EDN: FVESDY
УДК 517.9

Thermocapillary Convection of Immiscible Liquid in a Three-dimensional Layer at Low Marangoni Numbers

Viktor K. Andreev*
Institute of Computational Modelling SB RAS
Krasnoyarsk, Russian Federation

Received 26.08.2023, received in revised form 24.09.2023, accepted 20.12.2023

Abstract. The joint convection of two viscous heat-conducting liquids in a three-dimensional layer bounded by solid flat walls is studied. The upper wall is thermally insulated, and a non-stationary temperature field is set on the lower wall. Liquids are assumed to be immiscible and complex conjugation conditions are set at the flat interface between them. The evolution of this system is described by the Oberbeck-Boussinesq equations in each fluid. The solution of this problem is sought in the class of velocity fields linear in two coordinates, and temperature fields are quadratic functions of the same coordinates. In this case, the problem is reduced to a system of 10 nonlinear integro-differential equations. It is conjugate and inverse with respect to 4 functions of time. To find them, integral redefinition conditions are set. The physical meaning of these conditions is the closeness of the flow. The inverse initial-boundary value problem describes convection in a two-layer system that occurs near the temperature extremum point on the lower solid wall. For small Marangoni numbers, the problem is approximated by a linear one (the Marangoni number plays the role of the Reynolds number for the Navier-Stokes equations). A stationary solution to this problem has been found. The linear nonstationary problem is solved by the Laplace transform method, and the temperature can have discontinuities of the 1st kind (change by a jump). In Laplace images, the solution is obtained in quadratures. It is proved that with increasing time, it tends to stationary mode if the temperature on the lower wall stabilizes over time. The evolution of the behavior of the velocity field in the transformer oil-water system has been studied using the numerical inversion of the Laplace transform.

Keywords: Oberbeck-Boussinesq equations, interface, Marangoni number, thermocapillarity, inverse problem, Laplace transform.

Citation: Viktor K. Andreev, Thermocapillary Convection of Immiscible Liquid in a Three-dimensional Layer at Low Marangoni Numbers, J. Sib. Fed. Univ. Math. Phys., 2024, 17(2), 195–206. EDN: FVESDY.

We consider the solution of the Oberbeck–Boussinesq equations of the form

$$\mathbf{u} = ((f(z,t)+h(z,t))x, (f(z,t)-h(z,t))y, -2\int_0^z f(\xi,t)\,d\xi), \quad \bar{p} = \bar{p}(x,y,z,t),$$
$$\theta = a(z,t)x^2 + b(z,t)y^2 + q(z,t) \tag{1}$$

where $\bar{p}$ is modified pressure.

The initial idea to search for exact solutions of the Navier-Stokes equations with a linear dependence of the velocity components on two spatial variables, apparently, was first proposed in [1]. It was shown that the general three-dimensional system of viscous magnetic hydrodynamics equations is reduced to a closed system of one-dimensional equations. A similar result for the

*andr@icm.krasn.ru
© Siberian Federal University. All rights reserved

gas dynamics equations was obtained in [2]. A more special case of the velocity field representation (1) for the motion of a single fluid is considered in [3,4], and the pressure depended only on the vertical coordinate and time. The temperature is distributed according to the quadratic law (1) only at the free boundaries of the layer $z = \pm Z(t)$ and caused a thermocapillary effect. The numerical solution of the latter problem taking into account the general temperature distribution $\theta(x, y, z, t)$ in the layer $-Z(t) < z < Z(t)$ is carried out in the article [5]. A thorough review of the exact solutions of the Navier-Stokes system with a linear dependence of the velocity components on x and y is given in [6]. In [7], solution (1) was used to describe the slow convection of a single liquid in a layer with a free boundary. The paper [8] is devoted to the influence of interphase surface energy on stationary convection within the framework of solution (1). Unsteady creeping convection in the case of an isothermal interface for solution (1) was studied in the articles [9,10]. The nonlinear stationary problem of two liquid media convection is numerically investigated in [11]. Note that similar two-dimensional problems (solution (1) can be called a three-dimensional analogue of the well-known Himentz solution) in various formulations are studied in the monograph [12].

In this paper, the quadratic dependence of x and y temperatures in (1) is an additional assumption and it agrees well with the conditions on the interface.

1. Statement of problem

Substituting the solution (1) into the system of convection equations and further compatibility analysis leads to the conclusion that the modified pressure in the layers is also a quadratic function of the coordinates x and y. Further, this solution is used to describe two-layer thermocapillary convection in the layer $-l_1 < z < l_2$, $|x| < \infty$, $|y| < \infty$. The boundaries of the layer $z = l_1$, $z = l_2$ are solid fixed walls, and $z = 0$ is a fixed interface between the layer $-l_1 < z < 0$, and the layer $0 < z < l_2$ of liquids "1", "2". These heat-conducting viscous liquids have constants: densities ρ_j, kinematic viscosities ν_j, thermal conductivity χ_j, thermal expansion coefficients β_j, $j = 1, 2$. At the interface $z = 0$, the surface tension depends linearly on temperature $\sigma(\theta_1) = \sigma_0 - æ\theta_1(x, y, 0, t)$ with constants σ_0 and $æ > 0$.

Remark 1. *In order for the interface to be flat, it is enough to assume the smallness of the Bond $Bo = g(\rho_1 - \rho_2)l_1^2/\sigma_0$ and the capillary $Ca = \mu_1\chi_1/\sigma_0 l_1$ numbers, see [13].*

The unknowns, according to (1), are the functions $f_j(z, t)$, $h_j(z, t)$, $a_j(z, t)$, $b_j(z, t)$, $q_j(z, t)$, and $-l_1 \leqslant z \leqslant 0$ for $j = 1$, and $j = 2$ for $0 \leqslant z \leqslant l_2$. Suppose that the temperature is set on the substrate $z = -l_1$

$$\theta_1(x, y, -l_1, t) = \alpha_1(t)x^2 + \alpha_2(t)y^2 + \alpha_3(t) \tag{2}$$

with known functions $\alpha_i(t)$, $i = 1, 2, 3$, and the upper wall $z = l_2$ is thermally insulated: $\theta_{2z}(x, y, l_2, t) = 0$. For $\alpha_1(t) < 0$, $\alpha_2(t) < 0$, solution (1), (2) describes convection near the critical point $x = 0$, $y = 0$, when the temperature on the wall at this point has a maximum and with inverse values $\alpha_1(t)$, $\alpha_2(t)$ the temperature has a minimum.

Let $a^* = \max\limits_{t\geqslant 0}(|\alpha_1(t)|, |\alpha_2(t)|)$, $\theta^* = \max\limits_{t\geqslant 0}|\alpha_3(t)|$ is a characteristic temperature, so that $a^* l_1$ is a characteristic temperature gradient, $\tau = \chi_1 l_1^{-2} t$ is a characteristic thermal convection time.

