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Mellin Transforms for Rational Functions with Quasi-elliptic Denominators

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Abstract. The paper deals with residue representations of n–dimensional Mellin transforms for rational functions with quasi-elliptic denominators. These representations are given by integrals over $(n-1)$-dimensional relative cycles. The amount of representations (or cycles) equals to the number of facets of the Newton polytope for the denominator of the rational function.

Keywords: multidimensional Mellin transform, quasi-elliptic polynomial, Leray residue form, amoeba.

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Introduction

The fundamental property of the Mellin transform, which largely determines the scope of its applications, is the correspondence between the asymptotic behavior of the original function $g(x)$ and the singularities of the transformed function. The role of this fundamental correspondence for the Mellin transform of a function of one variable is noted in numerous papers by F. Flajolet, in particular, in [4] in relation with the calculation of harmonic sums.

We recall that the Mellin transform of the function $g(x)$ is defined by the integral

$$M[g](z) = \int_{\mathbb{R}^n_+} g(x)x^{z-I}dx, \tag{1}$$

where the differential form

$$x^{z-I}dx := x_1^{z_1} \cdot \ldots \cdot x_n^{z_n} \frac{dx_1}{x_1} \wedge \ldots \wedge \frac{dx_n}{x_n}$$

acts as a kernel. Inversion formulae for multidimensional Mellin transforms and classes of holomorphic functions that can be translated into each other by direct and inverse Mellin transforms are studied in [1]. In this paper we deal with the Mellin transform of rational functions with

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quasi-elliptic denominators f. Due to the specifics of the kernel, it suffices to consider the transform of the function $g(x) = 1/f(x)$. The concept of a quasi-elliptic polynomial was introduced by T. Ermolaeva and A. Tsikh in [3].

So, we consider the following polynomial in n variables

$$f(x) = f(x_1, \dots, x_n) = \sum_{\alpha \in A} c_\alpha x^\alpha = \sum_{\alpha \in A} c_{\alpha_1 \dots \alpha_n} x_1^{\alpha_1} \dots x_n^{\alpha_n} \tag{2}$$

with coefficients $c_\alpha \in \mathbb{C} \setminus \{0\}$ and the support $A \subset \mathbb{Z}^n_{\geqslant}$.

Definition 1. *A polynomial f is called to be quasi-elliptic if for any non-zero covector $a \in \mathbb{R}^{n*}$ its truncation f_a does not vanish in the torus $(\mathbb{R}\backslash 0)^n$.*

Recall that the *truncation* of the polynomial f in the direction $a \in \mathbb{R}^{n*}$ is determined to be the polynomial

$$f_a = \sum_{\alpha \in \Delta^a} c_\alpha x^\alpha,$$

where Δ^a is the face of the Newton polytope of f in the direction a. The *Newton polytope* $\mathcal{N}_f$ of a polynomial f is defined to be the convex hull in $\mathbb{R}^n$ of the support A of f.

Meromorphic continuations of the Mellin transforms for rational functions with quasi-elliptic denominators were studied by L. Nilsson and M. Passare in [9], where the following representation

$$M[1/f](z) = \Phi(z) \prod_{k=1}^{N} \Gamma\left(\nu^{(k)} - \left\langle \mu^{(k)}, z \right\rangle\right), \tag{3}$$

was proved. Here $\Phi(z)$ is an entire function, vectors $\mu^{(k)} \in \mathbb{Z}^n$ are primitive and define outward normal directions to facets of the Newton polytope $\mathcal{N}_f$, and $\nu^{(k)} \in \mathbb{Z}$. The Newton polytope can be given by the system of inequalities

$$\mathcal{N}_f = \bigcap_{k=1}^{N} \left\{ u \in \mathbb{R}^n : \left\langle \mu^{(k)}, u \right\rangle \leqslant \nu^{(k)} \right\},$$

so each $\nu^{(k)}$ is interpreted as the weighted power of the polynomial f with respect to the corresponding weight $\mu^{(k)}$. As it follows from the results of [3], the Mellin transform $M[1/f](z)$ is a holomorphic function in the tube domain over the interior of the Newton polytope $\mathcal{N}_f$, and formula (3) reveals that its polar set is the finite set of families of parallel hyperplanes. In each family, the hyperplanes are obtained by shifting of some facet of the Newton polytope of the polynomial f.

