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On the Non-standard Interpolations in $\mathbb{C}^n$ and Combinatorial Coefficients for Weil Polyhedra

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Abstract. Multidimensional non-standard interpolation has been recently presented in an article by D. Alpay and A. Yger. We are talking about algebraic interpolation where discrete roots of a system of polynomial equations serve as nodes. With the help of the Grothendieck residue duality, the problem of describing the desired interpolation space of functions is reduced to solving the affine-bilinear equation. To implement this reduction, algorithms for calculating local Grothendieck residues or their sums are required. In a fairly general situation, the calculation of these residues is based on the well-known Gelfond–Khovanskii formula. This article provides examples of calculating local residues or their sums. In 2-dimensional case we generalise the Gelfond–Khovanskii formula for Newton polyhedra that are not in the unfolded position. This is done using the concept of an amoeba of an algebraic set and the notion of an homological resolvent for the boundary of Weil polyhedron.

Keywords: Grothendieck residue, interpolation, amoeba, Homological resolvent

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Introduction

By classical, or standard, interpolations we understand the Lagrange, Hermite, or Newton interpolations. Let us consider the first two of them.

Problem (Lagrange)**.** *Given a set of distinct points $\{w_j\}_{j=1}^m \subset \mathbb{C}$ and the values $c_j \in \mathbb{C}$, find the polynomial $f(z)$ of degree $m-1$ with the property*

$$f(w_j) = c_j, \quad j = 1, \ldots, m.$$

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Note that the interpolation polynomial f is defined in terms of the polynomial $p(z) = (z - w_1) \cdot \ldots \cdot (z - w_m)$ by the formula:

$$f(z) = p(z) \sum_{j=1}^{m} \frac{c_j}{z - w_j} \operatorname*{res}_{w_j} \left(\frac{1}{p} \right).$$

Thus, specifying the interpolation nodes in the form of the null set of the polynomial p provides tools for constructing an interpolation polynomial by using residues. More general is the following

Problem (Hermite)**.** *Let $\{w_j\}_{j=1}^m \subset \mathbb{C}$ be a set of pairwise distinct points and the following values are given*

$$c_{j,\ell} \in \mathbb{C}, \quad \text{where } j = 1, \ldots, m, \quad \ell = 0, \ldots, \mu_j - 1.$$

It is necessary to find a polynomial $f(z)$ of minimal degree, which at points w_j has the given values of derivatives up to orders of $\mu_j - 1$ including, that is

$$f^{(\ell)}(w_j) = c_{j,\ell}, \quad j = 1, \ldots, m, \quad \ell = 0, \ldots, \mu_j - 1. \tag{1}$$

In the Hermite interpolation problem, it is advisable to enumerate the set of points w_j taking into account their multiplicities, thereby considering the set $\{w_j\}_{j=1}^m$ as an algebraic set $p^{-1}(0)$, where

$$p(z) = (z - w_1)^{\mu_1} \cdot \ldots \cdot (z - w_m)^{\mu_m}. \tag{2}$$

The Hermite interpolation polynomial can be represented as

$$\sum_{j=1}^{m} \frac{p(z)}{(z - w_j)^{\mu_j}} \sum_{\ell=0}^{\mu_j - 1} \frac{c_{j,\ell}}{\ell!} \left(\sum_{s=0}^{\mu_j - \ell - 1} (z - w_j)^{\ell + s} \operatorname*{res}_{w_j} \left(\frac{(z - w_j)^{\mu_j - 1 - s}}{p} \right) \right),$$

that is, again using residues.

Recently there have been papers on so-called non-standard interpolation (see [1,2]). They pose the problem of constructing a function f on an algebraic set $p^{-1}(0)$ (considered as an analytical space), whose values lie on a given hypersurface. More precisely, the article [1] discusses the following:

Problem. *Given the complex numbers $a_{j,k}$ ($j = 1, \ldots, m$; $k = 0, \ldots, \mu_j - 1$) and c. It is necessary to describe the set of all functions f which are analytic in the neighborhood $\Omega \subset \mathbb{C}$ of points $w_1, \ldots, w_m$ and satisfy the equation:*

$$\sum_{j=1}^{m} \sum_{k=0}^{\mu_j - 1} a_{j,k} f^{(k)}(w_j) = c. \tag{3}$$

Note that if f is a solution of (3), then $f + ph$ is also a solution, where

$$p(z) = \prod_{j=1}^{m} (z - w_j)^{\mu_j}, \quad h \in \mathcal{O}(\Omega).$$

In other words, we can work in the factor ring $\mathcal{O}(\Omega)/\langle p \rangle$ by the ideal generated by the polynomial p.

In a similar form, a non-standard problem can be posed in the multidimensional case, treating the interpolation nodes $w_1, \ldots, w_m \in \mathbb{C}^n$ as the zeros of an ideal $\langle \boldsymbol{p} \rangle = \langle p_1(\boldsymbol{z}), \ldots, p_n(\boldsymbol{z}) \rangle$ in the ring of polynomials in the variable $\boldsymbol{z}$ from $\mathbb{C}^n$. In the main part of the article, in particular, we will demonstrate a non-trivial example of such multidimensional interpolation.

1. Multidimensional non-standard interpolation

To formulate a multidimensional non-standard interpolation problem, we need the following definition of a Noetherian operator.

Definition 1 (Ehrenpreis [15], Palamodov [14]). *Let $I \subset \mathbb{C}[s_1, \ldots, s_n]$ be a primary ideal. A family of linear differential operators with polynomial coefficients $\partial_\ell(\boldsymbol{s}, D)$, $\ell = 1, \ldots, t$ is called a Noetherian operator for I, if the conditions*

$$\partial_\ell(\boldsymbol{s}, D)\varphi(\boldsymbol{s})|_{V(I)} = 0 \quad \forall \ell = 1, \ldots, t$$

are necessary and sufficient for the function $\varphi(\boldsymbol{s})$ to belong to ideal I.

In the one-dimensional case an arbitrary polynomial has the form:

$$p(s) = (s - w_1)^{\mu_1} \cdot \ldots \cdot (s - w_k)^{\mu_k},$$

and its generated ideal is decomposed into the intersection of primary ones

$$\rho_j = \langle (s - w_j)^{\mu_j} \rangle, \quad j = 1, \ldots, k.$$

A necessary and sufficient condition for a given function φ to belong to the primary component ρ_j is vanishing of φ by the following operators with constant coefficients:

$$\mathcal{L}_{j,0}, \mathcal{L}_{j,1}, \ldots, \mathcal{L}_{j,\mu_j - 1},$$

where $\mathcal{L}_{i,j}[\varphi(s)] = \left.\dfrac{d^j \varphi}{ds^j}\right|_{s = w_i}$.