For the first layer at $j = 1$ we put

$$\begin{gathered}
\xi = \frac{z}{l_1}, \quad -1 < \xi < 0, \quad f_1 = \frac{\chi_1}{l_1^2}\mathrm{M}F_1(\xi,\tau), \quad h_1 = \frac{\chi_1}{l_1^2}\mathrm{M}H_1(\xi,\tau), \\
a_1 = a^* A_1(\xi,\tau), \quad b_1 = a^* B_1(\xi,\tau), \quad q_1 = \theta^* Q_1(\xi,\tau), \quad s_j = \frac{\chi_1^2}{l_1^4}\mathrm{M}S_j(\tau), \\
\mathrm{M} = \frac{æ_1 a^* l_1^3}{\mu_1\chi_1}, \quad \mathrm{P}_1 = \frac{\nu_1}{\chi_1}, \quad L_1 = \frac{\rho_1\beta_1 g l_1^2}{æ_1}, \quad d = \frac{a^* l_1^2}{\theta^*},
\end{gathered} \tag{3}$$

and for the second layer $j = 2$ we put

$$\begin{gathered}
\xi = \frac{z}{l_2}, \quad 0 < \xi < 1, \quad f_2 = \frac{\chi_1}{l_1^2}\mathrm{M}F_2(\xi,\tau), \quad h_2 = \frac{\chi_1}{l_1^2}\mathrm{M}H_2(\xi,\tau), \\
a_2 = a^* A_2(\xi,\tau), \quad b_2 = a^* B_2(\xi,\tau), \quad q_2 = \theta^* Q_2(\xi,\tau), \quad s_i = \frac{\chi_1^2}{l_1^4}\mathrm{M}S_k(\tau), \quad i = 3,4, \\
\mathrm{P}_2 = \frac{\nu_2}{\chi_2}, \quad L_2 = \frac{\rho_1\beta_2 g l_1 l_2}{æ_1}, \quad \chi = \frac{\chi_1}{\chi_2}, \quad l = \frac{l_1}{l_2}, \quad \mu = \frac{\mu_1}{\mu_2},
\end{gathered} \tag{4}$$

where M is a Marangoni number, P_1 , P_2 are the Prandtl numbers.

Suppose that $|\mathrm{M}| \ll 1$ and we will look for a solution in the form

$$\begin{gathered}
F_j = F_j^{(0)} + \mathrm{M}F_j^{(1)} + \ldots, \quad H_j = H_j^{(0)} + \mathrm{M}H_j^{(1)} + \ldots, \\
A_j = A_j^{(0)} + \mathrm{M}A_j^{(1)} + \ldots, \quad B_j = B_j^{(0)} + \mathrm{M}B_j^{(1)} + \ldots, \quad Q_j = Q_j^{(0)} + \mathrm{M}Q_j^{(1)} + \ldots, \\
S_i = S_i^{(0)} + \mathrm{M}S_i^{(1)} + \ldots, \quad j = 1,2, \quad i = \overline{1,4}, \quad n = \overline{1,3}.
\end{gathered}$$

Assuming that $L_j = O(1)$ with $\mathrm{M} \to 0$, we get a linear inverse problem in the zero approximation (index "0" is omitted)

$$\begin{gathered}
F_{1\tau} = \mathrm{P}_1 F_{1\xi\xi} - \mathrm{P}_1 L_1 \int_0^\xi (A_1(\xi,\tau) + B_1(\xi,\tau))\, d\xi - S_1(\tau), \\
H_{1\tau} = \mathrm{P}_1 H_{1\xi\xi} - \mathrm{P}_1 L_1 \int_0^\xi (A_1(\xi,\tau) - B_1(\xi,\tau))\, d\xi - S_2(\tau), \\
A_{1\tau} = A_{1\xi\xi}, \quad B_{1\tau} = B_{1\xi\xi}, \quad Q_{1\tau} = Q_{1\xi\xi} + 2d(A_1 + B_1), \quad -1 < \xi < 0, \quad \tau \in [0,\tau_0],
\end{gathered} \tag{5}$$

$$\begin{gathered}
F_{2\tau} = \frac{\mathrm{P}_2 l^2}{\chi} F_{2\xi\xi} - \mathrm{P}_1 L_2 \int_0^\xi (A_2(\xi,\tau) + B_2(\xi,\tau))\, d\xi - S_3(\tau), \\
H_{2\tau} = \frac{\mathrm{P}_2 l^2}{\chi} H_{2\xi\xi} - \mathrm{P}_1 L_2 \int_0^\xi (A_2(\xi,\tau) - B_2(\xi,\tau))\, d\xi - S_4(\tau), \\
A_{2\tau} = \frac{l^2}{\chi} A_{2\xi\xi}, \quad B_{2\tau} = \frac{l^2}{\chi} B_{2\xi\xi}, \quad Q_{2\tau} = \frac{l^2}{\chi} Q_{2\xi\xi} + \frac{2d}{\chi}(A_2 + B_2), \quad 0 < \xi < 1, \quad \tau \in [0,\tau_0].
\end{gathered} \tag{6}$$

The boundary conditions on solid walls $\xi = -1$, $\xi = 1$ are ($\tau \in [0,\tau_0]$)

$$\begin{gathered}
F_1(-1,\tau) = H_1(-1,\tau) = 0, \quad A_1(-1,\tau) = \overline{\alpha}_1(\tau), \\
B_1(-1,\tau) = \overline{\alpha}_2(\tau), \quad Q_1(-1,\tau) = \overline{\alpha}_3(\tau),
\end{gathered} \tag{7}$$

$$F_2(1,\tau) = H_2(1,\tau) = 0, \quad A_{2\xi}(1,\tau) = B_{2\xi}(1,\tau) = Q_{2\xi}(1,\tau) = 0. \tag{8}$$

The conditions on interface are

$$\begin{gathered}
F_1(0,\tau) = F_2(0,\tau), \quad H_1(0,\tau) = H_2(0,\tau), \quad A_1(0,\tau) = A_2(0,\tau), \\
B_1(0,\tau) = B_2(0,\tau), \quad Q_1(0,\tau) = Q_2(0,\tau), \\
lF_{2\xi}(0,\tau) - \mu F_{1\xi}(0,\tau) = \mu(A_1(0,\tau) + B_1(0,\tau)), \\
lH_{2\xi}(0,\tau) - \mu H_{1\xi}(0,\tau) = \mu(A_1(0,\tau) - B_1(0,\tau)), \\
lA_{2\xi}(0,\tau) - kA_{1\xi}(0,\tau) = 0, \quad lB_{2\xi}(0,\tau) - kB_{1\xi}(0,\tau) = 0, \quad lQ_{2\xi}(0,\tau) - kQ_{1\xi}(0,\tau) = 0.
\end{gathered} \tag{9}$$

In addition, the initial data are

$$\begin{gathered} F_j(\xi,0)=F_{0j}(\xi), \quad H_j(\xi,0)=H_{0j}(\xi),\\ A_j(\xi,0)=A_{0j}(\xi), \quad B_j(\xi,0)=B_{0j}(\xi), \quad Q_j(\xi,0)=Q_{0j}(\xi), \end{gathered} \tag{10}$$

where for $j=1$ the variable $\xi\in(-1,0)$, for $j=2$ we have $\xi\in(0,1)$; and the redefinition conditions

$$\int_{-1}^{0} F_1(\xi,\tau)\,d\xi=\int_{-1}^{0} H_1(\xi,\tau)\,d\xi=0, \quad \int_{0}^{1} F_2(\xi,\tau)\,d\xi=\int_{0}^{1} H_2(\xi,\tau)\,d\xi=0, \quad \tau\in[0,\tau_0]. \tag{11}$$

The equalities (11), meaning the closure of the flow, allow us to determine the unknown functions $S_i(\tau)$, $i=\overline{1,4}$.