This approach was generalized for Euler–Mellin integrals in [2], and also found application in the theory of Feynman integrals, see [7, 8].

In this paper, we present alternative representations for the Mellin transform of rational functions of the specified class. Note that we can define the quasi-ellipticity concept on the set $\mathbb{R}^n_+$ by assuming that truncations f_a do not vanish on it, because the $\mathbb{R}^n_+$, being a connected component of the real torus $(\mathbb{R}\backslash 0)^n$, is its subgroup under the operation of coordinatewise multiplication.

Theorem 1. *Let us assume that the polynomial f is quasi-elliptic on $\mathbb{R}^n_+$. Then for each normal vector $\mu^{(k)}$ of the Newton polytope $\mathcal{N}_f$ there is a representation for the Mellin transform $M[1/f](z)$ of the following form*

$$M_k(z) = e^{-i\pi \langle \mu^{(k)}, z \rangle} \Gamma(-\langle \mu^{(k)}, z \rangle) \Gamma(1 + \langle \mu^{(k)}, z \rangle) \Phi_k(z), \tag{4}$$

in which

$$\Phi_k(z) = v.p. \int_{V_k} \operatorname{Res} \omega, \tag{5}$$

where $\mathrm{Res}\,\omega$ *is the Leray residue form of the integrand in (1),* V_k *is a surface of real dimension* $n-1$*, and v.p. denotes the principal value with respect to the set of singular points of the* V_k*.*

The function $\Phi_k(z)$ *defined by the integral (5) is holomorphic in the tube domain* $U_{[k]}+i\mathbb{R}^n$*, where*

$$U_{[k]}=\bigcap_{j\neq k}\left\{u\in\mathbb{R}^n:\left\langle\mu^{(j)},u\right\rangle<\nu^{(j)}\right\}.$$

1. Quasi-ellipticity and hypoellipticity

In this section, we characterize quasi-elliptic polynomials in more detail. First, note that the polytope $\mathcal{N}_f$ has only a finite number of faces, so the condition in Definition 1 needs to be verified only for a finite number of truncations f_a.

Following [3], we can single out two classes of polynomials that are quasi-elliptic in the sense of Definition 1. The first class consists of polynomials in which all monomials have positive coefficients and even powers α_i in each variable x_i. The second class consists of elliptic polynomials that do not vanish on $\mathbb{R}^n$. Recall that a polynomial f is called *elliptic* if its homogeneous polynomial of highest degree vanishes in $\mathbb{R}^n$ only at the point $x=0$.

Let us consider a few examples.

1. The polynomial

$$f(x)=1+2x_1+2x_2+(x_1-x_2)^2$$

is not quasi-elliptic in $\mathbb{R}^2_+$, because its truncation

$$f_{(1,1)}=(x_1-x_2)^2$$

vanishes on the diagonal $x_1=x_2$.

2. The polynomial $f=1-x_1+x_1^2-x_2+x_2^2-x_1x_2$ is quasi-elliptic in $\mathbb{R}^2$.

3. The polynomial $f=1+x_1+x_2$ is quasi-elliptic in $\mathbb{R}^2_+$ but not quasi-elliptic in $\mathbb{R}^2$.

The quasi-ellipticity property is related to the concept of hypoellipticity. A polynomial f is said to be *hypoelliptic* if for any multi-index $\alpha\neq 0$ the derivative $f^{(\alpha)}(x)$ satisfies the condition

$$\frac{f^{(\alpha)}(x)}{f(x)}\to 0$$

for $\parallel x\parallel\to\infty$ [6]. The following sufficient test for hypoellipticity holds.

