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Reduction of the Cosserat-type Nonlinear Equations to the System of Godunov's Form

Vladimir M. Sadovskii*
Institute of Computational Modelling SB RAS
Krasnoyarsk, Russian Federation

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Abstract. The complete system of equations for the dynamics of a Cosserat-type continuum with couple stresses under finite strains and particle rotations in Lagrangian variables is reduced to a compatible system of conservation laws in the Godunov sense. This system enables analyzing generalized solutions with surfaces of strong discontinuity of stresses and velocities and allows integral estimates that guarantee the uniqueness and continuous dependence of solutions of the Cauchy problem and boundary-value problems with dissipative boundary conditions on the initial data.

Keywords: elasticity, Cosserat continuum, couple stresses, curvature tensor.

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Introduction

Analysis of the correctness of setting boundary value problems for equations of a mathematical model is a fundamentally important step in the transition to the development of methods for numerical implementation. Special classes of equations and systems play an exceptional role in this process. One of these classes is formed by hyperbolic systems of conservation laws that are thermodynamically compatible according to Godunov.

Thermodynamically compatible systems were introduced by S. K. Godunov and applied by him with his followers for analysis the models of gas dynamics, theory of elasticity and some coupled problems [1–3]. Such form of equations assumes the setting so-called generating potentials $L^0(\boldsymbol{U})$ and $L^j(\boldsymbol{U})$ ($j = 1, \dots, n$, where n is the spatial dimension of a model) depending on the vector $\boldsymbol{U}$, whose components are projections of the velocity vector, components of the stress tensor and other thermodynamic state parameters. By means of generating potentials the system is written in divergent form as follows:

$$\frac{\partial \boldsymbol{\varphi}_0(\boldsymbol{U})}{\partial t} = \sum_{j=1}^{n} \frac{\partial \boldsymbol{\varphi}_j(\boldsymbol{U})}{\partial x_j}, \qquad \boldsymbol{\varphi}_0 = \frac{\partial L^0(\boldsymbol{U})}{\partial \boldsymbol{U}}, \quad \boldsymbol{\varphi}_j = \frac{\partial L^j(\boldsymbol{U})}{\partial \boldsymbol{U}}, \tag{1}$$

or in a more general form, including terms that are independent of derivatives. The additional conservation law

$$\frac{\partial}{\partial t}\Big(\boldsymbol{U}\cdot\boldsymbol{\varphi}_0 - L^0\Big) = \sum_{j=1}^{n} \frac{\partial}{\partial x_j}\Big(\boldsymbol{U}\cdot\boldsymbol{\varphi}_j - L^j\Big) \tag{2}$$

*sadov@icm.krasn.ru https://orcid.org/0000-0001-9695-0032

is valid for the system (1). The equation (2) may be a conservation law of energy or of entropy.

Thermodynamically compatible systems of the form (1), (2) turn out to be very useful for justifying the mathematical correctness of models. If the generating potential $L^0(\boldsymbol{U})$ is a strongly convex function, then the system (1) belongs to the hyperbolic type. It is characterized by a finite speed of propagation of disturbances and a limited area of dependence of solutions. Based on such formulation, a priori estimates of solutions in characteristic cones can be obtained, from which it follows the uniqueness and continuous dependence on initial data for the Cauchy problem and for boundary value problems with dissipative boundary conditions. It is intended for the integral generalization of the model, which allows to construct discontinuous solutions. For numerical analysis of the system (1), (2) the effective shock-capturing methods, such as Godunov's method, adapted to the computation of solutions with discontinuities, caused by concentrated and impulsive perturbations, may be applied.

The model of Cosserat continuum [4–6] is used to describe mechanical behavior of materials having microstructure (soil, rocks, granular and porous media, media with microcracks, liquid crystals), when subjected to deformation. In this model, in contrast to the classical theory of elasticity, where the medium is a continuum of material points, it is a continuum of particles, which are rigid bodies of small volume with translational and rotational degrees of freedom.

The authors of many studies have developed various methods to generalize constitutive equations of the Cosserat theory for finite strains and particle rotations. An exhaustive survey of the related research is given in [7].

In [8] the problem of waves propagation in a blocky medium is solved in the framework of the Cosserat theory by numerical methods on supercomputers of cluster architecture. In [9, 10] a plane system of equations is used to construct a model of the dynamics of a liquid crystal under the influence of weak mechanical, electrical and thermal disturbances. In a short note [11] equations of three-dimensional dynamics have been reduced to a thermodynamically compatible system of conservation laws. The scope of study encompasses a special case of a reduced Cosserat model with couple stresses assumed negligible and a version of a general model of physically and geometrically nonlinear medium with couple stresses, where curvature tensor has a rate of change equal to a gradient of an angular rate vector. The purpose of present study is a more complete presentation of the material and improvement of the nonlinear model by means of a special selection of equations for calculating the measure of curvature from the characteristics of the rotational motion of particles.

