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Mathematical Modeling of Three-dimensional Stress-strain State of Homogeneous and Composite Cylindrical Axisymmetric Shells

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Abstract. The study is devoted to the application of the asymptotic splitting method for solving static problems of deformation of homogeneous isotropic and composite cylindrical shells. The problem of deformation of a composite cylindrical shell subjected to an internal axisymmetric load is considered. The solution is constructed by expanding the components of the stress tensor and the displacement vector in powers of differential operators acting along the cylinder axis. A small parameter is the ratio of the shell thickness to its length. A governing differential system of equations describing the deformation of a cylindrical shell is obtained. It is shown that the developed mathematical model allows to compute all components of the stress tensor for both thick-walled and thin-walled cylindrical shells. The obtained analytic and numerical solutions are compared with the finite element solution of the 2D axisymmetric problem.

Keywords: cylindrical shells, stress-strain state, method of asymptotic splitting, linear theory of elasticity, axisymmetric problem, finite element method.

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Cylindrical composite shells are popular and important structural elements which are widely used in many high-tech industries, such as aircraft, rocket manufacturing, production of high-pressure composite vessels etc. In the current study we use the asymptotic splitting (AS) method to analyze three-dimensional stress-strain state (SSS) of homogeneous and composite cylindrical shells. The AS method was developed in [1] and was successfully used to analyze composite beams and plates [2–4]. Special attention is given to the study of short and thick-walled cylindrical shells, where the resulting SSS is complex along the entire length of the shell.

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1. Problem statement

Consider an axisymmetric problem of deformation of a cylindrical shell within the framework of the linear theory of elasticity. The Oz axis is directed along the cylinder axis, the Or axis is directed along the radius, as shown in Fig. (1). In the two-dimensional region describing the cross section of the shell, the equilibrium equations are expressed as follows

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{\partial \sigma_{rz}}{\partial z} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = 0, \quad \frac{\partial \sigma_{rz}}{\partial r} + \frac{\partial \sigma_{zz}}{\partial z} + \frac{\sigma_{rz}}{r} = 0, \tag{1}$$

where $\sigma_{rr}, \sigma_{\theta\theta}, \sigma_{zz}$ — radial, circumferential and longitudinal stresses, respectively; σ_{rz} — shear stresses acting in rOz plane.

A distributed axisymmetric load $p(z)$ acts on the inner surface of the cylinder. The outer side of the cylinder is free

$$\begin{gathered} \sigma_{rr}(r,z)|_{r=R_{in}} = -p(z), \quad \sigma_{rr}(r,z)|_{r=R_{out}} = 0, \\ \sigma_{rz}(r,z)|_{r=R_{in}} = 0, \quad \sigma_{rz}(r,z)|_{r=R_{out}} = 0, \end{gathered} \tag{2}$$

where R_{in}, R_{out} — inner and outer radius of the cylinder, respectively.

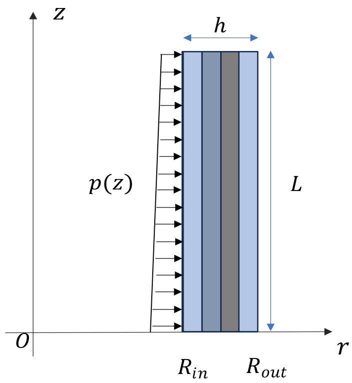

Fig. 1. Loading scheme of a composite cylindrical shell subjected to an internal load

Circumferential displacement is equal to zero. Thus, Cauchy equations for the linear strain tensor in cylindrical coordinate system are

$$e_{rr} = \frac{\partial u_r}{\partial r}, \quad e_{\theta\theta} = \frac{u_r}{r}, \quad e_{zz} = \frac{\partial u_z}{\partial z}, \quad e_{rz} = \frac{1}{2}(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}). \tag{3}$$

The shell is made of an arbitrary number of layers of constant thickness as schematically shown on figure 1 using a four layer shell as an example, where each layer has its own color. The designations h, L on figure 1 correspond to the thickness and length of the shell. The layers are counted from the inner surface from 1 to s, where s is the number of layers. On the interfaces between shell layers, the displacements and contact stresses are continuous

$$[\sigma_r]_i^j = 0, \quad [\sigma_{rz}]_i^j = 0, \quad (u_\alpha)_j = (u_\alpha)_i, \quad \alpha \in (r,\theta,z), \quad i,j = (1,s). \tag{4}$$

The material of each layer is assumed to be orthotropic and obeys generalized Hook's law

$$\begin{gathered} (\sigma_{\alpha\alpha})_i = E_{\alpha r}^{(i)}(e_{rr})_i + E_{\alpha\theta}^{(i)}(e_{\theta\theta})_i + E_{\alpha z}^{(i)}(e_{zz})_i, \quad \alpha \in (r,\theta,z), \\ \sigma_{rz} = 2G_{rz}^{(i)}(e_{rz})_i, \quad i = (1,s). \end{gathered} \tag{5}$$

The orthotropic axes of the material are aligned with the axes of the main coordinate system.