Functions $\overline{\alpha}_1(\tau)=\alpha_1(t)/a^*$, $\overline{\alpha}_2(\tau)=\alpha_2(t)/a^*$, $\overline{\alpha}_3(\tau)\alpha_3(t)/\theta^*$, $F_{0j(\xi)}$, $H_{0j}(\xi)$, $A_{0j}(\xi)$, $B_{0j}(\xi)$, $Q_{0j}(\xi)$ are defined on their definition domains. For a smooth solution, they must satisfy the compatibility conditions, for example,

$F_{01}(-1)=H_{01}(-1)=0, \quad F_{02}(1)=H_{02}(1)=0, \quad F_{01}(0)=F_{02}(0), \quad H_{01}(0)=H_{02}(0)$ and so on.

The modified pressures in the layers are determined by the formulas

$$\begin{gathered} \bar{p}_1=\frac{\rho_1\nu_1\chi_1}{l_1^2}\mathrm{M}\Pi_1(\xi,\tau),\\ \Pi_1(\xi,\tau)=\left[2L_1\int_0^\xi A_1(\zeta,\tau)\,d\zeta+\frac{1}{\mathrm{P}_1}(S_1(\tau)+S_2(\tau))\right]\frac{\bar{x}^2}{2}+\left[2L_1\int_0^\xi B_1(\zeta,\tau)\,d\zeta+\right.\\ \left.+\frac{1}{\mathrm{P}_1}(S_1(\tau)-S_2(\tau))\right]\frac{\bar{y}^2}{2}-\frac{2F_1}{\mathrm{P}_1}+\frac{1}{\mathrm{P}_1}\int_0^\xi(\xi-\zeta)F_{1\tau}\,d\zeta+\frac{L_1}{d}\int_0^\xi Q_1\,d\zeta+\Pi_1(\tau),\\ \bar{p}_2=\frac{\rho_2\nu_2\chi_2}{l_2^2}\mathrm{M}\Pi_2(\xi,\tau),\\ \Pi_2(\xi,\tau)=\left[\frac{2\chi\nu L_2}{l^2}\int_0^\xi A_2\,d\zeta+\frac{\chi^2}{l^2\mathrm{P}_2}(S_3(\tau)+S_4(\tau))\right]\frac{\bar{x}^2}{2}+\left[\frac{2\chi\nu L_2}{l^2}\int_0^\xi B_2\,d\zeta+\right.\\ \left.+\frac{\chi^2}{l^2\mathrm{P}_2}(S_3(\tau)-S_4(\tau))\right]\frac{\bar{y}^2}{2}-\frac{2\chi F_2}{l^2}+\frac{2\chi^2}{l^4\mathrm{P}_2}\int_0^\xi(\xi-\zeta)F_{2\tau}\,d\zeta+\frac{\chi\nu L_2}{dl^2}\int_0^\xi Q_2\,d\zeta+\Pi_2(\tau), \end{gathered}$$

where $\Pi_1(\tau),\Pi_2(\tau)$ are arbitrary functions.

2. Stationary flow in layers

Let's find a stationary solution to the last problem $F_j^c(\xi)$, $H_j^c(\xi)$, $A_j^c(\xi)$, $B_j^c(\xi)$, $Q_j^c(\xi)$, $F_j^c(\xi)$, S_i^c $(j=1,2; i=\overline{1,4})$. After some calculations , we get explicit expressions

$$\begin{gathered} A_1^c(\xi)=\alpha_1^c, \quad B_1^c(\xi)=\alpha_2^c, \quad Q_1^c(\xi)=\alpha_3^c+d(\alpha_1^c+\alpha_2^c)\left[-\xi^2+\frac{2\xi}{kl}+\frac{1}{3}\left(1+\frac{2}{kl}\right)\right],\\ F_1^c(\xi)=(\alpha_1^c+\alpha_2^c)\left[\left(\frac{3\xi^2}{4}+\xi+\frac{1}{4}\right)\gamma+L_1\left(\frac{\xi^3}{6}+\frac{3\xi^2}{16}-\frac{1}{48}\right)\right],\\ H_1^c(\xi)=(\alpha_1^c-\alpha_2^c)\left[\left(\frac{3\xi^2}{4}+\xi+\frac{1}{4}\right)\gamma+L_1\left(\frac{\xi^3}{6}+\frac{3\xi^2}{16}-\frac{1}{48}\right)\right],\\ \gamma=\frac{1}{\mu+l}\left(-\mu+\frac{lL_1}{12}+\frac{\nu}{12l}L_2\right), \quad -1\leqslant\xi\leqslant 0; \end{gathered} \tag{12}$$

$$A_2^c(\xi) = \alpha_1^c, \quad B_2^c(\xi) = \alpha_2^c, \quad Q_2^c(\xi) = \alpha_3^c + d(\alpha_1^c + \alpha_2^c)\left[\frac{2\xi - \xi^2}{l^2} + \frac{1}{3}\left(1 + \frac{2}{kl}\right)\right],$$

$$F_2^c(\xi) = (\alpha_1^c + \alpha_2^c)\left[\left(\frac{3\xi^2}{4} - \xi + \frac{1}{4}\right)\gamma + \left(\frac{\xi^3}{6} - \frac{\xi^2}{4} + \frac{\xi}{12}\right)\frac{\nu L_2}{l^2} - \left(\frac{\xi^2}{16} - \frac{\xi}{12} + \frac{1}{48}\right)L_1\right], \tag{13}$$

$$H_2^c(\xi) = (\alpha_1^c - \alpha_2^c)\left[\left(\frac{3\xi^2}{4} - \xi + \frac{1}{4}\right)\gamma + \left(\frac{\xi^3}{6} - \frac{\xi^2}{4} + \frac{\xi}{12}\right)\frac{\nu L_2}{l^2} - \left(\frac{\xi^2}{16} - \frac{\xi}{12} + \frac{1}{48}\right)L_1\right],$$
$$0 \leqslant \xi \leqslant 1;$$

$$\begin{aligned}
S_1^c &= \frac{3\mathrm{P}_1(\alpha_1^c + \alpha_2^c)}{2(\mu + l)}\left(-\mu + (4l + 3\mu)L_1 + \frac{\nu}{12l}L_2\right),\\
S_2^c &= \frac{3\mathrm{P}_1(\alpha_1^c - \alpha_2^c)}{2(\mu + l)}\left(-\mu + (4l + 3\mu)L_1 + \frac{\nu}{12l}L_2\right),\\
S_3^c &= -\frac{3\mathrm{P}_1 l^2(\alpha_1^c + \alpha_2^c)}{2(\mu + l)}\left(\rho + \frac{\rho}{12}L_1 + \frac{(4\mu + 3l)}{12l^2}L_2\right),\\
S_4^c &= -\frac{3\mathrm{P}_1 l^2(\alpha_1^c - \alpha_2^c)}{2(\mu + l)}\left(\rho + \frac{\rho}{12}L_1 + \frac{(4\mu + 3l)}{12l^2}L_2\right).
\end{aligned} \tag{14}$$