1. Kinematics in a medium with couple stresses

The translational motion of a particle in a medium possessing microstructure is described by an equation $\boldsymbol{x} = \boldsymbol{\xi} + \boldsymbol{u}$, connecting the Lagrangian $\boldsymbol{\xi}$ and Eulerian $\boldsymbol{x}$ vectors of centers of masses with the displacement vector $\boldsymbol{u}(\boldsymbol{\xi}, t)$. The independent rotation is defined by an orthogonal rotation tensor $\boldsymbol{R}(\boldsymbol{\xi}, t)$:

$$\boldsymbol{R} \cdot \boldsymbol{R}^* = \boldsymbol{I}, \quad \det \boldsymbol{R} = +1, \quad \dot{\boldsymbol{R}} \cdot \boldsymbol{R}^* + \boldsymbol{R} \cdot \dot{\boldsymbol{R}}^* = \boldsymbol{0}.$$

The antisymmetric tensor of angular velocity of a particle is calculated by the formula: $\boldsymbol{\Omega} = \dot{\boldsymbol{R}} \cdot \boldsymbol{R}^*$. As a measure of deformation of an infinitely small element, it is assumed to take the tensor $\boldsymbol{\Lambda} = \boldsymbol{R}^* \cdot \boldsymbol{x}_{\boldsymbol{\xi}}$, having such a property that if a medium moves as a rigid whole, when the distortion tensor $\boldsymbol{x}_{\boldsymbol{\xi}}$ coincides with the rotation tensor $\boldsymbol{R}$, then $\boldsymbol{\Lambda}$ equals to unit tensor, which conforms with the undeformed state of the element. By differentiating with respect to time, it is found that the latter tensor satisfies the equation:

$$\boldsymbol{R} \cdot \dot{\boldsymbol{\Lambda}} = \boldsymbol{v}_{\boldsymbol{\xi}} - \boldsymbol{\Omega} \cdot \boldsymbol{x}_{\boldsymbol{\xi}} \tag{3}$$

($\boldsymbol{v} = \dot{\boldsymbol{x}}$ is the vector of velocity of translational motion). The linear approximation of (3) precisely coincides with the kinematic equation for the strain rate tensor in the geometrically linear model by Cosserat. Moreover, it is possible to show that $\boldsymbol{\Lambda}$ is an invariant tensor unchangeable under rotation of the current configuration. This property is bound to be fulfilled in the Lagrangian description of motion.

Actually, if $\boldsymbol{O}$ is an orthogonal transformation of rotation of the current configuration, then

$$d\boldsymbol{x}' = \boldsymbol{O} \cdot d\boldsymbol{x} = \boldsymbol{O} \cdot \boldsymbol{x}_{\boldsymbol{\xi}} \cdot d\boldsymbol{\xi} = \boldsymbol{O} \cdot \boldsymbol{R} \cdot \boldsymbol{\Lambda} \cdot d\boldsymbol{\xi},$$

and, accordingly, $\boldsymbol{x}'_{\boldsymbol{\xi}} = \boldsymbol{R}' \cdot \boldsymbol{\Lambda}'$, where $\boldsymbol{R}' = \boldsymbol{O} \cdot \boldsymbol{R}$, $\boldsymbol{\Lambda}' = \boldsymbol{\Lambda}$.

Let $\boldsymbol{x}_{\boldsymbol{\xi}} = \boldsymbol{R}_e \cdot \boldsymbol{C}$ be the polar decomposition of the distortion tensor into a product of the orthogonal tensor $\boldsymbol{R}_e$, describing translatory rotation of a medium element, and the symmetric Cauchy – Green tensor $\boldsymbol{C}$, describing deformation of this element. Inasmuch as the particle rotation $\boldsymbol{R} = \boldsymbol{R}_e \cdot \boldsymbol{R}_r$ is the superposition of the relative $\boldsymbol{R}_r$ and translatory $\boldsymbol{R}_e$ rotations, the tensor $\boldsymbol{\Lambda} = \boldsymbol{R}^* \cdot \boldsymbol{R}_e \cdot \boldsymbol{C} = \boldsymbol{R}_r^* \cdot \boldsymbol{C}$, by structure, accounts for both the medium element distortion and the relative particle rotation. This property of the tensor $\boldsymbol{\Lambda}$ is in full compliance with the common view of the kinematics of the structurally inhomogeneous continuum composed of small-volume material particles.

If a particle makes complete revolution about fixed axis and returns to initial position, then the tensor $\boldsymbol{R}$ equals a unit tensor. Consequently, given such description, the complete revolution of a particle entails no change in a strain state, which is typical, e.g., for micropolar media representing large ensembles of magnetized particles in external magnetic field.

Based on the known theorem, it is valid to represent orthogonal tensors as:

$$\boldsymbol{R} = \boldsymbol{I} + \sin\psi\, \boldsymbol{P} + (1 - \cos\psi)\, \boldsymbol{P}^2, \tag{4}$$

where ψ is the angle of rotation, calculated in terms of the trace of the tensor $\boldsymbol{R}$ by the equation: $\cos\psi = (\operatorname{tr}\boldsymbol{R} - 1)/2$, and $\boldsymbol{P}$ is the antisymmetric tensor associated with the orientation of the instantaneous axis of rotation $\boldsymbol{p} = (p_1,\ p_2,\ p_3)$:

$$\boldsymbol{P} = \begin{pmatrix} 0 & -p_3 & p_2 \\ p_3 & 0 & -p_1 \\ -p_2 & p_1 & 0 \end{pmatrix}, \quad p_1^2 + p_2^2 + p_3^2 = 1.$$