1.1. Non-dimensional problem

Let us pass to non-dimensional variables and functions, for simplicity retaining the same designation. The exception is made for the variable r, which is replaced with non-dimensional variable x

$$\begin{gathered} x = \frac{r - R_{in}}{h}, \quad z = \frac{z}{L}, \quad x, z \in [0, 1], \\ u_\alpha = \frac{u_\alpha}{h}, \quad E^{(i)}_{\alpha\beta} = \frac{E^{(i)}_{\alpha\beta}}{\tilde{E}}, \quad \sigma_{\alpha\beta} = \frac{\sigma_{\alpha\beta}}{\tilde{E}}, \quad p = \frac{p}{\tilde{E}}, \end{gathered} \tag{6}$$

where $\tilde{E}$ – the characteristic value of Young's modulus.

After transformations we obtain the equilibrium equations (1) and Cauchy equations (3) in non-dimensional form

$$\begin{cases} \dfrac{\partial \sigma_{rr}}{\partial x} + \varepsilon \dfrac{\partial \sigma_{rz}}{\partial z} + \varepsilon_1 \dfrac{\sigma_{rr} - \sigma_{\theta\theta}}{1 + \varepsilon_1 x} = 0, \\ \dfrac{\partial \sigma_{rz}}{\partial x} + \varepsilon \dfrac{\partial \sigma_{zz}}{\partial z} + \varepsilon_1 \dfrac{\sigma_{rz}}{1 + \varepsilon_1 x} = 0, \end{cases} \tag{7}$$

$$e_{rr} = \frac{\partial u_r}{\partial x}, \quad e_{\theta\theta} = \varepsilon_1 \frac{u_r}{1 + \varepsilon_1 x}, \quad e_{zz} = \varepsilon \frac{\partial u_z}{\partial z}, \quad e_{rz} = \frac{1}{2}(\varepsilon \frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial x}). \tag{8}$$

There are two parameters in the system: $\varepsilon = h/L$ — characterizes the ratio of the shell thickness to its length and is assumed to be small for asymptotic analysis; $\varepsilon_1 = h/R_{in}$ — characterizes the ratio of the shell thickness to its inner radius and does not have to be small.

2. Method of asymptotic splitting

According to the main idea of the AS method the following approximation rules for the components of the displacement vector $\{u^\eta\}$ and stress tensor $[\sigma^\eta]$ are used

$$\begin{gathered} (u^\eta_\alpha)^{(n)}_i(x, z) = \sum_{k=0}^{n+3} (U^\eta_\alpha)^{(k)}_i) \frac{d^k \eta^{(n)}}{dz^k} \varepsilon^k, \quad (u^\eta_z)^{(n)}_i(x, z) = \sum_{k=0}^{n+2} (U^\eta_\beta)^{(k)}_i \frac{d^k \eta^{(n)}}{dz^k} \varepsilon^k, \\ (\sigma^\eta_{\alpha\alpha})^{(n)}_i(x, z) = \sum_{k=0}^{n+3} (\tau^\eta_{\alpha\alpha})^{(k)}_i \frac{d^k \eta^{(n)}}{dz^k} \varepsilon^k, \quad (\sigma^\eta_{zz})^{(n)}_i(x, z) = \sum_{k=0}^{n+1} (\tau^\eta_{zz})^{(k)}_i \frac{d^k \eta^{(n)}}{dz^k} \varepsilon^k, \\ (\sigma^\eta_{rz})^{(n)}_i(x, z) = \sum_{k=0}^{n+2} (\tau^\eta_{rz})^{(k)}_i \frac{d^k \eta^{(n)}}{dz^k} \varepsilon^k, \quad \alpha \in (r, \theta), \end{gathered} \tag{9}$$

where $(U^\eta_\alpha)^{(k)}_i(x)$, $(\tau^\eta_{\alpha\beta})^{(k)}_i(x)$ — stiffness functions describing distribution of displacements and stresses across the thickness of the shell.