For an arbitrary $n \geqslant 1$, the primary components ρ_j of a zero-dimensional polynomial ideal $\langle p_1, \ldots, p_n \rangle$ are attributed to the roots w_j. The Noetherian operators ρ_j are the arrangements of differential operators with constant coefficients.

$$\mathcal{L}_{w_j,\ell}(\partial/\partial s)|_{w_j}, \quad \ell \in A_{w_j}.$$

Here A_{w_j} is a finite subset in $\mathbb{N}^n$

Now we can proceed to the formulation of the multidimensional non-standard interpolation problem.

Problem 1 ([2]). *Let $\boldsymbol{p}^{-1}(0) = \{w_1, \ldots, w_m\}$ and U be an open subset of $\mathbb{C}^n$ containing $\boldsymbol{p}^{-1}(0)$. Fix $a_{j,l}$, $j = 1, \ldots, m$, $l \in A_{w_j}$ and c; all of them are complex numbers. We need to describe the space of holomorphic functions $f : U \to \mathbb{C}$ with the following property:*

$$\sum_{j=1}^{m} \sum_{\boldsymbol{\ell} \in A_{w_j}} a_{j,\boldsymbol{\ell}} \mathcal{L}_{w_j,\boldsymbol{\ell}}[f](w_j) = c. \tag{4}$$

The following monomial basis

$$\mathcal{B} = \{\boldsymbol{s}^{\beta_k}; k = 0, \ldots, N(\boldsymbol{p}) - 1\}$$

in the quotient space $\mathbb{C}[\boldsymbol{z}]/\langle p \rangle$ is one of ingredients for solving the interpolation problem. In fact, this factor is the space of remainders when dividing polynomials by the ideal $\langle \boldsymbol{p} \rangle$.

2. Grothendieck residue and its role in interpolation theory

The Grothendieck residue is a cornerstone of complex analysis and algebraic geometry and it plays the key role in the singularity theory and foliations theory. Assume that the sequence of germs

$$f_1, \ldots, f_n \in \mathbb{C}[z] = \mathbb{C}[z_1, \ldots, z_n]$$

have an isolated common zero at $a \in \mathbb{C}^n$. Consider a meromorphic differential n-form

$$\omega = \frac{1}{(2\pi i)^n} \frac{h(z)\, dz}{f_1(z) \ldots f_n(z)} \quad \text{(with } dz = dz_1 \wedge \cdots \wedge dz_n\text{)}.$$

Definition 2 ([4, 5]). *The Grothendieck residue, associated with $f = (f_1, \ldots, f_n)$ and h, is determined as an integral*

$$\underset{a\ f}{\operatorname{res}}(h) = \int_{\Gamma_a} \omega$$

of the form ω over a very special cycle

$$\Gamma_a = \{z \in U_a \colon |f_j(z)| = \varepsilon_j, j = 1, \ldots, n\},$$

where the neighborhood U_a of a and ε_j are chosen such that the closure $\overline{U}_a$ does not contain roots different from a and $\Gamma_a \subset\subset U_a$.

We call the integration set Γ_a a Grothendieck cycle. Note that this is the skeleton of the Weil polyhedron $\{z \in U_a \colon |f_j(z)| < \varepsilon_j, j = 1, \ldots, n\}$.

In the case of a finite set of zeros, the mapping $\boldsymbol{p} = (p_1, \ldots, p_n)$ can define the global Grothendieck residue as the sum of the local ones. The global residue is denoted by $\operatorname{Res}\begin{bmatrix} h(\boldsymbol{s})\, ds_1 \wedge \cdots \wedge ds_n \\ p_1(\boldsymbol{s}), \ldots, p_n(\boldsymbol{s}) \end{bmatrix}$, $h \in H(D)$ (D is the domain containing all zeros of the mapping $\boldsymbol{p}$).

Now, using the notation introduced above, we can state a theorem that gives the way to solve Problem 1.

Theorem 2.1 (Alpay, Yger [2]). *Let $\{w_1, \ldots, w_m\} = \boldsymbol{p}^{-1}(0)$, U be an open subset in $\mathbb{C}^n$ containing $\boldsymbol{p}^{-1}(0)$. Let the sequence*

$$\boldsymbol{a} = \{a_{j,\boldsymbol{\ell}}, j = 1, \ldots, m, \boldsymbol{\ell} \in A_{w_j}\}$$

and the complex number c be given. Let us denote the polynomials

$$h^{\boldsymbol{a}}_{w_j}(\boldsymbol{s}) = \sum_{\boldsymbol{\ell} \in A_{w_j}} a_{j,\boldsymbol{\ell}} (\boldsymbol{s} - w_j)^{\boldsymbol{\ell}} / \boldsymbol{\ell}!,$$

making up the sequence $\boldsymbol{h}^{\boldsymbol{a}}_{\boldsymbol{w}} = [h^{\boldsymbol{a}}_{w_1}, \ldots, h^{\boldsymbol{a}}_{w_m}]$, and let

$$\alpha[\boldsymbol{h}^{\boldsymbol{a}}_{\boldsymbol{w}}] = (\alpha_0[\boldsymbol{h}^{\boldsymbol{a}}_{\boldsymbol{w}}], \ldots, \alpha_{N(\boldsymbol{p})-1}[\boldsymbol{h}^{\boldsymbol{a}}_{\boldsymbol{w}}])$$

be the projection of this sequence onto the quotient space $\mathbb{C}[\boldsymbol{z}]/\langle p \rangle$.