2. Governing equations

The description of the strain state in a medium with couple stresses, aside from the tensor $\boldsymbol{\Lambda}$, uses a special curvature tensor $\boldsymbol{M}$, calculated in terms of the rotation tensor $\boldsymbol{R}$ and its derivatives in the Lagrangian coordinates $\boldsymbol{R}_{,k} = \partial\boldsymbol{R}/\partial\xi_k$ $(k = 1, 2, 3)$. Let $\boldsymbol{M}^{(k)} = \boldsymbol{R}_{,k} \cdot \boldsymbol{R}^*$ be the antisymmetric curvature tensors along the coordinate lines. The Darboux vectors fitting with these tensors are assigned by the columns of $\boldsymbol{M}$:

$$\boldsymbol{M} = \begin{pmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{pmatrix}, \quad \boldsymbol{M}^{(k)} = \begin{pmatrix} 0 & -M_{3k} & M_{2k} \\ M_{3k} & 0 & -M_{1k} \\ -M_{2k} & M_{1k} & 0 \end{pmatrix}.$$

Differentiating $\boldsymbol{M}^{(k)}$ with respect to time and $\boldsymbol{\Omega}$ with respect to the variables ξ_k yields kinematic equations $\dot{\boldsymbol{M}}^{(k)} = \boldsymbol{\Omega}_{,k} + \boldsymbol{\Omega}\cdot\boldsymbol{M}^{(k)} - \boldsymbol{M}^{(k)}\cdot\boldsymbol{\Omega}$ that admit, collectionwise, the tensor representation:

$$\dot{\boldsymbol{M}} = \boldsymbol{\omega}_{\boldsymbol{\xi}} + \boldsymbol{\Omega}\cdot\boldsymbol{M}. \tag{5}$$

The validity of this representation is readily tested with the componentwise writing of the tensors in the Cartesian coordinate system. It follows from (5) that $\boldsymbol{M}$ is neither an invariant nor an indifferent tensor, i.e., it changes both under rotation of the current configuration and under rotation of the original configuration. Therefore this tensor cannot be accepted in our model as an objective measure of curvature.

It can be shown that under rotation of the current configuration $d\boldsymbol{x}' = \boldsymbol{O}\cdot d\boldsymbol{x}$ this tensor transforms in accordance with the law: $\boldsymbol{M}' = \boldsymbol{O}\cdot\boldsymbol{M}$. In fact, since the rotation tensor $\boldsymbol{O}$ is independent on time, then

$$\boldsymbol{R}' = \boldsymbol{O}\cdot\boldsymbol{R}, \quad \boldsymbol{\Omega}' = \dot{\boldsymbol{R}}'\cdot\boldsymbol{R}'^{*} = \boldsymbol{O}\cdot\dot{\boldsymbol{R}}\cdot\boldsymbol{R}^{*}\cdot\boldsymbol{O}^{*} = \boldsymbol{O}\cdot\boldsymbol{\Omega}\cdot\boldsymbol{O}^{*}, \quad \boldsymbol{\omega}' = \boldsymbol{O}\cdot\boldsymbol{\omega}.$$

Consequently, Eq. (5) reduces to the equation $\dot{\boldsymbol{M}}' = \boldsymbol{O}\cdot\boldsymbol{\omega}_{\boldsymbol{\xi}} + \boldsymbol{O}\cdot\boldsymbol{\Omega}\cdot\boldsymbol{O}^{*}\cdot\boldsymbol{M}'$, having the solution: $\boldsymbol{M}' = \boldsymbol{O}\cdot\boldsymbol{M}$.

By the same law goes the distortion tensor $\boldsymbol{x}_{\boldsymbol{\xi}}$, e.g., which is used to determine the invariant strain measure $\boldsymbol{x}^{*}_{\boldsymbol{\xi}}\cdot\boldsymbol{x}_{\boldsymbol{\xi}}$, involved in the Lagrangian representation of motion in a classic elastic medium, and an indifferent measure $\boldsymbol{x}_{\boldsymbol{\xi}}\cdot\boldsymbol{x}^{*}_{\boldsymbol{\xi}}$, included in the Eulerian representation, [12]. The both measures are independent on rotation of a medium element as a rigid whole, which does not influence the potential energy of deformation. Similarly, the invariance is the property of the product $\boldsymbol{M}^{*}\cdot\boldsymbol{M}$, that will be used as an independent parameter of state to construct constitutive equations accounting for the couple properties of a medium, and that leads to a thermodynamically compatible system of conservation laws, as it will be illustrated below.

It is noteworthy that the selected curvature measure differs from the conventionally used measures [7], defined by nonsymmetric invariant tensors. As judged by the analogy with the strain measure, the symmetrized measure $\boldsymbol{M}^{*}\cdot\boldsymbol{M}$ eliminates "excessive" degrees of freedom, having no influence on potential energy of strain state.