The vertical velocities (dimensionless) are as follows

$$\begin{aligned}
W_1^c(\xi) &= -2\int_0^\xi F_1^c(\zeta)\,d\zeta = -2(\alpha_1^c + \alpha_2^c)\left[\left(\frac{\xi^3}{4} + \frac{\xi^2}{2} + \frac{\xi}{4}\right)\gamma + \left(\frac{\xi^4}{24} + \frac{\xi^3}{16} - \frac{\xi}{48}\right)L_1\right],\\
&\qquad -1 \leqslant \xi \leqslant 0,\\
W_2^c(\xi) &= -2\int_0^\xi F_2^c(\zeta)\,d\zeta = -2(\alpha_1^c + \alpha_2^c)\Big[\left(\frac{\xi^3}{4} - \frac{\xi^2}{2} + \frac{\xi}{4}\right)\gamma + \left(\frac{\xi^4}{24} - \frac{\xi^3}{12} + \frac{\xi^2}{24}\right)\frac{\nu L_2}{l^2} -\\
&\qquad - \left(\frac{\xi^3}{48} - \frac{\xi^2}{24} + \frac{\xi}{48}\right)L_1\Big], \quad 0 \leqslant \xi \leqslant 1,
\end{aligned} \tag{15}$$

Radial heating ($\alpha_1^c = \alpha_2^c$). When $H_1^c(\xi) = H_2^c(\xi) = 0$, $S_2^c = S_4^c = 0$ and the flow becomes axisymmetric. It is convenient to consider it in cylindrical coordinates (dimensionless) r, φ, ξ: $\mathbf{u}_j^c = (F_j^c(\xi)r, 0, W_j^c(\xi))$. Current functions $\Psi_j(r,\xi) : F_j^c(\xi)r = \dfrac{1}{r}\Psi_{j\xi}(r,\xi)$, $W_j^c(\xi) = -\dfrac{1}{r}\Psi_{jr}(r,\xi)$ and the mass conservation equation is fulfilled, so $\Psi_j(r,\xi) = r^2 W_j^c(\xi)$.

Remark 2. *For $\alpha_1 = -\alpha_2$ we have $F_1^c(\xi) = 0$, $F_2^c(\xi) = 0$, $W_1^c(\xi) = 0$, $W_2^c(\xi) = 0$ and the current will be plane parallel.*

Figs. 1–3 show the results of calculations of velocity fields for the transformer oil ($j = 1$)-water ($j = 2$) [15] system. Moreover, with $l_1 = l_2 = 5\,\mathrm{mm}$, $g = 9.8\,\mathrm{m/c^2}$ we have $L_1 = 0.6$, $L_2 = 2.3$ and $\mathrm{M} = 0.065a^*$. Fig. 1 shows the profile of the dimensionless vertical velocity component $W^c(\xi)$ (a) and velocity field in layers (b) at $\alpha_1^c = \alpha_2^c = 1°C/\mathrm{m}^2$, $l_1 = l_2 = 5\,\mathrm{mm}$, $g = 9.8\,\mathrm{m/c^2}$. It can be seen that in the first layer the flow is directed in the opposite direction of the z axis (ξ), and in the second layer the flow is directed in the direction of the z axis, as shown in Fig. 1b. A similar situation will occur for any case. Therefore, the results of calculations will be given below only for the vertical velocity component, since it gives an idea of the formed flows in the layers.

In Fig. 2 and the profiles of the dimensionless vertical velocity component $W^c(\xi) aregiven$ depending on α_2^c. If $\alpha_1^c + \alpha_2^c > 0$, then a return flow occurs in the first layer (the liquid moves

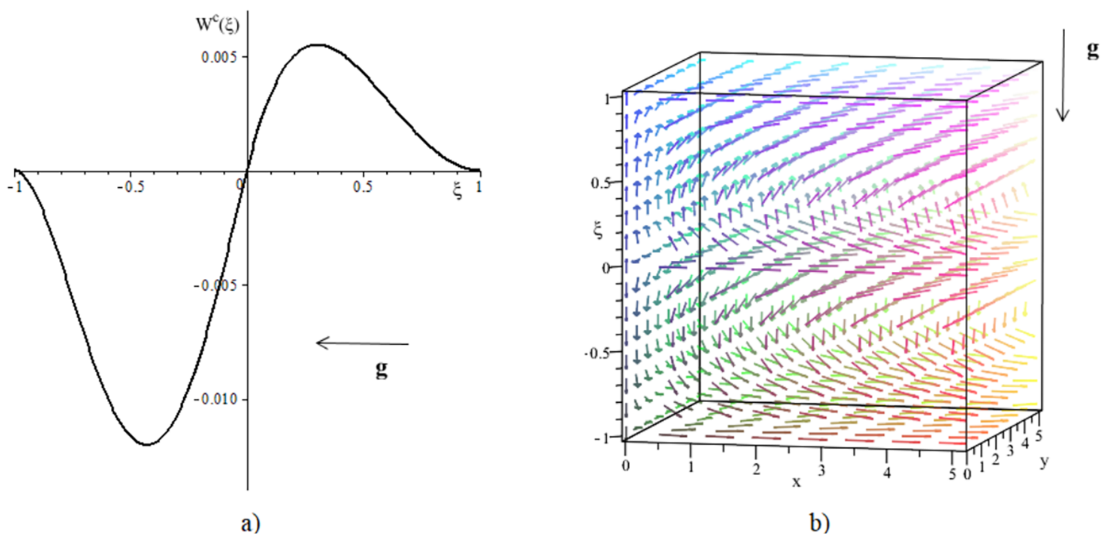


Fig. 1. Dimensionless vertical velocity component $W^c(\xi)$ (a) and the velocity field (b) at $\alpha_1^c = \alpha_2^c = 1$, $l_1 = l_2 = 5\,\text{мм}$, $g = 9.8\text{м}/\text{с}^2$, $a^* = 1\,{}^\circ C/\text{м}^2$

in the opposite direction of the z axis), and in the second layer the flow is directed along the z axis. If $\alpha_1^c + \alpha_2^c < 0$, then the direction of currents in the layers changes to the opposite.

In Fig. 2b shows the profiles of the dimensionless vertical component of the velocity $W^c(\xi)$ depending on the acceleration of gravity g. It can be seen that gravity affects only the intensity of the flow. Moreover, in the first layer, the intensity increases with the growth of g, and in the second it decreases. So $\max\limits_{\xi\in[-1,0]} |W^c(\xi, g = 0)| = 0.007$, $\max\limits_{\xi\in[-1,0]} |W^c(\xi, g = 9.8)| = 0.012$ и $\max\limits_{\xi\in[0,1]} |W^c(\xi, g = 0)| = 0.007$, $\max\limits_{\xi\in[0,1]} |W^c(\xi, g = 9.8)| = 0.005$.