- *If $\alpha[\boldsymbol{h}^{\boldsymbol{a}}_{\boldsymbol{w}}] = 0$, then the problem has no solution in the case $c \neq 0$, and any function $f \in \mathcal{O}(U)$ is a solution in the case $c = 0$;*

- *If* $\alpha[\boldsymbol{h}^{\boldsymbol{a}}_{\boldsymbol{w}}] \neq 0$*, then* $f \in \mathcal{O}(U)$ *satisfies the condition* (4) *iff*

$$\alpha[\boldsymbol{f}] \cdot \boldsymbol{Q}_{\boldsymbol{p}}[\mathcal{B}] \cdot \alpha[\boldsymbol{h}^{\boldsymbol{a}}_{\boldsymbol{w}}]^T = c,$$

where T *is the transposition sign, and* $\boldsymbol{Q}_{\boldsymbol{p}}[\mathcal{B}]$ *is the Grothendieck global residues matrix:*

$$\boldsymbol{Q}_{\boldsymbol{p}}[\mathcal{B}] = \operatorname{Res}\left[\frac{\boldsymbol{s}^{\beta_{k_1}+\beta_{k_2}}\,d\boldsymbol{s}}{p_1(\boldsymbol{s})\dots p_n(\boldsymbol{s})}\right]_{0\leqslant k_1,k_2\leqslant N\langle \boldsymbol{p}\rangle-1}$$

3. Amoeba and its complement

For further reasoning we will need to introduce the concept of the amoeba of the Laurent polynomial, as well as describe some of its properties.

Definition 3. *Given a Laurent polynomial* f *its amoeba* A_f *is the image of the hypersurface* $V = f^{-1}(0)$ *under the map*

$$\operatorname{Log}\colon (z_1,\dots,z_n) \to (\log|z_1|,\dots,\log|z_n|).$$

For the amoeba we will also use notation A_V*.*

Amoeba reflects the distribution of the algebraic set V. More precisely, one can say that the amoeba depicts hollows for V.

The shape of the amoeba is closely related to the Newton polytope Δ_f of the polynomial f. Recall that Δ_f is defined as the convex hull in $\mathbb{R}^n$ of the index set A in the experession

$$f(z_1,\dots,z_n) = \sum_{\alpha\in A} a_\alpha z^\alpha.$$

The set of integer points in Δ_f admits a natural partition $\Delta_f \cap \mathbb{Z}^n = \bigcup_\Gamma A_\Gamma$, where Γ is a face on Δ_f and A_Γ denotes the intersection of $\mathbb{Z}^n$ with the relative interior of Γ. We shall consider the dual cone C_ν of Δ_f at ν defined as

$$C_\nu = \left\{ s \in \mathbb{R}^n \colon \langle s,\nu\rangle = \max_{\alpha\in\Delta_f}\langle s,\alpha\rangle \right\}.$$

Notice that $\dim C_\nu = n - \dim\Gamma$ when $\nu \in A_\Gamma$. In particular, C_ν has nonempty interior if ν is a vertex of Δ_f, and it equals $\{0\}$ whenever ν is an interior point of Δ_f.

The following theorem allows us to introduce an order on the components of the amoeba complement.

Theorem 3.1 (Forsberg, Passare, Tsikh [13])**.** *On the set* $\{E\}$ *of connected components of* ${}^cA_f$ *there is an injective map (the order map)*

$$\nu\colon \{E\} \to \Delta_f \cap \mathbb{Z}^n$$

with the property that the dual cone $C_{\nu(E)}$ *is equal to the recession cone of* E*. That is, for any* $u \in E$ *one has* $u + C_\nu \in E$ *and no strictly larger cone is contained in* E *(Notice that if* ν *is the* k*-skeleton of* Δ_f *the* C_ν *has dimension* $n-k$*).*

Thus, connected components can be numbered as E_ν with integer $\nu \in \Delta_f$. See, for examples, the figures below for the polynomial $1 + z_1^2 z_2 + z_1 z_2^2 + 5 z_1 z_2$ (Fig. 1).

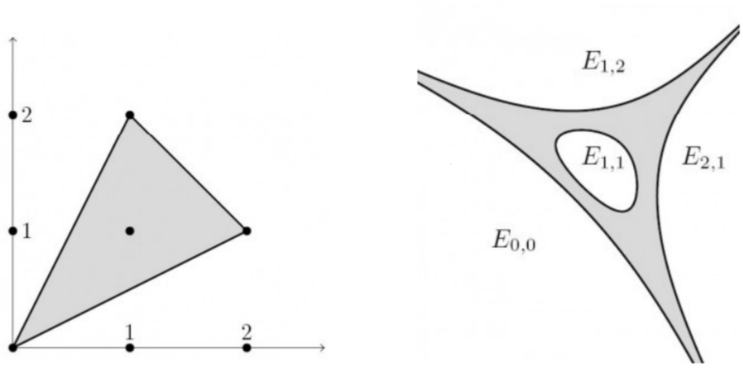

Fig. 1. An example of an amoeba

4. The Gelfond-Khovanskii formula

We say that the sequence of polytopes $\Delta_1, \ldots, \Delta_n \in \mathbb{R}^n$ is unfolded, if for each covector $v \in (\mathbb{R}^n)^*$ there is such number i, that for vectors $x \in \Delta = \Delta_1 + \ldots + \Delta_n$ (the Minkovskii sum of polytopes Δ_j) the scalar product (x, v) take its maximal value only in some vertex of Δ_i.

Theorem 4.1 (Gelfond–Khovanskii [16]). *Assume that the Newton polytopes $\Delta_1, \ldots, \Delta_n$ of polynomials $f_1, \ldots, f_n$ are unfolded. Then the sum of all local residues in $(\mathbb{C} \setminus 0)^n$ is calculated by the formula:*

$$\sum_{\{a\}} \operatorname*{res}_{a}{}_f (h) = \sum_{\nu \in \operatorname{Vert} \Delta} k_\nu \operatorname*{Res}_{E_\nu} \left(\frac{h}{f_1 \ldots f_n} \right),$$

where $\operatorname{Res}_{E_\nu}$ is the coefficient c_{-I} of the Laurent decomposition for $\dfrac{h}{f_1 \cdots f_n}$ in the connected component E_ν.

In fact one can prove that the sum $\sum\limits_{\{a\}} \Gamma_a$ of local Grothendieck cycles Γ_a is homologically equivalent to the sum

$$\sum_{\nu \in \operatorname{Vert} \Delta} k_\nu \operatorname{Log}^{-1}(u_\nu), \quad u_\nu \in E_\nu,$$

where k_ν are the combinatorial coefficients: We ascribe the combinatorial coefficient to each vertex ν of the sum Δ of unfolded polytopes. Each face $\Gamma \subset \Delta$ is a sum $\Gamma_1 + \ldots + \Gamma_n$ of faces $\Gamma_i \subset \Delta_i$.