The system of equations of the dynamics of a medium with couple stresses is constructed based on the integral laws of impulse, momentum and energy conservation in the Lagrangian form:

$$\begin{gathered}\frac{\partial}{\partial t}\int_V \rho_0\,\boldsymbol{v}\,dV = \int_S \boldsymbol{\sigma}\cdot\boldsymbol{\nu}\,dS + \int_V \boldsymbol{f}\,dV,\\ \frac{\partial}{\partial t}\int_V \big(\boldsymbol{J}\cdot\boldsymbol{\omega} + \rho_0\,\boldsymbol{x}\times\boldsymbol{v}\big)\,dV = \int_S (\boldsymbol{x}\times\boldsymbol{\sigma} + \boldsymbol{m})\cdot\boldsymbol{\nu}\,dS + \int_V \big(\boldsymbol{x}\times\boldsymbol{f} + \boldsymbol{g}\big)\,dV,\\ \frac{\partial}{\partial t}\int_V \Big(\rho_0\,\frac{\boldsymbol{v}\cdot\boldsymbol{v}}{2} + \frac{1}{2}\,\boldsymbol{\omega}\cdot\boldsymbol{J}\cdot\boldsymbol{\omega} + \Phi\Big)dV = \int_S (\boldsymbol{v}\cdot\boldsymbol{\sigma} + \boldsymbol{\omega}\cdot\boldsymbol{m} - \boldsymbol{q})\cdot\boldsymbol{\nu}\,dS + \\ + \int_V (\boldsymbol{v}\cdot\boldsymbol{f} + \boldsymbol{\omega}\cdot\boldsymbol{g} + Q)\,dV.\end{gathered} \tag{6}$$

Here V is an arbitrary domain with a piecewise-smooth boundary S, delineated when the medium was in the original (undeformed) state, $\boldsymbol{\nu}$ is the vector of external normal to the boundary, ρ_0 is the initial density, $\boldsymbol{J}$ is the symmetric and positively definite inertia tensor, $\boldsymbol{\sigma}$ is the Piola–Kirchhoff stress tensor, $\boldsymbol{m}$ is the couple stress tensor, Φ is the internal energy of a medium per unit volume, $\boldsymbol{q}$ is the heat flux vector, $\boldsymbol{f}$ and $\boldsymbol{g}$ are the bulk densities of mass forces and couple forces, Q is the intensity of internal heat sources.

As a medium moves, the domain V composed of material particles passes into deformed state V_t, the mass of the substance remains unaltered: $\rho_0\, dV = \rho\, dV_t$, the density changes in conformity with the law: $\rho = \rho_0/\det \boldsymbol{x_\xi}$, and the inertia tensor of particles in a unit volume transforms as is given by the formula: $\boldsymbol{J}_t = (\rho/\rho_0)\,\boldsymbol{J}$. The tensor $\boldsymbol{J}$ related to the initial state changes with time in accordance with the equation: $\boldsymbol{J} = \boldsymbol{R}\cdot\boldsymbol{J}^0\cdot\boldsymbol{R}^*$, which can be evaluated by passing to an associated coordinate system connected with a rotating particle. The time differentiation yields the following equation for the inertia tensor:

$$\dot{\boldsymbol{J}} = \boldsymbol{\Omega}\cdot\boldsymbol{J} - \boldsymbol{J}\cdot\boldsymbol{\Omega}. \tag{7}$$

For the continuous motions, the integral conservation laws are equivalent to the differential equations derivable from (6) using the Green formula:

$$\begin{gathered}\rho_0\,\dot{\boldsymbol{v}} = \operatorname{div}_{\boldsymbol{\xi}}\boldsymbol{\sigma} + \boldsymbol{f}, \quad \frac{\partial}{\partial t}\big(\boldsymbol{J}\cdot\boldsymbol{\omega}\big) = \operatorname{div}_{\boldsymbol{\xi}}\boldsymbol{m} + 2\,(\boldsymbol{\sigma}\cdot\boldsymbol{x}_{\boldsymbol{\xi}}^*)^a + \boldsymbol{g},\\ \dot{\Phi} = \boldsymbol{\sigma}^* : (\boldsymbol{v_\xi} - \boldsymbol{\Omega}\cdot\boldsymbol{x_\xi}) + \boldsymbol{m}^* : \boldsymbol{\omega_\xi} - \operatorname{div}_{\boldsymbol{\xi}}\boldsymbol{q} + Q.\end{gathered} \tag{8}$$

Hereinafter $\operatorname{div}_{\boldsymbol{\xi}}$ is the operator of divergence with respect to Lagrangian variables, the superscript "a" denotes a vector corresponding to the antisymmetric part of a tensor. Derivation of (8) made using the equality $\boldsymbol{\omega}\cdot\dot{\boldsymbol{J}}\cdot\boldsymbol{\omega} = 0$, which is a corollary of Eq. (7).

For the reversible processes, the state of which is characterized with the thermodynamic parameters represented by the strain measure $\boldsymbol{\Lambda}$, curvature measure $\boldsymbol{M}$ and entropy s, the latter equation in the system (8), rewritted with regard to (3) and (5) as

$$\frac{\partial\Phi}{\partial\boldsymbol{\Lambda}^*} : \dot{\boldsymbol{\Lambda}} + \frac{\partial\Phi}{\partial\boldsymbol{M}^*} : \dot{\boldsymbol{M}} + T\dot{s} = \boldsymbol{\sigma}^* : \big(\boldsymbol{R}\cdot\dot{\boldsymbol{\Lambda}}\big) + \boldsymbol{m}^* : \big(\dot{\boldsymbol{M}} - \boldsymbol{\Omega}\cdot\boldsymbol{M}\big) - \operatorname{div}_{\boldsymbol{\xi}}\boldsymbol{q} + Q,$$