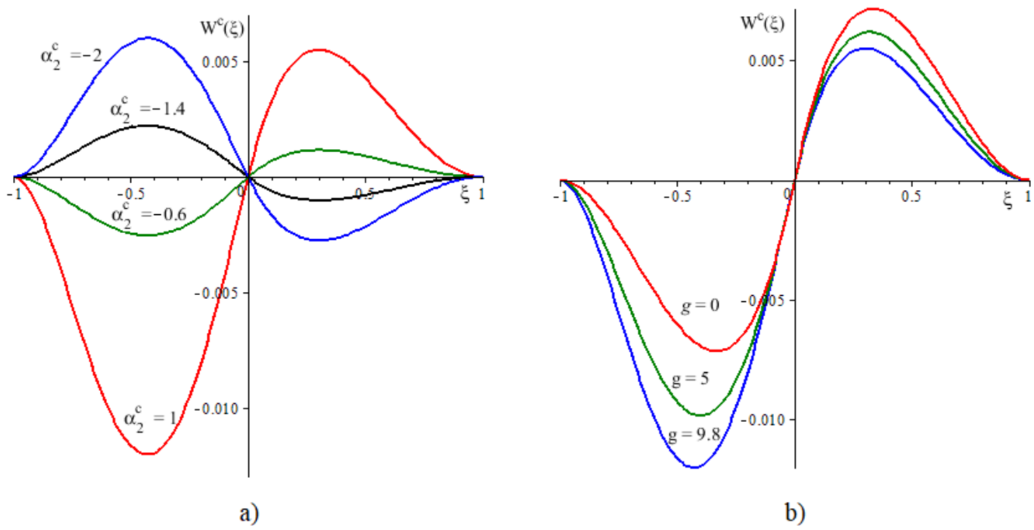


Fig. 2. Dimensionless vertical velocity component $W^c(\xi)$ depending on α_2^c(a) ($\alpha_1^c = 1$, $g = 9.8\text{m}/\text{c}^2$) and gravity acceleration g (b) ($\alpha_1^c = \alpha_2^c = 1$) at $l_1 = l_2 = 5\,\text{мм}$

Fig. 3a shows the profiles of the dimensionless vertical velocity component $W^c(\xi)$ depending on the thickness of the first layer l_1. It can be seen that with a decrease in l_1, the flow in the first layer remains recurrent (only the intensity changes), and in the second layer, at $l_1 < 0.1$ mm, the flow becomes two-vortex (a return flow occurs near a solid wall). At $l_1 < 10^{-6}$ mm, the flow in the second layer becomes completely reversible.

Fig. 3b shows the profiles of the dimensionless vertical velocity component $W^c(\xi)$ depending on the thickness of the second layer l_2. It can be seen that with a decrease in l_2, the intensity of flows in the layers decreases. So $\max\limits_{\xi\in[-1,0]} |W^c(\xi, l_2 = 5)| - \max\limits_{\xi\in[-1,0]} |W^c(\xi, l_2 = 0.03)| = \max\limits_{\xi\in[0,1]} |W^c(\xi, l_2 = 5)| - \max\limits_{\xi\in[0,1]} |W^c(\xi, l_2 = 0.03)| \approx 0.006$.

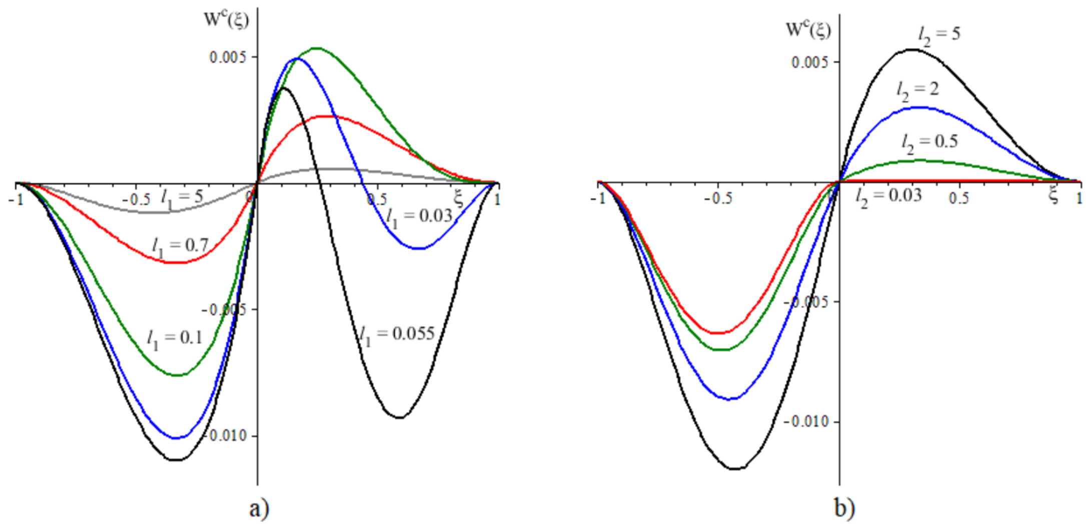


Fig. 3. Dimensionless vertical velocity component $W^c(\xi)$ depending on l_1 ($l_2 = 5$ мм)(a) and l_2 ($l_1 = 5$ мм) (b) at $\alpha_1 = \alpha_2 = 0.1$, $g = 9.8$м/с^2

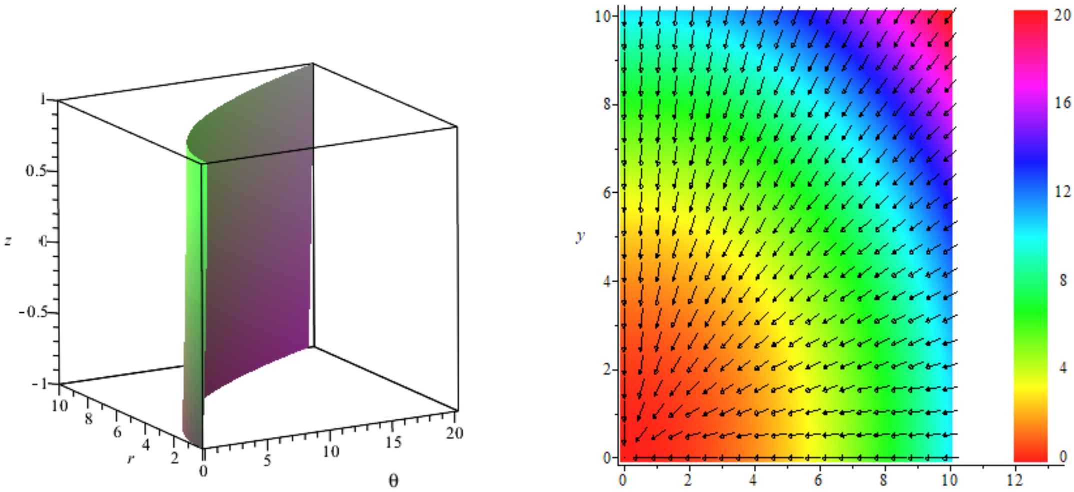
Fig. 4 shows a dimensionless temperature field in layers at $\alpha_1 = \alpha_2$ (radial heating). In this case, the temperature at the point $x = 0$, $y = 0$ is minimal. The surface tension decreases in the direction of the axes x, y and the flow on the interface is directed in the direction opposite to the direction of the axes x, y (Fig. 4b).