Theorem 2 (E. Zubchenkova)**.** *If f is a quasi-elliptic polynomial and its Newton polytope is full, then f is hypoelliptic polynomial.*

Regarding the convergence of A-hypergeometric integrals, see articles [10] and [11]. The fullness of the polytope means that its projections on all coordinate planes belong to it. This condition in Theorem 2 is essential, as the following example confirms. The polynomial $f(x_1,x_2)=x_1^8x_2^2+x_1^4+1$ is quasi-elliptic, but its Newton polytope is not full. The hypoellipticity condition is not satisfied for it, since for $\alpha=(4,0)$

$$\left.\frac{f^{(\alpha)}(x)}{f(x)}\right|_{x_1=0}=24\not\to 0.$$

We note also that the hypoellipticity condition does not imply the quasi–ellipticity one. For instance, the polynomial $f(x_1,x_2)=(x_1^2-1)^2+x_2^4$ is elliptic, and therefore hypoelliptic, but it is not quasi–elliptic, since the truncation $f_{(0,-1)}=(x_1^2-1)^2$ has zeros in the torus $(\mathbb{R}\backslash 0)^2$.

Following [3], we now formulate the condition for the convergence of the integral of a rational function over $\mathbb{R}^n$ with a quasi-elliptic denominator .

Theorem 3 (Ermolaeva–Tsikh). *If Q is a quasi-elliptic polynomial non-vanishing in $\mathbb{R}^n$, then the integral*

$$\int_{\mathbb{R}^n} \frac{P(x_1, \ldots, x_n)}{Q(x_1, \ldots, x_n)} dx_1 \ldots dx_n$$

is absolutly convergent if and only if

$$I + \mathcal{N}_P \subset (\mathcal{N}_Q)^\circ,$$

that is, the translation of $\mathcal{N}_P$ by $I = (1, \ldots, 1) \in \mathbb{R}^n$ lies in the interior $(\mathcal{N}_Q)^\circ$ of $\mathcal{N}_Q$.

Regarding the convergence of A-hypergeometric integrals, see articles [10] and [11].

2. Sets V_k

The ortant $\mathbb{R}^n_+$ is a group with respect to the operation of coordinatewise multiplication. This is a connected component of the real torus $(\mathbb{R}\backslash 0)^n$. Any torus $(\mathbb{R}\backslash 0)^n$ automorphism , as well as an automorphism of the $\mathbb{R}^n_+$, is defined by a monomial transformation

$$y \to x = y^\eta = (y^{\eta_1}, \ldots, y^{\eta_n}),$$

where $\eta_1, \ldots, \eta_n$ are rows of some integer unimodular matrix η ($\det\eta = \pm 1$). The automorphism allows to integrate over $\mathbb{R}^n_+$ with fibers on shifts of one-parameter subgroups in $\mathbb{R}^n_+$.

Let us define the construction of sets V_k. For each outward normal $\mu^{(k)}$ of the Newton polytope $\mathcal{N}_f$ of the polynomial (2), we define a one-parameter subgroup

$$\gamma^{(k)} = \left\{ y_1^{\mu^{(k)}} := (y_1^{\mu_1^{(k)}}, \ldots, y_1^{\mu_n^{(k)}}) \in \mathbb{R}^n_+ : y_1 \in \mathbb{R}_+ \right\}.$$

Next, we foliate the orthant $\mathbb{R}^n_+$ into shifts (cosets with respect to the subgroup $\gamma^{(k)}$) as follows

$$c \odot \gamma^{(k)} = (c_1 y_1^{\mu_1^{(k)}}, \ldots, c_n y_1^{\mu_n^{(k)}}).$$

The set of all shifts can be given as $c = (y')^{\eta'}$, where $y' := (y_2, \ldots, y_n)$, η' is an integer $(n \times (n-1))$-matrix such that $\eta := (\mu^{(k)}, \eta')$ is a unimodular $(n \times n)$-matrix. The existence of such a matrix is ensured by the condition the vector $\mu^{(k)}$ to be primitive [13, Prop. 4.2.13].

Consider a section of the complex hypersurface $V := \{x \in \mathbb{C}^n : f(x_1, \ldots, x_n) = 0\}$ by a family of shifts of the subgroup $\gamma^{(k)}$. As a result, we get the set

$$V_k = \bigcup_{y' \in \mathbb{R}^{n-1}_+} \left(V \bigcap \left\{ x = y_1^{\mu^{(k)}} \odot (y')^{\eta'^T} \right\} \right),$$

of the real dimension $n-1$. This observation allows us to apply Fubini's theorem doing integration over arbitrary one-parameter fibers.