where $T = \partial\Phi/\partial s$ is the absolute temperature, decomposes, due to linear independence of the values $\dot{\boldsymbol{\Lambda}}$, $\dot{\boldsymbol{M}}$ (variations of strain state), into the constitutive equations:

$$\boldsymbol{R}^*\cdot\boldsymbol{\sigma} = \frac{\partial\Phi}{\partial\boldsymbol{\Lambda}}, \quad \boldsymbol{m} = \frac{\partial\Phi}{\partial\boldsymbol{M}}, \tag{9}$$

heat influx equation

$$T\dot{s} = -\operatorname{div}_{\boldsymbol{\xi}}\boldsymbol{q} + Q \tag{10}$$

and a complementary equation $\boldsymbol{m}^* : \big(\boldsymbol{\Omega}\cdot\boldsymbol{M}\big) = 0$.

In view of the linear independence of the projections of the angular velocity vector, the complementary equation reduces to the symmetry condition for the tensor $\boldsymbol{m}\cdot\boldsymbol{M}^*$, confining general relationship between the elastic potential Φ and the curvature tensor $\boldsymbol{M}$. With a pass to the coordinate representation, it is possible to show that this condition holds true only when Φ is a function of the symmetric tensor $\boldsymbol{N} = \boldsymbol{M}^*\cdot\boldsymbol{M}$. Actually, the symmetry condition in expanded form results in superdefinite system of equations in terms of first-order partial derivatives relative to the function Φ:

$$\begin{aligned}M_{31}\frac{\partial\Phi}{\partial M_{21}} + M_{32}\frac{\partial\Phi}{\partial M_{22}} + M_{33}\frac{\partial\Phi}{\partial M_{23}} &= M_{21}\frac{\partial\Phi}{\partial M_{31}} + M_{22}\frac{\partial\Phi}{\partial M_{32}} + M_{23}\frac{\partial\Phi}{\partial M_{33}},\\ M_{11}\frac{\partial\Phi}{\partial M_{31}} + M_{12}\frac{\partial\Phi}{\partial M_{32}} + M_{13}\frac{\partial\Phi}{\partial M_{33}} &= M_{31}\frac{\partial\Phi}{\partial M_{11}} + M_{32}\frac{\partial\Phi}{\partial M_{12}} + M_{33}\frac{\partial\Phi}{\partial M_{13}},\\ M_{21}\frac{\partial\Phi}{\partial M_{11}} + M_{22}\frac{\partial\Phi}{\partial M_{12}} + M_{23}\frac{\partial\Phi}{\partial M_{13}} &= M_{11}\frac{\partial\Phi}{\partial M_{21}} + M_{12}\frac{\partial\Phi}{\partial M_{22}} + M_{13}\frac{\partial\Phi}{\partial M_{23}}.\end{aligned}$$

Each of the equations can be solved using the method of characteristics. For the first equation, the system of characteristic equations

$$\frac{dM_{21}}{M_{31}} = \frac{dM_{22}}{M_{32}} = \frac{dM_{23}}{M_{33}} = -\frac{dM_{31}}{M_{21}} = -\frac{dM_{32}}{M_{22}} = -\frac{dM_{33}}{M_{23}} = \frac{d\Phi}{0}$$

has six functionally independent integrals

$$M_{21}^2 + M_{31}^2 = C_1, \quad M_{22}^2 + M_{32}^2 = C_2, \quad M_{23}^2 + M_{33}^2 = C_3,$$

$$M_{21}\, M_{22} + M_{31}\, M_{32} = C_4, \quad M_{21}\, M_{23} + M_{31}\, M_{33} = C_5, \quad \Phi = C_6.$$

The general solution of the equation $\Phi = \Phi(C_1, C_2, C_3, C_4, C_5)$ depends on the values M_{jk} with the index $j = 1$ as on parameters. The analysis of the general solutions of the two remaining equations with selecting a universal dependence $\Phi(\boldsymbol{M})$, satisfying all of the three equations of the system, yields $\Phi = \Phi(\boldsymbol{N})$. In this case, the constitute equation (9) for the couple stresses takes the form: $\boldsymbol{m} = 2\,\boldsymbol{M}\cdot\partial\Phi/\partial\boldsymbol{N}$, and the symmetry condition $\boldsymbol{m}\cdot\boldsymbol{M}^* = \boldsymbol{M}\cdot\boldsymbol{m}^*$ is fulfilled automatically.

It is worthy of noting that linearization of Eq. (5) in case of the infinitely small curvature yields the equation: $\dot{\boldsymbol{M}} = \boldsymbol{\omega}_{\boldsymbol{\xi}}$, and its correctness in description of finite strains and rotations has been discussed above. Such equation for the curvature tensor is used in the classical theory of the Cosserat continuum with small strains and rotations. Notwithstanding the resultant constraint, the stress potential in the classical theory has a term represented by the quadratic form of all components of the curvature tensor $\boldsymbol{M}$. Inasmuch as independent development of a geometrically linear model entails no complementary equation – the symmetry condition for the tensor $\boldsymbol{m}\cdot\boldsymbol{M}^*$, this shows no direct disagreement with the principles of thermodynamics. Nevertheless, the potential cannot be taken in such form for the correct generalization of the linear model since the nonlinearity is only possible, when the quadratic form is independent on the combinations different from the components of the symmetric tensor $\boldsymbol{N}$. Accordingly, in the case of isotropic Cosserat continuum, the curvature-dependent term of the quadratic stress potential is to be proportional to the first invariant of the tensor $\boldsymbol{N}$ and to equal $\gamma\, M_{jk}\, M_{jk}$. The respective equation for the couple stresses, $m_{jk} = 2\,\gamma\, M_{jk}$, contains a single elastic coefficient γ instead of three independent coefficients β, γ and ε of the classical theory. For an anisotropic continuum, this term should be defined by the quadratic form $\Gamma_{kl}\, M_{jk}\, M_{jl}$ with the symmetric tensor $\boldsymbol{\Gamma}$ of the second rank rather than of the fourth rank.