The approximation rules are considered in two directions $\eta \in (v_z, v_r)$, where $v_z(z)$ is the average displacement of the shell along the z axis, $v_r(z)$ is the average displacement of the shell along the r axis.

The stresses and displacements could be determined as the sum of the components of each of the approximations according to the superposition rule

$$(u_\alpha)_i^{(n)}(x,z) = (u_\alpha^{v_z})_i^{(n)} + (u_\alpha^{v_r})_i^{(n)}, \quad (u_z)_i^{(n)}(x,z) = (u_z^{v_z})_i^{(n)} + (u_z^{v_r})_i^{(n)},$$
$$(\sigma_{\alpha\alpha})_i^{(n)}(x,z) = (\sigma_{\alpha\alpha}^{v_z})_i^{(n)} + (\sigma_{\alpha\alpha}^{v_r})_i^{(n)}, \quad (\sigma_{zz})_i^{(n)}(x,z) = (\sigma_{zz}^{v_z})_i^{(n)} + (\sigma_{zz}^{v_r})_i^{(n)}, \tag{10}$$
$$(\sigma_{rz})_i^{(n)}(x,z) = (\sigma_{rz}^{v_z})_i^{(n)} + (\sigma_{rz}^{v_r})_i^{(n)}, \quad \alpha \in (r,\theta).$$

2.1. BVPs across the thickness of the shell

Let us substitute expressions (9) into the equilibrium equations (1) and equate the terms at identical degree of the small parameter ε. Then, we obtain a series of differential equations across the thickness of the shell for different values of variables k and $\eta \in (v_r, v_z)$

$$\begin{cases} \dfrac{d(\tau_{rr}^\eta)_i^{(k)}}{dx} + (\tau_{rz}^\eta)_i^{(k-1)} + \varepsilon_1 \dfrac{(\tau_{rr}^\eta)_i^{(k)} - (\tau_{\theta\theta}^\eta)_i^{(k)}}{1+\varepsilon_1 x} = 0, \\ \dfrac{d(\tau_{rz}^\eta)_i^{(k)}}{dx} + (\tau_{zz}^\eta)_i^{(k-1)} + \varepsilon_1 \dfrac{(\tau_{rz}^\eta)_i^{(k)}}{1+\varepsilon_1 x} = 0. \end{cases} \tag{11}$$

The boundary conditions on the outer and inner surfaces are derived from the necessary conditions for solvability of equations (11) and absence of load on the outer surface of the cylinder

$$(\tau_{rr}^\eta)_i^{(k)}(0) = -B_{rr}^{\eta,(k)}, \quad (\tau_{rr}^\eta)_i^{(k)}(1) = 0, \quad (\tau_{rz}^\eta)_i^{(k)}(0) = -B_{rz}^{\eta,(k)}, \quad (\tau_{rz}^\eta)_i^{(k)}(1) = 0, \tag{12}$$

where stiffness coefficients $B_{rr}^{\eta,(k)}, B_{rz}^{\eta,(k)}$ are expressed as follows

$$B_{rr}^{\eta,(k)} = \int_0^1 (\varepsilon_1 (\tau_{\theta\theta}^\eta)_i^{(k)} - (1+\varepsilon_1 x)(\tau_{rz}^\eta)_i^{(k-1)})\, dx, \quad B_{rz}^{\eta,(k)} = -\int_0^1 (1+\varepsilon_1 x)(\tau_{zz}^\eta)_i^{(k-1)}. \tag{13}$$

From the generalized Hooke's law (5) it follows that the stiffness functions of stress tensor $(\tau_{\alpha\beta}^\eta)_i^{(k)}$ are related to the stiffness functions of displacement vector $(U_\alpha^\eta)_i^{(k)}$ in the following way

$$(\tau_{\alpha\alpha}^\eta)_i^{(k)} = E_{\alpha r}^{(i)} \frac{d(U_r^\eta)_i^{(k)}}{dx} + E_{\alpha\theta}^{(i)} \varepsilon_1 \frac{(U_r^\eta)_i^{(k)}}{1+\varepsilon_1 x} + E_{\alpha z}^{(i)} (U_z^\eta)_i^{(k-1)},$$
$$(\tau_{rz}^\eta)_i^{(k)} = G_{rz}^{(i)} (\frac{d(U_z^\eta)_i^{(k)}}{dx} + (U_r^\eta)_i^{(k-1)}). \tag{14}$$