Let us describe this construction using the example of a complex hyperplane

$$V = \left\{ x \in (\mathbb{C} \setminus 0)^2 : 1 + x_1 + x_2 = 0 \right\}.$$

The Newton polytope of the defining polynomial is a triangle, it has outward normals $\mu^{(1)} = (-1, 0)$, $\mu^{(2)} = (0, -1)$, $\mu^{(3)} = (1, 1)$ (Fig. 1 on the left). Fig. 2 shows the real part of V and its sections by shifts of the one-parameter subgroups $\gamma^{\mu^{(1)}}$ (red ray), $\gamma^{\mu^{(2)}}$ (green ray) and $\gamma^{\mu^{(3)}}$ (blue segment). These are the sets V_1, V_2, V_3 respectively. Their logarithmic images

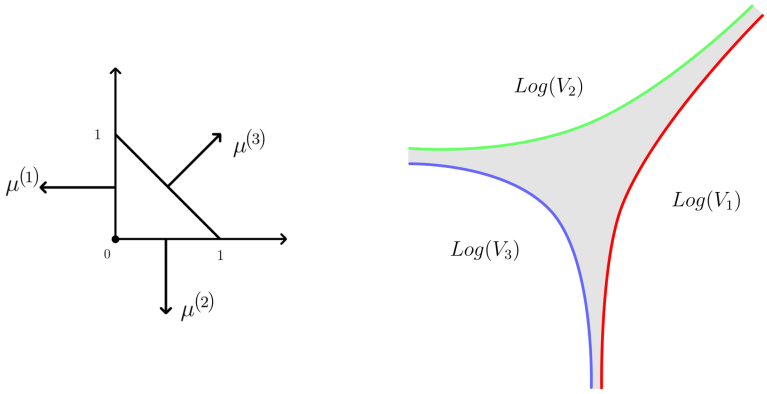

Fig. 1. Newton polytope (left) and amoeba $\mathcal{A}_V$ (right) for $f = 1 + x_1 + x_2$

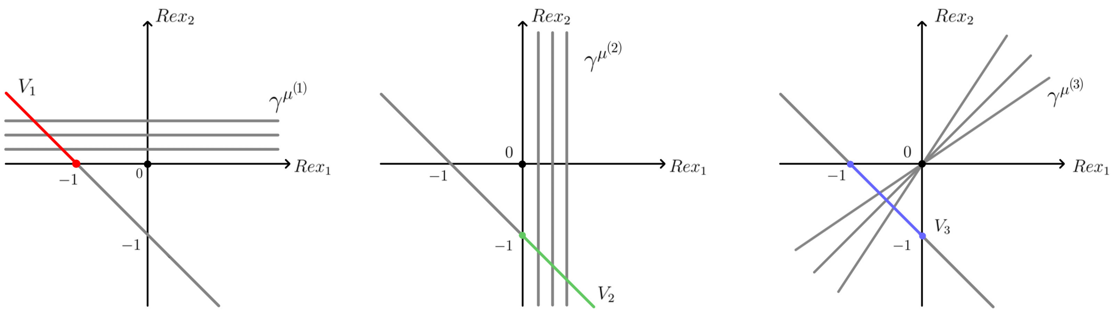

Fig. 2. Sets V_1, V_2, V_3

are connected components of the contour of the amoeba $\mathcal{A}_V$ of the hyperplane V (see Fig. 1 on the right). Recall that the amoeba of an algebraic hypersurface V is defined to be its image under the mapping

$$\mathrm{Log} : (x_1, \dots, x_n) \to (\log|x_1|, \dots, \log|x_n|),$$

see, for example, [5]. The contour of the amoeba is determined as the set of critical values of the specified projection $\mathrm{Log}|_V$, i.e. the set $\mathrm{Log}(\gamma^{-1}(\mathbb{RP}^{n-1}))$, where $\gamma : V \to \mathbb{CP}^{n-1}$ is the logarithmic Gauss mapping [12].