3. Canonical form of equations

In the adiabatic approximation of the model ($\boldsymbol{q} = 0$, $Q = 0$), a closed system consists of equations of translational and rotational motion from (8), constitutive equations (9), equation $\dot{\boldsymbol{R}} = \boldsymbol{\Omega}\cdot\boldsymbol{R}$ for the tensor of rotation and equation $\dot{s} = 0$ for the entropy.

Let $\boldsymbol{\tau} = \boldsymbol{R}^*\cdot\boldsymbol{\sigma}$ be a stress tensor making a dual couple with $\boldsymbol{\Lambda}$,

$$\Psi(\boldsymbol{\tau}, \boldsymbol{m}, s) = \boldsymbol{\tau}^* : \boldsymbol{\Lambda} + \boldsymbol{m}^* : \boldsymbol{M} - \Phi(\boldsymbol{\Lambda}, \boldsymbol{M}, s)$$

be a dual potential equal to the Legendre transform from internal energy. Written in terms of the dual potential, the constitutive equations (9) are given in inverted form:

$$\boldsymbol{R}\cdot\boldsymbol{\Lambda} = \boldsymbol{R}\cdot\frac{\partial\Psi(\boldsymbol{\tau}, \boldsymbol{m}, s)}{\partial\boldsymbol{\tau}} = \frac{\partial\Psi(\boldsymbol{R}^*\cdot\boldsymbol{\sigma}, \boldsymbol{m}, s)}{\partial\boldsymbol{\sigma}}, \quad \boldsymbol{M} = \frac{\partial\Psi(\boldsymbol{R}^*\cdot\boldsymbol{\sigma}, \boldsymbol{m}, s)}{\partial\boldsymbol{m}}.$$

Using (3) and (5), the equations are reduced to the differential equations

$$\frac{\partial}{\partial t}\,\frac{\partial\Psi(\boldsymbol{R}^*\cdot\boldsymbol{\sigma},\boldsymbol{m},s)}{\partial\boldsymbol{\sigma}}=\boldsymbol{v_\xi},\quad \frac{\partial}{\partial t}\,\frac{\partial\Psi(\boldsymbol{R}^*\cdot\boldsymbol{\sigma},\boldsymbol{m},s)}{\partial\boldsymbol{m}}=\boldsymbol{\omega_\xi}+\boldsymbol{\Omega}\cdot\frac{\partial\Psi(\boldsymbol{R}^*\cdot\boldsymbol{\sigma},\boldsymbol{m},s)}{\partial\boldsymbol{m}}.$$

This allows representing the model by a thermodynamically compatible system of the laws of conservation in the following sense [13]: it is possible to indicate generating potentials L^0 and L^j, the use of which modifies the complete system of equations in the Cartesian coordinates:

$$\begin{gathered}\rho_0\,\dot v_i=\sigma_{ij,j}+f_i,\\ \frac{\partial}{\partial t}\big(J_{ij}\,\omega_j\big)=m_{ij,j}+\varepsilon_{ijk}\,\sigma_{kl}\,\frac{\partial\Psi(\boldsymbol{R}^*\cdot\boldsymbol{\sigma},\boldsymbol{m},s)}{\partial\sigma_{jl}}+g_i,\\ \frac{\partial}{\partial t}\,\frac{\partial\Psi(\boldsymbol{R}^*\cdot\boldsymbol{\sigma},\boldsymbol{m},s)}{\partial\sigma_{ij}}=v_{i,j},\\ \frac{\partial}{\partial t}\,\frac{\partial\Psi(\boldsymbol{R}^*\cdot\boldsymbol{\sigma},\boldsymbol{m},s)}{\partial m_{ij}}=\omega_{i,j}+\varepsilon_{ikl}\,\omega_k\frac{\partial\Psi(\boldsymbol{R}^*\cdot\boldsymbol{\sigma},\boldsymbol{m},s)}{\partial m_{lj}},\\ \dot R_{ij}=\varepsilon_{ikl}\,\omega_k\,R_{lj},\quad \dot s=0,\quad J_{ij}=J^0_{kl}\,R_{ik}\,R_{jl}\end{gathered}\tag{11}$$

(ε_{ijk} is the discriminant tensor) and makes it uniform:

$$\frac{\partial}{\partial t}\,\frac{\partial L^0(\boldsymbol{D}\boldsymbol{U})}{\partial\boldsymbol{U}}=\frac{\partial}{\partial\xi_j}\,\frac{\partial L^j(\boldsymbol{U})}{\partial\boldsymbol{U}}+\boldsymbol{F}(\boldsymbol{D},\boldsymbol{U}),\quad \frac{\partial\boldsymbol{D}}{\partial t}=\boldsymbol{G}(\boldsymbol{D},\boldsymbol{U}).\tag{12}$$