2. Unsteady convection in layers

The inverse problem (6)–(11) in Laplace images is solved in quadratures, which allows us to obtain quantitative information about the solution. Let $U(\xi, \tau)$ be the original, $\tau \in [0, \infty)$, $\xi \in [-1, 0]$ (or $\xi \in [0, 1]$), its Laplace transform (image) is integral

$$\hat{U}(\xi, s) = \in_0^\infty U(\xi, \tau)e^{-s\tau}\, d\tau.$$

The definition and properties of the Laplace transform are described in many manuals, see for example [14]. It is applicable to a wide class of functions, in particular, having a finite number of discontinuity points of the first kind with respect to the variable τ.



a) b)

Fig. 4. The dimensionless temperature field θ in coordinates r, z in layers (a) and the temperature distribution on the interface $z = 0$ (b) at $\alpha_1^c = \alpha_2^c = 1$, $g = 9.8\,\text{м/с}^2$

The problem for $A_j(\xi,\tau)$ in Laplace images will be written like this

$$
\begin{gathered}
\hat{A}_{1\xi\xi} - s\hat{A}_1 = -A_{01}(\xi), \quad \xi \in [-1,0], \\
\hat{A}_{2\xi\xi} - \frac{\chi}{l^2} s\hat{A}_2 = -\frac{\chi}{l^2} A_{02}(\xi), \quad \xi \in [0,1], \\
\hat{A}_1(0,s) = \hat{A}_2(0,s), \quad l\hat{A}_{2\xi}(0,s) - k\hat{A}_{1\xi}(0,s) = 0, \\
\hat{A}_1(-1,s) = \hat{\hat{\alpha}}_1(s), \quad \hat{A}_{2\xi}(1,s) = 0.
\end{gathered}
$$

The solution of this boundary value problem is written out without difficulty

$$
\begin{gathered}
\hat{A}_1(\xi,s) = \frac{1}{\Delta}\Big[\Big(\hat{\hat{\alpha}}_1(s) + \frac{1}{\sqrt{s}}\int_{-1}^{0} A_{01}(\xi)\,\mathrm{sh}\,\sqrt{s}(1+\xi)\,d\xi\Big)\Big(\mathrm{ch}\sqrt{\frac{s\chi}{l^2}}\,\mathrm{ch}\,\sqrt{s}\xi - \\
-\frac{\sqrt{\chi}}{k}\,\mathrm{sh}\sqrt{\frac{s\chi}{l^2}}\,\mathrm{sh}\,\sqrt{s}\xi\Big) - \frac{\chi}{kl\sqrt{s}}\,\mathrm{sh}[\sqrt{s}(1+\xi)]\int_0^1 A_{02}(\xi)\,\mathrm{ch}\sqrt{\frac{s\chi}{l^2}}(1-\xi)\,d\xi\Big] - \\
-\frac{1}{\sqrt{s}}\int_0^{\xi} A_{01}(\zeta)\,\mathrm{sh}\,\sqrt{s}(\xi-\zeta)\,d\zeta, \\
\hat{A}_2(\xi,s) = \frac{1}{\Delta}\Big[\Big(\hat{\hat{\alpha}}_1(s) + \frac{1}{\sqrt{s}}\int_{-1}^{0} A_{01}(\xi)\,\mathrm{sh}\,\sqrt{s}(1+\xi)\,d\xi\Big)\,\mathrm{ch}\sqrt{\frac{s\chi}{l^2}}(\xi-1) + \\
+\sqrt{\frac{\chi}{l^2 s}}\Big(\mathrm{sh}\sqrt{\frac{s\chi}{l^2}}\xi\,\mathrm{ch}\,\sqrt{s} + \frac{\sqrt{\chi}}{k}\,\mathrm{sh}\,\sqrt{s}\,\mathrm{ch}\sqrt{\frac{s\chi}{l^2}}\xi\Big)\int_0^1 A_{02}(\xi)\,\mathrm{ch}\sqrt{\frac{s\chi}{l^2}}(1-\xi)\,d\xi\Big] - \\
-\sqrt{\frac{\chi}{l^2 s}}\int_0^{\xi} A_{02}(\zeta)\,\mathrm{sh}\sqrt{\frac{s\chi}{l^2}}(\xi-\zeta)\,d\zeta, \\
\Delta = \mathrm{ch}\sqrt{\frac{s\chi}{l^2}}\,\mathrm{ch}\,\sqrt{s} + \frac{\sqrt{\chi}}{k}\,\mathrm{sh}\sqrt{\frac{s\chi}{l^2}}\,\mathrm{sh}\,\sqrt{s}.
\end{gathered} \tag{16}
$$

The Laplace transform $\hat{B}_1(\xi,s)$, $\hat{B}_2(\xi,s)$ is defined by the formulas (16) with the replacement of $\hat{\hat{\alpha}}_1(s)$ and $A_{0j}(\xi)$ by $\hat{\hat{\alpha}}_2(s)$ and $B_{0j}(\xi)$, $j = 1,2$ respectively. As for the functions $\hat{Q}_j(\xi,s)$, they are from (16) with the replacement of $A_{01}(\xi)$ and $A_{02}(\xi)$ by $Q_{01}(\xi)-2d(\hat{A}_1(\xi,s)+\hat{B}_1(\xi,s))$ and $\big[\chi Q_{20}(\xi)-2d(\hat{A}_2(\xi,s)+\hat{B}_2(\xi,s))\big]/l^2$ respectively.

Inverse problem for functions $\hat{F}_j(\xi,s)$, $\hat{S}_1(s)$, $\hat{S}_2(s)$ has the form

$$\begin{gathered}
\hat{F}_{1\xi\xi}-\frac{s}{\mathrm{P}_1}\hat{F}_1=\frac{1}{\mathrm{P}_1}\hat{S}_1(s)+L_1\hat{\Psi}_1(\xi,s)-\frac{1}{\mathrm{P}_1}F_{01}(\xi),\quad \xi\in[-1,0],\\
\hat{F}_{2\xi\xi}-\frac{s\chi}{\mathrm{P}_2l^2}\hat{F}_2=\frac{\chi}{\mathrm{P}_2l^2}\hat{S}_3(s)+\frac{\nu}{l^2}L_2\hat{\Psi}_2(\xi,s)-\frac{\chi}{\mathrm{P}_2l^2}F_{02}(\xi),\quad \xi\in[0,1],\\
\hat{F}_1(-1,s)=0,\quad \hat{F}_2(1,s)=0,\quad \hat{F}_1(0,s)=\hat{F}_2(0,s),\\
\mu\hat{F}_{1\xi}(0,s)-l\hat{F}_{2\xi}(0,s)=-\mu\hat{\Psi}_{1\xi}(0,s),\\
\int_{-1}^{0}\hat{F}_1(\xi,s)\,d\xi=0,\quad \int_{0}^{1}\hat{F}_2(\xi,s)\,d\xi=0,
\end{gathered}\tag{17}$$

where the functions $\hat{\Psi}_j$ are given by formulas $\hat{\Psi}_j=\int\limits_0^\xi(\hat{A}_j(\xi,s)+\hat{B}_j(\xi,s))\,d\xi$ with already known $\hat{A}_j$, $\hat{B}_j$, $j=1,2$. After some transformations, we find a solution to the problem (17):

