_0}\|d\|_1^{p-1}\frac{(r(d), g) + \sqrt{G_2}}{\sqrt{G_2}(2(r(d), g) + \sqrt{G_2})}\right)|\mu_1 - \mu_2|.$$

Theorem is proved. □

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Об устойчивости решений некоторых обратных задач для эллиптических уравнений

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Аннотация. В работе рассматриваются обратные задачи отыскания неизвестного младшего коэффициента в линейном и нелинейном эллиптических уравнениях второго порядка с интегральными условиями переопределения на границе исследуемой области. Для линейного уравнения доказана непрерывная зависимость сильного решения обратной задачи от ее исходных данных в случае смешанного граничного условия. Для нелинейного уравнения установлена непрерывная зависимость сильного решения обратной задачи с граничным условием первого рода от данных переопределения.

Ключевые слова: обратная задача, эллиптическое уравнение, интегральное переопределение, непрерывная зависимость от входных данных.