Together, the expressions (11), (12), (14) form boundary value problems (BVPs) across the shell thickness to determine the stiffness functions $(\tau_{\alpha\alpha}^\eta)_i^{(k)}, (U_\alpha^\eta)_i^{(k)}$. The sought-for functions $(U_r^\eta)_i^{(k)}, (U_z^\eta)_i^{(k)}$ are defined up to a constant. Therefore, additional conditions are needed, called normalization rules. Depending on their choice the function $\eta(z)$ have a different physical meaning. In the current study the normalization rules are chosen in such a way that $\eta = v_r$ is the average displacement of the shell along the r axis for approximation in the radial direction

$$v_r = \int_0^1 (u_r^{v_r})\, dx, \quad \int_0^1 (u_z^{v_r})\, dx = 0.$$

The normalization rules in the radial direction are the following one

$$\int_0^1 (U_r^{v_r})_i^{(0)} dx = 1, \quad \int_0^1 (U_z^{v_r})_i^{(0)} dx = 0, \quad \int_0^1 (U_r^{v_r})_i^{(k)} dx = \int_0^1 (U_z^{v_r})_i^{(k)} dx = 0, \quad k \geqslant 1.$$

For approximation in the longitudinal direction the function $\eta = u_0$ is the average displacement of the shell along the z axis

$$\int_0^1 (u_r^{v_z})\, dx = 0, \quad v_z = \int_0^1 (u_z^{v_z})\, dx.$$

The normalization rules in the longitudinal direction are the following one

$$\int_0^1 (U_r^{v_z})_i^{(0)} dx = 1, \quad \int_0^1 (U_z^{v_z})_i^{(0)} dx = 0, \quad \int_0^1 (U_r^{v_z})_i^{(k)} dx = \int_0^1 (U_z^{v_z})_i^{(k)} dx = 0, \quad k \geqslant 1.$$

In particular cases BVPs across the shell thickness could be solved analytically. In the current study the open source finite element package Fenics Project [5] is used to solve BVPs numerically in general manner. For this purpose a weak formulation of BVPs is considered. Quadratic Lagrangian elements for mesh discretization are used. Additional integral normalization rules are implemented using Lagrange multipliers.

2.2. Governing differential equations of an axisymmetrically loaded cylindrical shell

Let us consider the equilibrium equations of a cylindrical shell in dimensionless variables, which follow from the equations (1)

$$\varepsilon \frac{dQ_r}{dz} - \varepsilon_1 \frac{N_\theta}{1+\varepsilon_1 x} + p(z) = 0, \quad \varepsilon \frac{dN_z}{dz} = 0, \quad \varepsilon \frac{dM_r}{dz} = Q_r, \tag{15}$$

where M_r — bending moment in the plane Orz, Q_r — shear force in radial direction, N_z and N_θ — normal forces in longitudinal and circumferential directions, respectively,

$$\begin{aligned} M_r &= \int (1+\varepsilon_1 x)\sigma_{zz}\, x\, dx, \quad Q_r = \int (1+\varepsilon_1 x)\sigma_{rz}\, dx, \\ N_z &= \int (1+\varepsilon_1 x)\sigma_{zz}\, dx, \quad N_\theta = \int (1+\varepsilon_1 x)\sigma_{\theta\theta}\, dx. \end{aligned} \tag{16}$$

Let us consider the approximation for $n = 1$ and substitute it into expressions (16) and into the equilibrium equations (15). We arrive at the following governing system of equations for the unknowns v_r, v_z

$$\begin{cases} B_{rr}^{v_r,(0)} v_r + B_{rr}^{v_r,(2)} \dfrac{d^2 v_r}{dz^2}\varepsilon^2 + B_{rr}^{v_r,(4)} \dfrac{d^4 v_r}{dz^4}\varepsilon^4 + B_{rr}^{v_z,(1)} \dfrac{dv_z}{dz}\varepsilon + B_{rr}^{v_z,(3)} \dfrac{d^3 v_z}{dz^3}\varepsilon^3 = -p(z), \\ B_{rz}^{v_r,(1)} \dfrac{dv_r}{dz}\varepsilon + B_{rz}^{v_r,(3)} \dfrac{d^3 v_r}{dz^3}\varepsilon^3 + B_{rz}^{v_z,(2)} \dfrac{d^2 v_z}{dz^2}\varepsilon^2 = 0, \end{cases} \tag{17}$$

where stiffness functions $B_{rr}^{\eta,(k)}, B_{rz}^{\eta,(k)}$ are to be found by formulas (13). Let us eliminate a third derivative from equations (17). We differentiate the second equation, express the third derivative with respect to v_z and substitute it into the first equation.