Definition 4 ([16]). *Combinatorial coefficient k_A is the local degree of the germ*

$$(\partial\Delta, A) \to (\partial\mathbb{R}^n_+, 0)$$

of the characteristic map $(h_1, \ldots, h_n) \colon \partial\Delta \to \partial\mathbb{R}^n_+$, where each component h_i is zero precisely on that face of Γ, for which the term Γ_i is a vertex of Δ_i.

5. The homological resolvent

Let $\mathcal{U} = \{U_i\}$ be a finite covering of some manifold X. Denote $S_q^{\mathcal{U}}$ the group generated by all singular simplices of dimension q which supports belong to some element U_i of the covering $\mathcal{U}$.

Let $C_*(\mathcal{U}, S^*)$ be the complex (which can be called the Čech–de Rham complex in homological version) formed by the group $S_q^{\mathcal{U}}$, that is bigraduated groups

$$C_{p,q} = \bigoplus_{i_0<i_1<\ldots<i_p} S_q(U_{i_0} \cap U_{i_1} \cap \ldots \cap U_{i_p}), \qquad p, q = 0, 1, \ldots$$

We will need the following definition later to calculate local Grothendieck residues. The definition given below differs slightly from Gleason's definition [17].

Definition 5. *The sequence of $\mathcal{U}$-chains $\{\xi_p\}_{p=0}^r$, $\xi_p \in C_{p,r-p}$ we will call the $\mathcal{U}$-resolvent of the cycle $\xi \in Z_r(S_*^{\mathcal{U}})$ if the following two conditions are met:*

1. $\varepsilon\xi_0 = \xi$.
2. $\delta\xi_p = \partial\xi_{p-1}, \quad p = 1, \ldots, r.$

Here $\varepsilon : C_{0,*} \to S_*^{\mathcal{U}}$ is an inclusion operator $U_i \subset X$ the action of which is determined by the formula of the alternated sum:

$$\varepsilon\sigma = \sum_{i \in I} \sigma(i).$$

Boundary operator $\partial : C_{*,q} \to C_{*,q-1}$ is defined as

$$(\partial\sigma)(i_0, i_1, \ldots, i_p) = \partial(\sigma(i_0, i_1, \ldots, i_p)).$$

The Čech coboundary operator: Inclusions

$$U_{i_0} \cap U_{i_1} \cap \ldots \cap U_{i_p} \hookrightarrow U_{i_0} \cap U_{i_1} \cap \ldots [i_k] \ldots \cap U_{i_p}, \quad k = 0, \ldots, p,$$

induce the operator $\delta : C_{p,*} \to C_{p-1,*}$ determined using the alternated sum formula:

$$(\delta\sigma)(i_0, i_1, \ldots, i_{p-1}) = \sum_{i \in I} \sigma(i, i_0, \ldots, i_{p-1}).$$

6. A generalisation of the Gelfond-Khovanskii formula in 2-dimensional case

Let us consider an example of the polynomial system of equations:

$$\begin{aligned} F_1 &= 3z_1^2 z_2 + z_2^4 + 2z_1 z_2^3 = 0, \\ F_2 &= z_1^3 + 4z_1 z_2^3 + 3z_1^2 z_2^2 = 0, \end{aligned} \tag{5}$$

for which the Newton polytopes are not in unfolded position: Δ_1 and Δ_2 have parallel edges.

For convenience, we introduce the notation:

$$\begin{aligned} F_1 &= z_2(3z_1^2 + z_2^3 + 2z_1 z_2^2) = z_2 f_1, \\ F_2 &= z_1(z_1^2 + 4z_2^3 + 3z_1 z_2^2) = z_1 f_2. \end{aligned}$$

Amoebas of A_{f_1} and A_{f_2} are shown in Fig. 2. In the complement of amoeba $A_{f_1 f_2}$ there are 6 connected components: E_{51}, E_{35}, E_{17} corresponds to the vertices of $\Delta = \Delta_1 + \Delta_2$, and E_{13}, E_{26}, E_{34} correspond to integer points in the relative interiors of the edges.

We will find the formula for the resolvent of the boundary ∂W of the Weil polyhedron $W = \text{Log}^{-1}(z\Delta)$, which is defined by a homothetic dilatation of the triangle Δ which contains

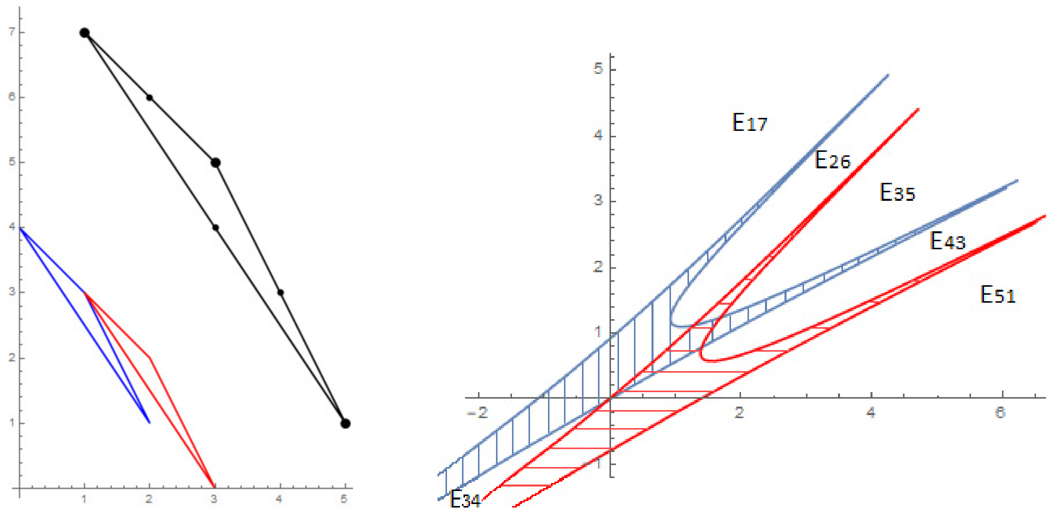

Fig. 2. Newton polytopes $\Delta_1, \Delta_2, \Delta$ and amoebas $\mathcal{A}_{f_1}$, $\mathcal{A}_{f_2}$

the intersection of the amoebas Δ_1 and Δ_2. So, W contains all roots of the system $F_1 = F_2 = 0$ in the torus $(\mathbb{C}\backslash\{0\})^2$.