Here $\boldsymbol{U}$ is the column–vector composed of unknown functions, except for the entropy, namely, projections of vectors of velocity of translational motion and angular velocity, components of tensors of stresses and couple stresses, and components of tensor of rotation; $\boldsymbol{D}$ is the nonsingular matrix, non-zero and non-unit elements of which are given by the values R_{ij}; $\boldsymbol{F}$ and $\boldsymbol{G}$ are the preset vector–function and matrix–function readily determinable from the form of the equation. The generating potentials are equal to:

$$L^0(\boldsymbol{D}\boldsymbol{U})=\rho_0\,\frac{v_i\,v_i}{2}+\frac{1}{2}\,(\boldsymbol{R}^*\cdot\boldsymbol{\omega})_i\,J^0_{ij}\,(\boldsymbol{R}^*\cdot\boldsymbol{\omega})_j+\Psi(\boldsymbol{R}^*\cdot\boldsymbol{\sigma},\boldsymbol{m},s),\quad L^j(\boldsymbol{U})=v_i\,\sigma_{ij}+\omega_i\,m_{ij}.$$

The equation for the entropy from (11) is not included in the system (12), as it automatically yields an equivalent equation in the form of auxiliary law of conservation:

$$\begin{gathered}\frac{\partial}{\partial t}\Big(\boldsymbol{U}\cdot\frac{\partial L^0(\boldsymbol{D}\boldsymbol{U})}{\partial\boldsymbol{U}}-L^0(\boldsymbol{D}\boldsymbol{U})\Big)=\frac{\partial}{\partial\xi_j}\Big(\boldsymbol{U}\cdot\frac{\partial L^j(\boldsymbol{U})}{\partial\boldsymbol{U}}-L^j(\boldsymbol{U})\Big)+\\ +\boldsymbol{U}\cdot\boldsymbol{F}-\frac{\partial L^0(\boldsymbol{D}\boldsymbol{U})}{\partial\boldsymbol{U}}\cdot\boldsymbol{D}^{-1}\boldsymbol{G}\boldsymbol{U},\end{gathered}\tag{13}$$

the validity of which is checked using the differentiation formula

$$\frac{\partial L^0(\boldsymbol{D}\boldsymbol{U})}{\partial t}=\frac{\partial L^0(\boldsymbol{D}\boldsymbol{U})}{\partial\boldsymbol{U}}\cdot\Big(\frac{\partial\boldsymbol{U}}{\partial t}+\boldsymbol{D}^{-1}\frac{\partial\boldsymbol{D}}{\partial t}\,\boldsymbol{U}\Big).\tag{14}$$

The system of Eqs. (12) possesses some essential properties reflective of mathematical correctness of the model. It has a divergent form and can serve to describe generalized solutions with discontinuous velocities and stresses – shock waves and contact discontinuities at interfaces of media having different mechanical properties. Solving of such systems involves effective computational algorithms adapted to calculation of discontinuities [14].

Application of the formula (14) to the derivative $\partial L^0(\boldsymbol{D}\boldsymbol{U})/\partial \boldsymbol{U}$ brings the system (12) into symmetric form:

$$\begin{pmatrix} \boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{A} \end{pmatrix} \frac{\partial}{\partial t} \begin{pmatrix} \boldsymbol{D} \\ \boldsymbol{U} \end{pmatrix} = \begin{pmatrix} \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{B}^j \end{pmatrix} \frac{\partial}{\partial \xi_j} \begin{pmatrix} \boldsymbol{D} \\ \boldsymbol{U} \end{pmatrix} + \begin{pmatrix} \boldsymbol{G} \\ \boldsymbol{H} \end{pmatrix}. \tag{15}$$

Here

$$\boldsymbol{A}(\boldsymbol{D},\boldsymbol{U}) = \frac{\partial^2 L^0(\boldsymbol{D}\boldsymbol{U})}{\partial \boldsymbol{U}^2}, \quad \boldsymbol{B}^j(\boldsymbol{U}) = \frac{\partial^2 L^j(\boldsymbol{U})}{\partial \boldsymbol{U}^2}, \quad \boldsymbol{H}(\boldsymbol{D},\boldsymbol{U}) = \boldsymbol{F} - \boldsymbol{A}\boldsymbol{D}^{-1}\boldsymbol{G}\boldsymbol{U}.$$

The matrices $\boldsymbol{A}$ and $\boldsymbol{B}^j$ are symmetric, and, moreover, when the potential $L^0(\boldsymbol{D}\boldsymbol{U})$ is strongly convex, the matrix $\boldsymbol{A}$ is positively definite. Therefore, the system of equations (12) is of hyperbolic type. The strong convexity condition L^0 is fulfilled, when the dual potential $\Psi(\boldsymbol{\tau},\boldsymbol{m},s)$ is a strongly convex function with respect to the set of variables $\boldsymbol{\tau}$ and $\boldsymbol{m}$.