$$\begin{cases} B_{rr}^{v_r,(0)} v_r + \tilde{B}_{rr}^{v_r,(2)} \dfrac{d^2 v_r}{dz^2}\varepsilon^2 + \tilde{B}_{rr}^{v_r,(4)} \dfrac{d^4 v_r}{dz^4}\varepsilon^4 + B_{rr}^{v_z,(1)} \dfrac{dv_z}{dz}\varepsilon = -p(z), \\ B_{rz}^{v_r,(1)} \dfrac{dv_r}{dz}\varepsilon + B_{rz}^{v_r,(3)} \dfrac{d^3 v_r}{dz^3}\varepsilon^3 + B_{rz}^{v_z,(2)} \dfrac{d^2 v_z}{dz^2}\varepsilon^2 = 0, \end{cases} \tag{18}$$

where stiffness corrections $\tilde{B}_{rr}^{v_r,(2)}, \tilde{B}_{rr}^{v_r,(4)}$ could be found using the formulas

$$\tilde{B}_{rr}^{v_r,(2)} = B_{rr}^{v_r,(2)} - B_{rr}^{v_z,(3)} \frac{B_{rz}^{v_r,(1)}}{B_{rz}^{v_z,(2)}}, \quad \tilde{B}_{rr}^{v_r,(4)} = B_{rr}^{v_r,(4)} - B_{rr}^{v_z,(3)} \frac{B_{rz}^{v_r,(3)}}{B_{rz}^{v_z,(2)}}.$$

Boundary conditions at the ends are expressed in terms of generalized forces and average values of displacements and rotation angles

$$\text{Hinged:} \quad v_r = N_z = M_r = 0, \quad \text{Clamped:} \quad v_r = v_z = \phi_r = 0. \tag{19}$$

The rotation angle of the normal is determined using the least squares method, minimizing the standard deviation of the straight line given by the rotation angle ϕ_r from longitudinal displacements [2]

$$\phi_r = \frac{1}{J_r} \int u_z(x - 0.5)\, dx, \quad J_r = \int (x - 0.5)^2\, dx.$$

For comparison, consider a governing system of differential equations the the theory of composite cylindrical shells based on Kirchhoff–Love hypothesis

$$\begin{cases} B_{22} v_r + D_{11} \dfrac{d^4 v_r}{dz^4} \varepsilon^4 + B_{21} \dfrac{d v_z}{dz} \varepsilon = -p(z), \\ B_{12} \dfrac{d v_r}{dz} \varepsilon + B_{11} \dfrac{d^2 v_z}{dz^2} \varepsilon^2 = 0, \end{cases} \tag{20}$$

where $B_{ij},\ D_{11}$ — laminate stiffness coefficients.

The system (18) contains terms for the second derivative with respect to v_r in the first equation and for the third derivative with respect to v_r in the second equation, which are not present in the system (20). Both systems have the same differential order of 6. However as will be noted further the solution of the (18) system allows one to find all components of the stress tensor in contrast to the Kirchhoff–Love theory. The stress tensor components in each layer could be found using the formulas (9), (10).

3. Numerical examples

This section provides examples of numerical calculations in non-dimensional form for homogeneous and composite cylindrical shells subjected to an internal pressure.

The results of finite element modeling are presented to verify the accuracy of the AS method. The open finite element analysis package CacluliX is used to solve the original axisymmetric two-dimensional problem. Six node finite element CAX6 [6] is used.

Comparison of the AS method with calculations based on Kirchhoff–Love theory (KL) and the Andreev–Nemirovsky (AN) theory [7] is given. The governing system of differential equations for the AS method (11) is solved using the collocation method in the python library Scipy [8].

3.1. Homogeneous cylindrical shell subjected to an internal pressure

Let us consider a homogeneous isotropic cylindrical shell subjected to an internal pressure. At the ends of the shell boundary conditions assumed to be clamped. The parameters of the problem are equal to the following ones: $h = 1$, $p = 1$, $E = 1$, $\nu = 0.32$. The parameters $\varepsilon = h/L$, $\varepsilon_1 = h/R_{in}$ is varied by changing the values of the shell length L and the inner radius R_{in}.