Note that the sets

$$U_1 = \left\{(\mathbb{C}\backslash\{0\})^2\right\} \backslash \{z : F_1(z) = 0\},$$
$$U_2 = \left\{(\mathbb{C}\backslash\{0\})^2\right\} \backslash \{z : F_2(z) = 0\}.$$

form a covering $\mathcal{U}$ of the complement $(\mathbb{C}\backslash\{0\})^2\backslash\{z : F_1(z) = F_2(z) = 0\}$. We want to construct the resolvent for the cycle $\xi \in Z_3(S_*^{\mathcal{U}})$ which is the boundary of the polyhedron W.

1^{st} step: Decomposition $\xi = \sigma_1 + \sigma_2$ by blue and red chains with supports *supp* $\sigma_j \subset U_j$ (see Fig. 3). Therefore we can take $\xi_0 \in C_{0,3}$ in Definition 5 as the following:

$$\begin{cases} \xi_0(1) = \sigma_1, \\ \xi_0(2) = \sigma_2 \end{cases}$$

(Note that each support of σ_j consists of 2 connected components).

2^{nd} step: Computation of the boundary of chain ξ_0 gives

$$\begin{cases} (\partial\xi_0)(1) = \partial(\xi_0(1)) = \partial\sigma_1 = \Gamma_{26} - \Gamma_{35} + \Gamma_{43} - \Gamma_{34}, \\ (\partial\xi_0)(2) = \partial(\xi_0(2)) = \partial\sigma_2 = \Gamma_{34} - \Gamma_{26} + \Gamma_{35} - \Gamma_{43}. \end{cases}$$

Now let $\xi_1 \in C_{1,2}$ be our chain from the resolvent. Then by inclusions $U_i \cap U_j \hookrightarrow U_i$ we get the following system:

$$\begin{cases} (\delta\xi_1)(1) = \sum_{i\in I} \xi_1(i,1) = \xi_1(1,1) + \xi_1(2,1) = -\xi_1(1,2), \\ (\delta\xi_1)(2) = \sum_{i\in I} \xi_1(i,2) = \xi_1(1,2) + \xi_1(2,2) = \xi_1(1,2). \end{cases}$$

Therefore if we take into account the fact that $\delta\xi_1$ should be equal to $\partial\xi_0$ then from the system

$$\begin{cases} -\xi_1(1,2) = (\delta\xi_1)(1) = (\partial\xi_0)(1) = \Gamma_{26} - \Gamma_{35} + \Gamma_{43} - \Gamma_{34}, \\ \xi_1(1,2) = (\delta\xi_1)(2) = (\partial\xi_0)(2) = \Gamma_{34} - \Gamma_{26} + \Gamma_{35} - \Gamma_{43} \end{cases}$$

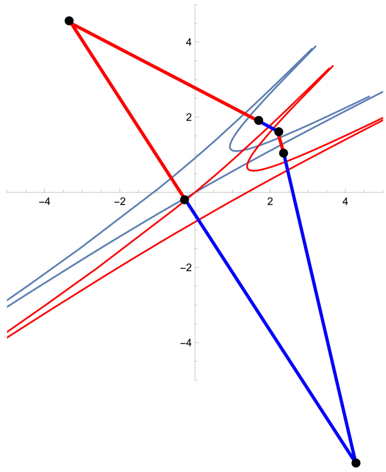

Fig. 3. The boundary of the Weil polyhedron

we have the following expression for the resolvent:

$$\xi_1(1,2) = \Gamma_{34} - \Gamma_{26} + \Gamma_{35} - \Gamma_{43}.$$

Thus we get the following formula for the sum of the Grothendieck residues:

$$\sum_{\{a\}} \operatorname{res}_{F \atop a}(h) = \operatorname*{Res}_{E_{34}}\left(\frac{h}{F_1F_2}\right) - \operatorname*{Res}_{E_{26}}\left(\frac{h}{F_1F_2}\right) + \operatorname*{Res}_{E_{35}}\left(\frac{h}{F_1F_2}\right) - \operatorname*{Res}_{E_{43}}\left(\frac{h}{F_1F_2}\right).$$

Obviously, the following statement in dimension 2 is obtained by similar reasoning.

Theorem 6.1. *Assume that the system $F_1 = F_2 = 0$ has in $(\mathbb{C}\backslash\{0\})^2$ a finite number of roots. Then the sum of Grothendieck residues in the torus $(\mathbb{C}\backslash\{0\})^2$ is calculated by the formula*

$$\sum_{\{a\}} \operatorname{res}_{F \atop a}(h) = \sum_{\nu\in\mathbb{Z}^2\cap\partial\Delta} k_\nu \operatorname*{Res}_{E_\nu}\left(\frac{h}{F_1F_2}\right), \text{ where } k_\nu \in \{0,1,-1\}.$$

7. Example in dimension 3

Let us consider an example of non-standard interpolation when the single point $a = 0$ is defined as an isolated zero of the polynomial system

$$P_1 = z_1^3 - z_2z_3 = 0,$$
$$P_2 = z_2^3 - z_1z_3 = 0,$$
$$P_3 = z_3^3 - z_1z_2 = 0.$$

We have an open covering $\mathcal{U}$ for the punctured neighborhood U of the origin:

$$U_i = U\backslash\{z : P_i(z) = 0\}, \qquad i = 1,2,3.$$

Now we want to construct the resolvent for a multicoloured cycle $\xi \in Z_5(S_*^{\mathcal{U}})$ (in the picture 4 below) which is homeomorphic to the sphere S^5.