$$\begin{gathered}
\hat{F}_1(\xi,s)=-\frac{\hat{S}_1}{s}+D_1\operatorname{sh}\sqrt{\frac{s}{\mathrm{P}_1}}\xi+D_2\operatorname{ch}\sqrt{\frac{s}{\mathrm{P}_1}}\xi+\\
+\sqrt{\frac{\mathrm{P}_1}{s}}\int_0^\xi\Big[L_1\hat{\Psi}_1(\zeta,s)-\frac{1}{\mathrm{P}_1}F_{01}(\zeta)\Big]\operatorname{sh}\sqrt{\frac{s}{\mathrm{P}_1}}(\xi-\zeta)\,d\zeta,\quad \xi\in[-1,0],\\
\hat{F}_2(\xi,s)=-\frac{\hat{S}_3}{s}+D_3\operatorname{sh}\beta\xi+D_4\operatorname{ch}\beta\xi+\frac{1}{\beta}\int_0^\xi\Big[\frac{\nu}{l^2}L_2\hat{\Psi}_2(\zeta,s)-\frac{\chi}{\mathrm{P}_2l^2}F_{02}(\zeta)\Big]\operatorname{sh}\beta(\xi-\zeta)\,d\zeta,\\
\xi\in[-1,0],\quad \beta=\sqrt{\frac{\chi s}{\mathrm{P}_2l^2}},
\end{gathered}\tag{18}$$

$$\begin{gathered}
\hat{S}_1=-sD_1\operatorname{sh}\sqrt{\frac{s}{\mathrm{P}_1}}+sD_2\operatorname{ch}\sqrt{\frac{s}{\mathrm{P}_1}}+\sqrt{\mathrm{P}_1s}\int_{-1}^0\Big[L_1\hat{\Psi}_1(\xi,s)-\frac{1}{\mathrm{P}_1}F_{01}(\xi)\Big]\operatorname{sh}\sqrt{\frac{s}{\mathrm{P}_1}}(1+\xi)\,d\xi,\\
\hat{S}_3=sD_3\operatorname{sh}\beta+sD_4\operatorname{ch}\beta+\frac{s}{\beta}\int_0^1\Big[\frac{\nu}{l^2}L_2\hat{\Psi}_2(\xi,s)-\frac{\chi}{\mathrm{P}_2l^2}F_{02}(\xi)\Big]\operatorname{sh}\beta(1-\xi)\,d\xi.
\end{gathered}\tag{19}$$

The values $D_i(s), i=\overline{1,4}$ are determined from the boundary conditions (7)–(9), (11), they have a bulky appearance and are not listed here.

The solution of the inverse problem for $\hat{H}_1$, $\hat{H}_2$, $\hat{S}_2$, $\hat{S}_4$ is determined by the formulas (18), (19) with obvious substitutions of $F_{01}(\xi)$, $F_{02}(\xi)$ and $\hat{\Psi}_1$, $\hat{\Psi}_2$ by $H_{01}(\xi)$, $H_{02}(\xi)$ and $\hat{\Psi}_3=$ $=\int\limits_0^\xi(\hat{A}_1(\xi,s)-\hat{B}_1(\xi,s))\,d\xi$, $\hat{\Psi}_4=\int\limits_0^\xi(\hat{A}_2(\xi,s)-\hat{B}_2(\xi,s))\,d\xi$ respectively.

Let there be limits $\lim\limits_{\tau\to\infty}\bar{\alpha}_j(\tau)=\alpha_j^c$, $j=1,2,3$. Then [9] $\lim\limits_{s\to0}s\hat{\bar{\alpha}}_j(s)=\alpha_j^c$. Using the asymptotic equalities $\operatorname{sh}x=x+x^3/3+O(x^5)$, $\operatorname{ch}x=1+x^2/2+O(x^4)$, $x\to0$ and the obtained formulas (16), (18), (19), it can be proved that

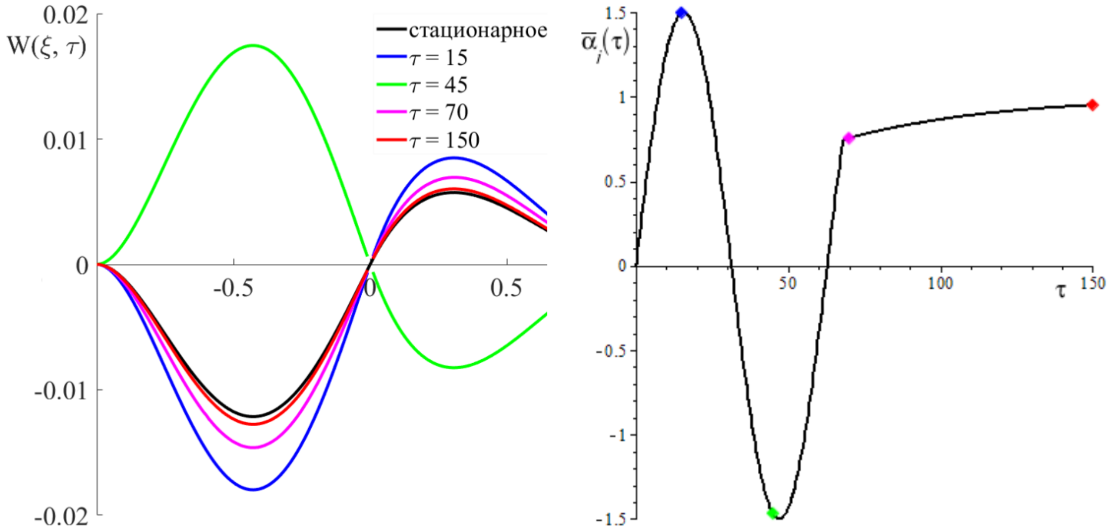
$$\begin{gathered}
\lim_{\tau\to\infty}A_j(\xi,\tau)=\lim_{s\to0}s\hat{A}_j(\xi,s)=\alpha_1^c,\quad \lim_{\tau\to\infty}B_j(\xi,\tau)=\lim_{s\to0}s\hat{B}_j(\xi,s)=\alpha_2^c,\\
\lim_{\tau\to\infty}F_j(\xi,\tau)=\lim_{s\to0}s\hat{F}_j(\xi,s)=F_j^c(\xi),\quad \lim_{\tau\to\infty}H_j(\xi,\tau)=\lim_{s\to0}s\hat{H}_j(\xi,s)=H_j^c(\xi),\\
\lim_{\tau\to\infty}Q_j(\xi,\tau)=\lim_{s\to0}s\hat{Q}_j(\xi,s)=Q_j^c(\xi),\quad \lim_{\tau\to\infty}S_i(\tau)=\lim_{s\to0}s\hat{S}_i(s)=S_i^c,\quad j=1,2,\quad i=\overline{1,4},
\end{gathered}$$

which confirms the theoretical conclusions.