In Fig. 2 the values of radial displacements on the middle surface of the shell are shown for four different sets of parameters $\varepsilon,\ \varepsilon_1$:

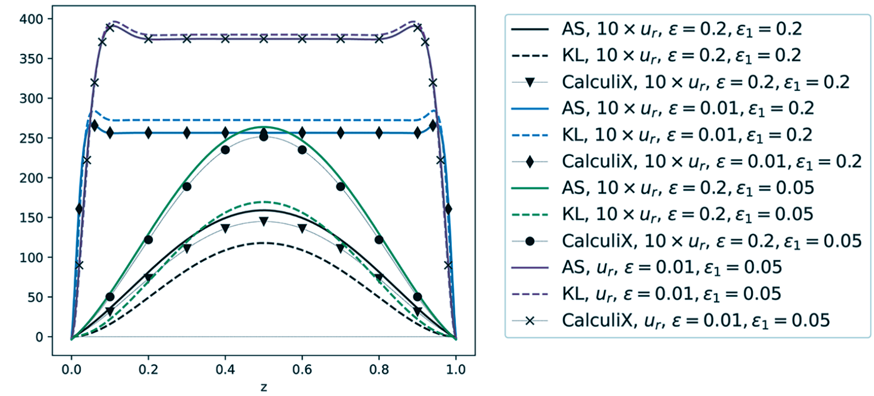

Fig. 2. Radial displacements of the shell middle surface for different values of ε, ε_1

- the first set of parameters (black color) is extreme in the sense that in this case the shell is short enough ($\varepsilon = 0.2$) and thick-walled ($\varepsilon_1 = 0.2$);
- for the second set of parameters (green color), the shell still remains short ($\varepsilon = 0.2$) and could be considered thin-walled ($\varepsilon_1 = 0.05$);
- the third set of parameters (blue color) corresponds to a long shell ($\varepsilon = 0.01$) and at the same time thick-walled ($\varepsilon_1 = 0.2$);
- the fourth set of parameters (purple color) describes a long ($\varepsilon = 0.01$) and thin-walled shell ($\varepsilon_1 = 0.05$).

The results are obtained using the first approximation of the AS method, Kirchov–Love (KL) theory and using two-dimensional finite element analysis in the CalculiX package. Based on the results obtained one could conclude that for short shells KL theory significantly underestimates the radial displacements in the shell (up to 40 %). For short shells the AS method estimates radial displacements much better (within 10 % percent for thick-walled and 5 % for thin-walled). The difference in the results could be explained by presence of boundary layers at the ends of the shell. One could notice that the selected parameter values are not small compared to unity, thus the asymptotic method begins to diverge from the exact solution of the problem. For long shells the KL theory predicts radial displacement within 10 % percent. For long thick-walled shells the KL theory slightly overestimates the real values of radial displacements. In turn for long shells the AS method gives excellent agreement for both thick-walled and thin-walled shells (< 1 %).

One of the advantages of the AS method is that it allows to compute all components of the stress tensor. Fig. 3 shows the values of the stress tensor components for $\varepsilon = 0.2$, $\varepsilon_1 = 0.05$. The values of the longitudinal σ_{zz} and circumferential $\sigma_{\theta\theta}$ components are computed on the outer surface of the shell. Shear σ_{rz} and radial σ_{rr} stresses are computed on the middle surface of the shell. It could be concluded that for short cylindrical shells the stress-strain state is strongly dependent on bending moment along the entire length of the shell in contrast to long shells. Longitudinal and circumferential components are calculated with good accuracy using both the KL theory (within 5 %) and the AS method (within 1 %) except for a narrow region at the ends where stress concentrators arise. The finite element mesh was refined near the ends of the shell in order to correctly show that stress concentrators appear at the ends of the shell. It could be seen that the AS method predicts shear and radial stresses in the main part of the cylindrical shell