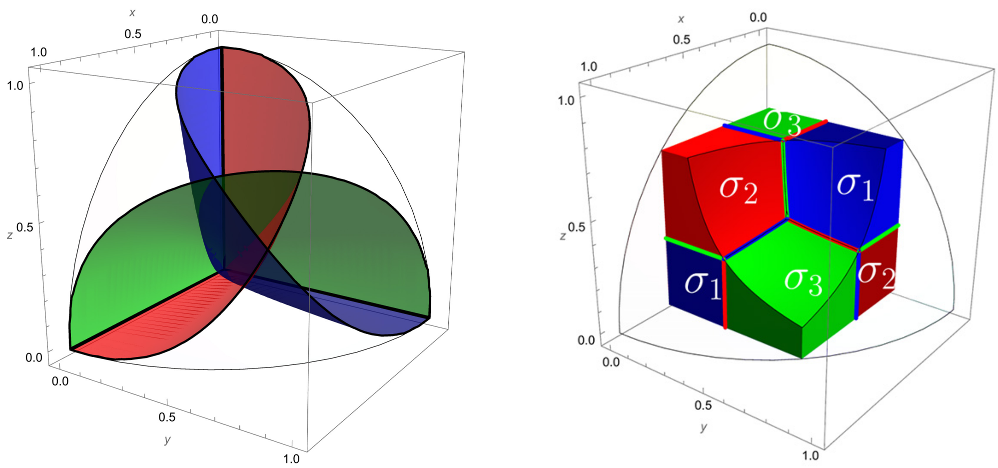

Fig. 4. Hypersurfaces (left) and toric polyhedron (right) on the Reinhardt diagram

1[st] step: Decomposition $\xi = \sigma_1 + \sigma_2 + \sigma_3$ by blue, red and green chains (see Fig. 4) with supports *supp* $\sigma_j \subset U_j$. Therefore we can take $\xi_0 \in C_{0,5}$ as the following chain:

$$\{\xi_0(1) = \sigma_1,\ \xi_0(2) = \sigma_2,\ xi_0(3) = \sigma_3\}$$

(Note that each support of σ_j consists of 2 connected components).

2[nd] step: Computation of the boundary of chain ξ_0 (see Fig. 5):

$$\begin{cases} (\partial\xi_0)(1) = \partial(\xi_0(1)) = \partial\sigma_1 = c + b - f - g - d + h, \\ (\partial\xi_0)(2) = \partial(\xi_0(2)) = \partial\sigma_2 = -a - c + e + d - i + g, \\ (\partial\xi_0)(3) = \partial(\xi_0(3)) = \partial\sigma_3 = -h - e + f + i - b + a. \end{cases}$$

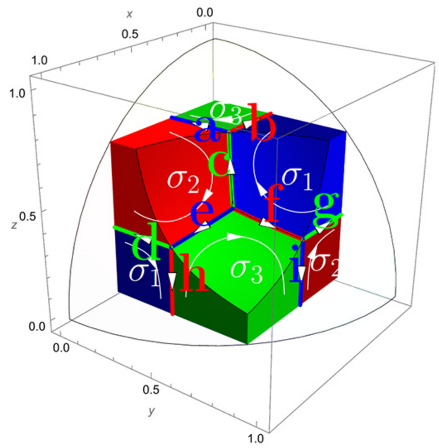

Fig. 5. Orientation and indexing of chains

Now let $\xi_1 \in C_{1,4}$ be our chain from the resolvent. Then by inclusions $U_i \cap U_j \hookrightarrow U_i$ we get the following system:

$$\begin{cases} (\delta\xi_1)(1) = \sum\limits_{i\in I} \xi_1(i,1) = \xi_1(1,1) + \xi_1(2,1) + \xi_1(3,1) = -\xi_1(1,2) - \xi_1(1,3), \\ (\delta\xi_1)(2) = \sum\limits_{i\in I} \xi_1(i,2) = \xi_1(1,2) + \xi_1(2,2) + \xi_1(3,2) = \xi_1(1,2) - \xi_1(2,3), \\ (\delta\xi_1)(3) = \sum\limits_{i\in I} \xi_1(i,3) = \xi_1(1,3) + \xi_1(2,3) + \xi_1(3,3) = \xi_1(1,3) + \xi_1(2,3). \end{cases}$$

Note that chain $\xi_1(1,2)$ can contain only green segments (d,c,g) [only these segments belongs to both sets U_1 and U_2], chain $\xi_1(1,3)$ can contain only red segments (b,f,h) chain $\xi_1(2,3)$ can contain only blue segments (a,e,i). Therefore if we take this fact into account together with the fact that $\delta\xi_1$ should be equal to $\partial\xi_0$ then from system:

$$\begin{cases} -\xi_1(1,2) - \xi_1(1,3) = (\delta\xi_1)(1) = (\partial\xi_0)(1) = c + b - f - g - d + h, \\ \xi_1(1,2) - \xi_1(2,3) = (\delta\xi_1)(2) = (\partial\xi_0)(2) = -a - c + e + d - i + g, \\ \xi_1(1,3) + \xi_1(2,3) = (\delta\xi_1)(3) = (\partial\xi_0)(3) = -h - e + f + i - b + a. \end{cases}$$

we have the following solution:

$$\begin{cases} \xi_1(1,2) = d - c + g, \\ \xi_1(1,3) = -b + f - h, \\ \xi_1(2,3) = a - e + i. \end{cases}$$

3rd step: Compute the boundary of the chain ξ_1 (see Fig. 6):

$$\begin{cases} (\partial\xi_1)(1,2) = \partial(\xi_1(1,2)) = \partial(d - c + g) = \partial d - \partial c + \partial g = -\Gamma_2 + \Gamma_0 - \Gamma_1 - \Gamma_3, \\ (\partial\xi_1)(1,3) = \partial(\xi_1(1,3)) = \partial(-b + f - h) = -\partial b + \partial f - \partial h = \Gamma_1 + \Gamma_3 - \Gamma_0 + \Gamma_2, \\ (\partial\xi_1)(2,3) = \partial(\xi_1(2,3)) = \partial(a - e + i) = \partial a - \partial e + \partial i = -\Gamma_1 + \Gamma_0 - \Gamma_2 - \Gamma_3. \end{cases}$$

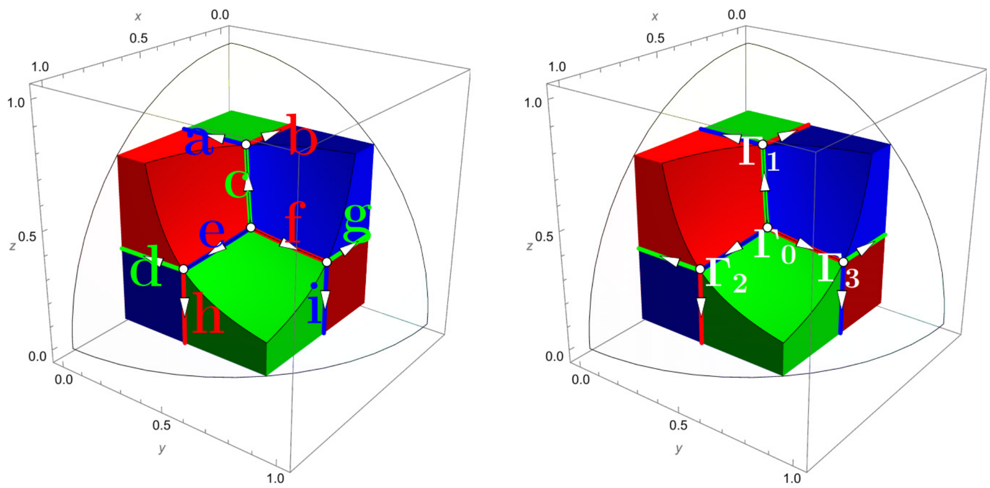

Fig. 6. 4-dimensional chains $a, b, \ldots$ (left) and toric cycles $\Gamma_0, \Gamma_1, \ldots$ (right)