Fig. 5 shows, for example, dimensionless velocity profiles $W(\xi,\tau)$ (Fig. 5a) for the case when

$$\bar{\alpha}_1(\tau) = \bar{\alpha}_2(\tau) = \begin{cases} \varepsilon_1 \sin \varepsilon_2 \tau & \text{при}\, 0 \leqslant \tau \leqslant \tau_1; \\ \alpha_1^c + e^{-\varepsilon_3 \tau} & \text{при}\, \tau > \tau_1 \end{cases}$$

and $\varepsilon_1 = 1.5$, $\varepsilon_2 = 0.1$, $\varepsilon_3 = 0.02$, $\tau_1 = 68$. At $0 < \tau \leqslant 68$, the temperature gradients on the lower wall change their sign, and at $\tau > 68$ they reach a constant value of $\alpha_1^c = \alpha_2^c = 1$ (Fig. 5 b). With $\tau = 20$, the real time is $t \approx 3500$ with.



a) b)

Fig. 5. Vertical component of the velocity vector $W(\xi, \tau)$ (a) and temperature gradients $\bar{\alpha}_j(\tau)$ (b)

Conclusion

A linear model describing slow two-layer convection in a 3D layer is constructed. It takes into account both the influence of thermocapillary forces and the change in buoyancy forces in the layers. From a mathematical point of view, the resulting initial-boundary value problem is the inverse. It's stationary solution has been found, which makes it possible to trace the influence of dimensionless parameters on the structure of flows in layers. The solution of the non-stationary problem is obtained in Laplace images in the form of quadratures. It is proved that if the set temperature on the lower wall stabilizes with time, then the non-stationary solution goes to a stationary mode with increasing time. Which means it's stability.

This work is supported by the Krasnoyarsk Mathematical Center and financed by the Ministry of Science and Higher Education of the Russian Federation in the framework of the establishment and development of regional Centers for Mathematics Research and Education (Agreement No. 075–02–2023–912).

References

[1] C.C.Lin, Note on a class of exact solutions in magnetohydrodynamics, *Arch. Rational Mech. Anal.*, **1**(1958), 391–395.

[2] A.F.Sidorov, On two classes of solutions of equations of fluid and gas mechanics and their connection with the theory of traveling waves, *Applied Mechanics and Technical Physics*, (1989), no. 2, 34–40.

[3] V.V.Pukhnachev, Model of viscous layer deformation by thermocapillary forces, *Eur. Journal Appl. Math.*, **13**(2002), no. 2, 205–224. DOI: 10.1017/S0956792501004776

[4] V.K.Andreev, Yu.A.Gaponenko, O.N.Goncharova, V.V.Pukhnachev, Mathematical Models of Convection, Berlin/Boston: De Gruyter, 2020.

[5] E.V.Rezanova, Numerical medelling of heat transfer in the layer of viscous incompressible liquid with free boundaries, *EPJ Web of Conference*, (2017), no. 159, 00047.
DOI: 10.1051/epjconf/201715900047

[6] S.N.Aristov, D.V.Knyazev, A.D.Polyanin, Exact solutions of the Navier-Stokes equations with linear dependence of the velocity components on two spatial variables, *Theoretical Foundations of Chemical Engineering*, **43**(2009), no. 5, 547–566. DOI:10.1134/S0040579509050066

[7] A.A.Azanov, V.K.Andreev, Solving the problem of the creeping motion of a fluid with a free boundary with a special velocity field in a three-dimensional strip, Some actual problems of modern mathematics and mathematical education. Herzen Readings — 2021. Materials of the scientific conference. St. Petersburg: Publishing House of the A.I. Herzen State Pedagogical University. Publishing House of the VVM, 2021, 42–54.

[8] V.K.Andreev, E.N.Lemeshkova, Two-layer steady creeping thermocapillary flow in a three-dimensional channel, *Journal of Applied Mechanics and Technical Physics*, **63**(2022), no. 1, 82–88. DOI:10.1134/S0021894422010138

[9] V.K.Andreev, On a creeping 3D Convective Motion of Fluids with an Isothermal Interface, *Journal of Siberian Federal University. Mathematics & Physics*, **13**(2020), no. 6, 661–669.
DOI: 10.17516/1997-1397-2020-13-6-661-669

[10] V.K.Andreev, The solution of three-dimensional equations of thermal convection and its interpretation, Some actual problems of modern mathematics and mathematical education. Herzen Readings — 2020. Materials of the scientific conference. St. Petersburg: Publishing House of the A. I. Herzen State Pedagogical University, VVM, 2021, 4–8.

[11] V.K.Andreev, E.N.Lemeshkova, Thermal convection of two immiscible liquids in a three-dimensional channel with a velocity field of a special type, *PMM*, **87**(2023), no. 2, 200–210 (in Russian). DOI: 10.31857/S0032823523020029

[12] V.K.Andreev, E.N.Lemeshkova, Linear problems of convective motions with interface surfaces, Krasnoyarsk: Sib. Fed. Univ, 2018.

[13] R.H.Zeitunyan, The problem of thermocapillary instability of Benara-Marangoni, *UFN*, **168**(1998), 259–286 (in Russian). DOI: 10.3367/UFNr.0168.199803b.0259

[14] M.A.Lavrentiev, B.V.Shabat, Methods of the theory of functions of a complex variable, Moscow, Nauka, 1973.

[15] N.B.Vargaftik, Handbook of Thermophysical Properties of Gases and Liquids, Nauka, 1972.

Термокапиллярная конвекция несмешивающихся жидкостей в трехмерном слое при малых числах Марангони

Виктор К. Андреев
Институт вычислительного моделирования СО РАН
Красноярск, Российская Федерация

Аннотация. Изучается совместная конвекция двух вязких теплопроводных жидкостей в трёхмерном слое, ограниченном твердыми плоскими стенками. Верхняя стенка теплоизолирована, а на нижней стенке задано нестационарное поле температур. Жидкости предполагаются несмешивающимися, и на плоской границе раздела между ними заданы сложные условия сопряжения. Эволюция этой системы описывается уравнениями Обербека-Буссинеска в каждой жидкости. Решение указанной задачи ищется в классе полей скоростей, линейных по двум координатам, а поля температур — квадратичные функции тех же координат. В этом случае задача редуцируется к системе 10-ти нелинейных интегродифференциальных уравнений. Она является сопряженной и обратной относительно 4-х функций времени. Для их нахождения ставятся интегральные условия переопределения, имеющие ясный физический смысл — замкнутость потока. Поставленная обратная начально-краевая задача описывает конвекцию в двухслойной системе, возникающую вблизи точки экстремума температуры на нижней твердой стенке. При малых числах Марангони задача аппроксимируется линейной (число Марангони играет роль числа Рейнольдса для уравнений Навье-Стокса). Найдено стационарное решение этой задачи. Линейная нестационарная задача решена методом преобразования Лапласа, причем температура может иметь разрывы 1-го рода — изменяться скачком. В образах по Лапласу решение получено в квадратурах. Доказано, что с ростом времени оно выходит на стационарный режим, если температура на нижней стенке стабилизируется со временем. С помощью численного обращения преобразования Лапласа изучена эволюция поведения поля скоростей в системе трансформаторное масло – вода.

Ключевые слова: уравнения Обербека-Буссинеска, поверхность раздела, число Марангони, термокапиллярность, обратная задача, преобразование Лапласа.