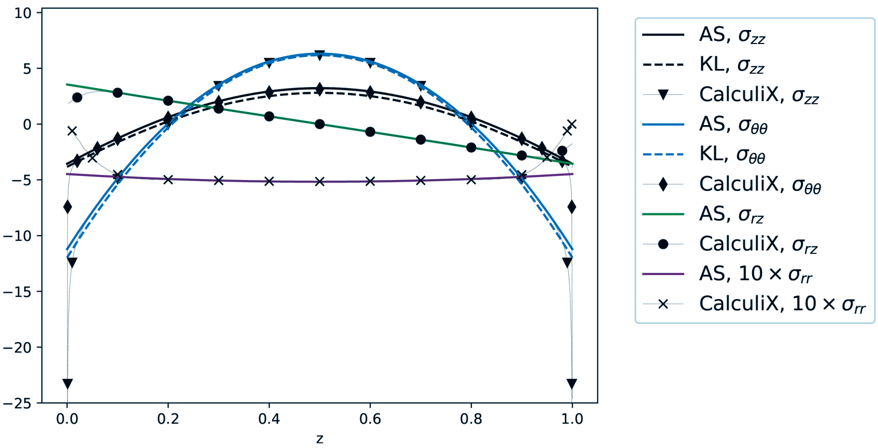

Fig. 3. Stress tensor components at $\varepsilon = 0.2,\ \varepsilon_1 = 0.05$

with good accuracy. Analysis of the results for the AS method revealed that boundary layers appear near the ends, where the values of the obtained radial and shear stresses begin to differ from the solution of the original problem. The size of the boundary layer directly depends on the length of the shell. The area of influence of the boundary layer on shear and radial stresses will only decrease as the length of the shell increases. In turn, the KL theory does not allow to directly compute shear and radial stresses. For short shells the contribution of shear stresses is of the same order as for longitudinal and circumferential stresses. Therefore, radial displacements in the KL theory for short shells are significantly underestimated.

3.2. Three-layer composite cylindrical shell subjected to an internal pressure

Let us consider a three-layer composite cylindrical shell subjected to an internal pressure p. The layers are assumed to be made from isotropic materials. The parameters of the problem are

$$\begin{gathered} R_{mid}/L = 1.5, \quad E_1/E_2 = E_3/E_2 = 10, \quad \nu_i = 0.3, \quad h_1 = h_3 = 0.1, \quad h_2 = 0.8, \\ p = 1, \quad h = 1, \quad E_2 = 1, \end{gathered} \tag{21}$$

where R_{mid} — shell middle surface radius, E_i, ν_i — Young s modulus and Poisson coefficients of i-th layer, h_i — thickness of i-th layer. In this section the parameter ε_1 is defined in different manner as the ratio of the shell thickness to the radius of the middle surface of the shell $\varepsilon = h/R_{mid}$. Due to the parameters chosen, the following relation between ε and ε_1 is valid: $\varepsilon = 1.5 \times \varepsilon_1$.

The boundary conditions for a cylindrical shell with rigid ends are expressed as follows

$$v_r(0) = v_r(1) = 0, \quad \phi_r(0) = \phi_r(1) = 0, \quad v_z(0) = 0, \quad N_z(1) = \frac{pR_{mid}}{2}. \tag{22}$$

To verify the results of the AS method a comparison is made with the results obtained by the theory of composite shells based on the Kirchhoff–Love (KL) hypotheses and the Andreev–Nemirovsky (AN) theory. AN theory allows to take into account shear deformations as described in [7]. A reference numerical solution of a 2D axisymmetric problem is obtained using finite element method and the CalculiX software.

In the Tab. 1 the maximum radial displacements of the middle surface for a three-layer cylindrical shell with rigid ends are shown. It could be seen that the difference between the AS method and finite element solution is within 11 % for small values of $1/\varepsilon_1$ and within 5 % for large values. The results for simplified scheme of the AS method (AS*) are also shown in the table 1. Simplified scheme (AS*) means that the factor $1 + \varepsilon_1 x$ is assumed to be equal to unity. In this case, BPVs across the thickness of the shell are significantly simplified. It could be seen that AS and AS* give approximately the same results. Therefore, the factor $1/\varepsilon_1$ could be neglected when solving BVPs across the shell thickness.

Table 1. Maximum radial displacement of the middle surface for a three-layer cylindrical shell

	$10^{-4} \times u_r(0.5)$				
$1/\varepsilon_1$	CacluliX	AS, %	AS*, %	KL, %	AN, %
10	0.0131	0.0146 (11 %)	0.0144 (11 %)	0.0072 (45 %)	0.0130 (1 %)
20	0.0834	0.0886 (6 %)	0.0869 (6 %)	0.0740 (11 %)	0.0846 (1 %)
30	0.2388	0.2468 (4 %)	0.2429 (4 %)	0.2327 (3 %)	0.2489 (4 %)
55	0.9697	0.9788 (1 %)	0.9683 (1 %)	0.9765 (1 %)	0.9815 (1 %)

At small values of $1/\varepsilon_1$ the theory of composite shells based on the KL hypotheses significantly underestimates radial displacements (up to 45 % at $1/\varepsilon_1 = 10$). AN theory determines radial movements with good accuracy (within 5 %). However, we note that the order of the governing differential system in the AN theory is 8, which is 2 orders higher than in the AS method and the theory of composite shells based on the KL hypotheses.