Now let $\xi_2 \in C_{2,3}$ be our chain from the resolvent. Then by inclusions $U_1 \cap U_2 \cap U_3 \hookrightarrow U_i \cap U_j$

we get the following system:

$$(\delta\xi_2)(1,2) = \sum_{i\in I} \xi_2(i,1,2) = \xi_2(1,1,2) + \xi_2(2,1,2) + \xi_2(3,1,2) = \xi_2(1,2,3),$$
$$(\delta\xi_2)(1,3) = \sum_{i\in I} \xi_2(i,1,3) = \xi_2(1,1,3) + \xi_2(2,1,3) + \xi_2(3,1,3) = -\xi_2(1,2,3),$$
$$(\delta\xi_2)(2,3) = \sum_{i\in I} \xi_2(i,2,3) = \xi_2(1,2,3) + \xi_2(2,2,3) + \xi_2(3,2,3) = \xi_2(1,2,3).$$

Therefore by resolvent condition we have:

$$\begin{cases} \xi_2(1,2,3) = (\delta\xi_2)(1,2) = (\partial\xi_1)(1,2) = -\Gamma_2 + \Gamma_0 - \Gamma_1 - \Gamma_3, \\ -\xi_2(1,2,3) = (\delta\xi_2)(1,3) = (\partial\xi_1)(1,3) = \Gamma_1 + \Gamma_3 - \Gamma_0 + \Gamma_2, \\ \xi_2(1,2,3) = (\delta\xi_2)(2,3) = (\partial\xi_1)(2,3) = -\Gamma_1 + \Gamma_0 - \Gamma_2 - \Gamma_3. \end{cases}$$

That is why $\xi_2(1,2,3) = \Gamma_0 - \Gamma_1 - \Gamma_2 - \Gamma_3$, and therefore the Grothendieck cycle admits a representation

$$\Gamma = \Gamma_{222} - \Gamma_{511} - \Gamma_{151} - \Gamma_{115}$$

by toric cycles. This fact helps us to construct the matrix $\boldsymbol{Q_p}[\mathcal{B}]$ from Theorem 2.1. For example, let us compute the following integral.

$$\begin{aligned}
&\frac{1}{(2\pi i)^3}\int\limits_{\Gamma_{222}} \frac{c z_1^\alpha z_2^\beta z_3^\gamma\, d\boldsymbol{z}}{P_1P_2P_3} = \frac{c}{(2\pi i)^3}\int\limits_{\Gamma_{222}} \frac{z_1^\alpha z_2^\beta z_3^\gamma\, dz_1\, dz_2\, dz_3}{(z_1^3 - z_2z_3)(z_2^3 - z_1z_3)(z_3^3 - z_1z_2)} = \\
&= \frac{c}{(2\pi i)^3}\int\limits_{\Gamma_{222}} \frac{z_1^\alpha z_2^\beta z_3^\gamma\, dz_1\, dz_2\, dz_3}{(-z_2z_3)(-z_1z_3)(-z_1z_2)(1-\frac{z_1^3}{z_2z_3})(1-\frac{z_2^3}{z_1z_3})(1-\frac{z_3^3}{z_1z_2})} = \\
&= \frac{-c}{(2\pi i)^3}\int\limits_{\Gamma_{222}} \frac{z_1^{\alpha-2} z_2^{\beta-2} z_3^{\gamma-2}\, dz_1\, dz_2\, dz_3}{(1-\frac{z_1^3}{z_2z_3})(1-\frac{z_2^3}{z_1z_3})(1-\frac{z_3^3}{z_1z_2})} = \\
&= \frac{-c}{(2\pi i)^3}\int\limits_{\Gamma_{222}} \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{l=0}^{\infty} \left(\frac{z_1^3}{z_2z_3}\right)^m \left(\frac{z_2^3}{z_1z_3}\right)^n \left(\frac{z_3^3}{z_1z_2}\right)^l z_1^{\alpha-2} z_2^{\beta-2} z_3^{\gamma-2}\, d\boldsymbol{z} = \\
&= -c\sum_{m,n,l\geqslant 0} \frac{1}{(2\pi i)^3}\int\limits_{\Gamma_{222}} z_1^{3m-n-l+\alpha-2} z_2^{3n-m-l+\beta-2} z_3^{3l-m-n+\gamma-2}\, d\boldsymbol{z} = -ck.
\end{aligned}$$

Here k is the number of sets of non-negative integers $\{m,n,l\}$ for which the following system of equalities holds:

$$\begin{cases} 3m - n - l + \alpha - 2 = -1, \\ -m + 3n - l + \beta - 2 = -1, \\ -m - n + 3l + \gamma - 2 = -1. \end{cases}$$

Let us consider an example of non-standard interpolation when the single point $a = 0$ is defined as an isolated zero of the polynomial system

$$P_1 = z_1^3 - z_2z_3 = 0,$$
$$P_2 = z_2^3 - z_1z_3 = 0,$$
$$P_3 = z_3^3 - z_1z_2 = 0.$$