Suppose that the matrices–coefficients of the system (15) and the right-hand sides $\boldsymbol{G}$ and $\boldsymbol{H}$ meet the Lipschitz condition relative to $\boldsymbol{D}$ and $\boldsymbol{U}$. In this case, for the difference of two solutions derived for such system in the space-time domain W in the form of a truncated cone with its bases represented by hyperplanes $t = t_0$ and $t = t_1$ and the lateral surface equation $h(\boldsymbol{\xi}, t) = 0$ complying with the Hamilton – Jacobi inequality [1]: $\dot{h} + c(h_{\boldsymbol{\xi}}) \geqslant 0$, where $h_{\boldsymbol{\xi}}$ is the gradient of h and $c(\boldsymbol{\nu})$ is a minimum root of the characteristic equation $\det(c\boldsymbol{A} + \nu_j \boldsymbol{B}^j) = 0$, a priori estimate is valid:

$$\big\|(\boldsymbol{D}',\boldsymbol{U}') - (\boldsymbol{D},\boldsymbol{U})\big\|(t_1) \leqslant \big\|(\boldsymbol{D}',\boldsymbol{U}') - (\boldsymbol{D},\boldsymbol{U})\big\|(t_0)\exp a(t_1 - t_0). \tag{16}$$

Here a is a constant governed, generally speaking, by the both solutions and by their derivatives with respect to time and with respect to spatial variables, the double brackets denote the energy norm:

$$\big\|(\boldsymbol{D},\boldsymbol{U})\big\|^2(t) = \frac{1}{2}\int_{W_\xi} \big(\mathrm{tr}\,\boldsymbol{D}^*\boldsymbol{D} + \boldsymbol{U}\boldsymbol{A}\boldsymbol{U}\big)\, dV$$

(tr is the trace of a matrix), calculated as an integral taken over the section W_ξ of the conical domain W by the hyperplane $t = \mathrm{const}$. It follows that the Cauchy problem solution

$$\boldsymbol{D}|_{t=t_0} = \boldsymbol{D}^0(\xi), \quad \boldsymbol{U}|_{t=t_0} = \boldsymbol{U}^0(\xi)$$

is unique in W and continuously dependent on initial data. Furthermore, the estimate (16) shows boundedness of the domains of dependence and influence of the solutions — finiteness of perturbation velocities in the model under analysis.

An analogous estimate is valid in truncated cones, adjoining the problem solution region in case that the boundary conditions set at the region boundaries are dissipative. The dissipativity precisely means that for any two solutions at the boundary the following inequality holds true:

$$\big(\boldsymbol{U}' - \boldsymbol{U}\big)\,\boldsymbol{B}^j\big(\boldsymbol{U}' - \boldsymbol{U}\big)\,\nu_j \leqslant 0,$$

where $\boldsymbol{\nu}$ is the vector of the external normal. In this case, the integral estimate yields the unique and continuous dependence of the solutions of boundary-value problems on the initial data. Considering structure of the matrices $\boldsymbol{B}^j$, it is possible to show that the condition of dissipativity for the model of the couple-stress elasticity theory reduces to the inequality:

$$(v_i' - v_i)(\sigma_{ij}' - \sigma_{ij})\,\nu_j + (\omega_i' - \omega_i)(m_{ij}' - m_{ij})\,\nu_j \leqslant 0. \tag{17}$$

The dissipative boundary conditions include conditions in terms of velocities: $v_i = \bar{v}_i$, $\omega_i = \bar{\omega}_i$, and conditions in terms of stresses: $\sigma_{ij}\,\nu_j = \bar{\sigma}_i$, $m_{ij}\,\nu_j = \bar{m}_i$, as well as their combinations. For example, at a certain section of a boundary, the vectors of angular velocity $\bar{\omega}_i$ and stresses $\bar{\sigma}_i$, or the vectors of linear velocity $\bar{v}_i$ and couple stresses $\bar{m}_i$ may be set. It is allowable to impose mixed boundary conditions, when normal velocities and tangential stresses, or, conversely, normal stresses and tangential velocities are given.

Conclusion

The problem of reducing the governing equations to a thermodynamically compatible system of conservation laws in the nonlinear continuum mechanics is extremely complicated. This problem can be solved relatively easy only within the framework of a geometrically linear approximation of models. Nevertheless, research in this direction is actively developing (see, [15, 16]), since this form of the equations guarantees the mathematical correctness of the model and allows the use of well-developed computational algorithms for numerical implementation. The canonical form (12) is a simple generalization of the thermodynamically compatible system (1). Such form also provides reliable properties. In present paper, the equations of the nonlinear elastic Cosserat continuum with a special measure of curvature are reduced to canonical form. The question of reducing the equations with other curvature measures, different from (5), to the thermodynamically compatible form remains open.

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Приведение нелинейных уравнений типа Коссера к системе в форме Годунова

Владимир М. Садовский
Институт вычислительного моделирования СО РАН
Красноярск, Российская Федерация

Аннотация. Полная система уравнений динамики моментной среды типа Коссера при конечных деформациях и вращениях частиц в лагранжевых переменных приводится к термодинамически согласованной по Годунову системе законов сохранения. Такая форма системы позволяет анализировать обобщенные решения с поверхностями сильного разрыва скоростей и напряжений, а также получать интегральные оценки, гарантирующие единственность и непрерывную зависимость решений задачи Коши и краевых задач с диссипативными граничными условиями от начальных данных.

Ключевые слова: упругость, континуум Коссера, моментные напряжения, тензор кривизны.