Comparison of maximum values of the stress tensor components for composite shell with rigid ends is difficult due to the presence of stress concentrators near the ends. In order to exclude consideration of concentrators, the stresses were calculated near the ends with a small offset about 10 % of the shell thickness. In the Tab. 2 the values of the maximum circumferential and shear stresses for a composite shell with rigid ends are shown. The AS method predicts circumferential stresses within 10 %. The accuracy of the method increases with increasing the value $1/\varepsilon_1$. In the Tab. 2 results for longitudinal stresses are shown if the shell is hinged at the ends. In this case the maximum normal stresses are achieved away from the ends and the AS method gives good enough agreement for all values of the parameter $1/\varepsilon_1$ for maximum longitudinal stresses (within 10 %).

Table 2. Maximum $\sigma_{\theta\theta}$, σ_{rz} and σ_{zz} stresses in a three-layer cylindrical shell

	Rigid ends						Hinged ends	
	$10^{-2} \times \sigma_{\theta\theta}$				$10^{-2} \times \sigma_{rz}$		$10^{-2} \times \sigma_{zz}$	
$1/\varepsilon_1$	CalculiX	AS	KL	AN	CalculiX	AS	CalculiX	AS
10	0.1997	0.2175	0.1679	0.2136	1.8976	2.2589	0.2313	0.2509
20	0.5856	0.6154	0.5804	0.5936	3.4213	3.7342	0.3529	0.3584
30	1.0450	1.0750	1.0700	1.0728	4.3800	4.6633	0.4429	0.4442
55	2.1369	2.1542	2.1885	2.1567	6.1037	6.3649	0.7866	0.7911

Conclusions

A mathematical model for calculating three-dimensional SSS of cylindrical axisymmetric shells has been developed based on the first approximation of the AS method. The first approximation of the AS method effectively solves the problem of deformation of composite cylindrical shells if the boundary conditions at the ends are not absolutely rigid. The governing differential system has the same order of 6 as in the KL theory. At the same time its solution allows one to find all components of the stress tensor with good enough accuracy for both thin and thick-walled shells. For short cylindrical shells AS method gives much better predictions compared to the theory of composite shells based on KL hypotheses. The AN theory determines the components of displacements and stresses (except for radial ones) with good accuracy (within 5 %). At the same time the order of the governing differential system in the AN theory is 8, which is 2 orders of magnitude higher than in the AS method and the theory of composite shells based on the KL hypotheses.

For thick-walled and short cylindrical shells or shells with a large difference in elastic moduli (more than 100 times), the first approximation of the AS method cannot exactly satisfy the absolutely rigid boundary conditions at the ends of the shell. It is advisable to study the second approximation of the AS method to take into account the boundary conditions at the shell ends more precisely.

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Математическая модель расчета трехмерного напряженно-деформированного состояния однородных и композитных цилиндрических оболочек в осесимметричной постановке

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Аннотация. Работа посвящена применению метода асимптотического расщепления для решения статических задач деформирования однородных изотропных и композитных цилиндрических оболочек. Рассмотрена задача упругого деформирования композитной цилиндрической оболочки под действием внутренней осесимметричной нагрузки. Решение построено путем разложения компонент тензора напряжений и вектора перемещений по степеням дифференциальных операторов, действующих вдоль оси цилиндра. При этом малым параметром выступает отношение толщины оболочки к длине цилиндра. Получена разрешающая система уравнений деформирования цилиндрической оболочки. Показано, что разработанная математическая модель позволяет восстанавливать все компоненты тензора напряжений как для толстостенных, так и для тонких цилиндрических оболочек. Произведено сравнение полученных аналитических и численных решений с конечно-элементным решением исходной осесимметричной задачи.

Ключевые слова: цилиндрические оболочки, напряженно-деформированное состояние, метод асимптотического расщепления, линейная теория упругости, осесимметричная задача, метод конечных элементов.