The multiplicity at 0 equals 11.
The Grothendieck cycle admits a representation

$$\Gamma_0 = \Gamma_{222} - \Gamma_{511} - \Gamma_{151} - \Gamma_{115}.$$

Proposition 1. *The list of Noetherian operators for the ideal $I_0\langle \boldsymbol{P}\rangle$ is:*

$$\{\boldsymbol{\mathcal{L}}_{\mathbf{0},\ell}\} = \Bigg\{\boldsymbol{\mathcal{L}}_{\mathbf{0},000} = \left(-\partial^0 - \frac{\partial^3}{\partial z_1 \partial z_2 \partial z_3} - \frac{1}{4!}\frac{\partial^4}{\partial z_1^4} - \frac{1}{4!}\frac{\partial^4}{\partial z_2^4} - \frac{1}{4!}\frac{\partial^4}{\partial z_3^4}\right);$$
$$\boldsymbol{\mathcal{L}}_{\mathbf{0},100} = \left(-\frac{1}{3!}\frac{\partial^3}{\partial z_1^3} - \frac{\partial^2}{\partial z_2 \partial z_3}\right); \boldsymbol{\mathcal{L}}_{\mathbf{0},010} = \left(-\frac{1}{3!}\frac{\partial^3}{\partial z_2^3} - \frac{\partial^2}{\partial z_1 \partial z_3}\right);$$
$$\boldsymbol{\mathcal{L}}_{\mathbf{0},001} = \left(-\frac{1}{3!}\frac{\partial^3}{\partial z_3^3} - \frac{\partial^2}{\partial z_1 \partial z_2}\right); \boldsymbol{\mathcal{L}}_{\mathbf{0},110} = \left(-\frac{\partial}{\partial z_3}\right); \boldsymbol{\mathcal{L}}_{\mathbf{0},101} = \left(-\frac{\partial}{\partial z_2}\right);$$
$$\boldsymbol{\mathcal{L}}_{\mathbf{0},011} = \left(-\frac{\partial}{\partial z_1}\right); \boldsymbol{\mathcal{L}}_{\mathbf{0},200} = \left(-\frac{1}{4}\frac{\partial^2}{\partial z_1^2}\right); \boldsymbol{\mathcal{L}}_{\mathbf{0},020} = \left(-\frac{1}{4}\frac{\partial^2}{\partial z_2^2}\right);$$
$$\boldsymbol{\mathcal{L}}_{\mathbf{0},002} = \left(-\frac{1}{4}\frac{\partial^2}{\partial z_3^2}\right); \boldsymbol{\mathcal{L}}_{\mathbf{0},111} = (-\partial^0); \boldsymbol{\mathcal{L}}_{\mathbf{0},300} = \left(-\frac{1}{3!}\frac{\partial}{\partial z_1}\right);$$
$$\boldsymbol{\mathcal{L}}_{\mathbf{0},030} = \left(-\frac{1}{3!}\frac{\partial}{\partial z_2}\right); \boldsymbol{\mathcal{L}}_{\mathbf{0},003} = \left(-\frac{1}{3!}\frac{\partial}{\partial z_3}\right); \boldsymbol{\mathcal{L}}_{\mathbf{0},400} = \left(-\frac{1}{4!}\partial^0\right);$$
$$\boldsymbol{\mathcal{L}}_{\mathbf{0},040} = \left(-\frac{1}{4!}\partial^0\right); \boldsymbol{\mathcal{L}}_{\mathbf{0},004} = \left(-\frac{1}{4!}\partial^0\right)\Bigg\}.$$

Proposition 2. *The monomial basis for the factor-space $\mathcal{O}_{\mathbf{0}}/I_0\langle \boldsymbol{P}\rangle$ is*

$$\{1, z_1, z_2, z_3, z_1^2, z_2^2, z_3^2, z_1 z_2, z_1 z_3, z_2 z_3, z_1 z_2 z_3\}.$$

Now we can formulate the local non-standard interpolation problem and its solution.

Problem 2. *Let the complex numbers $\{a_\ell\}_{\ell \in A_{\mathbf{0}}}$ and c be given. Let $U_{\mathbf{0}}$ is an open subset of $\mathbb{C}^3$, containing the point $\mathbf{0} = (0,0,0)$ which is an isolated zero of mapping $\boldsymbol{P} = (z_1^3 - z_2 z_3, z_2^3 - z_1 z_3, z_3^3 - z_1 z_2)$. It is necessary to describe the space of holomorphic functions $f : U_{\mathbf{0}} \to \mathbb{C}$, with the property:*

$$\sum_{\ell \in A_{\mathbf{0}}} a_\ell \boldsymbol{\mathcal{L}}_{\mathbf{0},\ell}[f](\mathbf{0}) = c.$$

Theorem 7.1. *If $\boldsymbol{\alpha}[\boldsymbol{h}_{\boldsymbol{w}}^{\boldsymbol{a}}] \neq \mathbf{0}$, then the holomorphic function $f(\boldsymbol{s})$ satisfies the Alpay–Yger problem for single point ($m = 1$) iff the coordinatization of f satisfies the following condition:*

$$\left(a_{000} + a_{111} - \frac{a_{400} + a_{040} + a_{004}}{24}\right)\alpha_1[f] + \left(a_{011} + \frac{a_{300}}{6}\right)\alpha_2[f] +$$
$$+ \left(a_{101} + \frac{a_{030}}{6}\right)\alpha_3[f] + \left(a_{110} + \frac{a_{003}}{6}\right)\alpha_4[f] + \frac{a_{200}}{2}\alpha_5[f] +$$
$$+ \frac{a_{020}}{2}\alpha_6[f] + \frac{a_{002}}{2}\alpha_7[f] + a_{001}\alpha_8[f] + a_{010}\alpha_9[f] + a_{001}\alpha_{10}[f] + a_{000}\alpha_{11}[f] = -c.$$

This means that the coordinate vector of f in the local algebra lies in the prescribed affine hyperplane $\Pi_a \subset \mathbb{C}^{11}$.

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О нестандартных интерполяциях в $\mathbb{C}^n$ и комбинаторных коэффициентах для многогранников Вейля

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Аннотация. Многомерная нестандартная интерполяция была недавно представлена в статье Д. Алпая и А. Ижера. Речь идет об алгебраической интерполяции, в которой узлами служат дискретные корни системы полиномиальных уравнений. С помощью двойственности вычета Гротендика задача описания искомого интерполяционного пространства функций редуцируется к решению аффинно-билинейного уравнения. Для реализации этой редукции требуются алгоритмы вычисления локальных вычетов Гротендика или их сумм. В достаточно общей ситуации вычисление указанных вычетов основано на известной формуле Гельфонд–Хованского. В данной статье приведены примеры вычисления локальных вычетов или их сумм. В двумерном случае мы обобщаем формулу Гельфонд–Хованского для многогранников Ньютона, которые не находятся в развернутом положении. Это делается с использованием понятия амебы алгебраического множества и понятия гомологической резольвенты для границы многогранника Вейля.

Ключевые слова: вычет Гротендика, интерполяция, амёба, гомологическая резольвента.