3. Proof of Theorem 1

Consider a polynomial (2) that has no multiple irreducible factors, i.e. $df \not\equiv 0$ on each irreducible component of V. According to (1), the Mellin transform of the function $1/f$ is expressed by the integral

$$M[1/f](z) = \int\limits_{\mathbb{R}^n_+} \frac{x^{z-I}}{f(x)} dx.$$

Fix a normal vector $\mu^{(k)} = (\mu_1^{(k)}, \dots, \mu_n^{(k)})$, $k = 1, \dots, N$, of the polytope $\mathcal{N}_f$. Let us construct an integer unimodular matrix η in which the vector $\mu^{(k)}$ is the first column:

$$\eta = \begin{array}{c} \begin{array}{cccc} \eta^{(1)} & \eta^{(2)} & \dots & \eta^{(n)} \end{array} \\ \begin{pmatrix} \mu_1^{(k)} & \eta_1^{(2)} & \dots & \eta_1^{(n)} \\ \dots & \dots & \dots & \dots \\ \mu_n^{(k)} & \eta_n^{(2)} & \dots & \eta_n^{(n)} \end{pmatrix} \end{array}.$$

The monomial transform $x = y^\eta$ with coordinates

$$x_1 = y_1^{\mu_1^{(k)}} y_2^{\eta_1^{(2)}} \dots y_n^{\eta_1^{(n)}},$$
$$\dots$$
$$x_n = y_1^{\mu_n^{(k)}} y_2^{\eta_n^{(2)}} \dots y_n^{\eta_n^{(n)}}$$

is an automorphism of $\mathbb{R}_+^n$ due to the unimodularity of the matrix η. Let us write in variables y the expression $\dfrac{dx}{x}$. The result looks as follows

$$\frac{dx}{x} = J \prod_{j=1}^{n} (y_1^{\mu_j^{(k)}} \dots y_n^{\eta_j^{(n)}})^{-1} dy_1 \wedge \dots \wedge dy_n,$$

where

$$J = \begin{vmatrix} \mu_1^{(k)} y_1^{\mu_1^{(k-1)}} \dots y_n^{\eta_1^{(n)}} & \eta_1^{(2)} y_1^{\mu_1^{(k)}} y_2^{\eta_1^{(2)}-1} \dots y_n^{\eta_1^{(n)}} & \dots & \eta_1^{(n)} y_1^{\mu_1^{(k)}} \dots y_n^{\eta_1^{(n)}-1} \\ \dots & \dots & \dots & \dots \\ \mu_n^{(k)} y_1^{\mu_n^{(k)}-1} \dots y_n^{\eta_n^{(n)}} & \eta_n^{(2)} y_1^{\mu_n^{(k)}} y_2^{\eta_n^{(2)}-1} \dots y_n^{\eta_n^{(n)}} & \dots & \eta_n^{(n)} y_1^{\mu_n^{(k)}} \dots y_n^{\eta_n^{(n)}-1} \end{vmatrix}.$$

is the Jacobian of the monomial mapping. Multiplying the jth row of the Jacobian by $\left(y_1^{\mu_j^{(k)}} y_2^{\eta_j^{(2)}} \dots y_n^{\eta_j^{(n)}} \right)^{-1}$, we get the representation

$$\frac{dx}{x} = \begin{vmatrix} \mu_1^{(k)} y_1^{-1} & \eta_1^{(2)} y_2^{-1} & \dots & \eta_1^{(n)} y_n^{-1} \\ \dots & \dots & \dots & \dots \\ \mu_n^{(k)} y_1^{-1} & \eta_n^{(2)} y_2^{-1} & \dots & \eta_n^{(n)} y_n^{-1} \end{vmatrix} dy_1 \wedge \dots \wedge dy_n.$$

Further, from the j-th column of the determinant we take out the multiplier y_j^{-1}. As a result, we get

$$\frac{dx}{x} = \det \eta \frac{dy}{y},$$

where $det\, \eta = \pm 1$, i. e. the matrix η is unimodular.

As a result of the change of variables, the Mellin transform is represented by the integral

$$M[1/f(y^\eta)](z) = \det \eta \int_{\mathbb{R}_+^n} \frac{1}{f(y^\eta)} y_1^{\langle \mu^{(k)}, z \rangle} y_2^{\langle \eta^{(2)}, z \rangle} \dots y_n^{\langle \eta^{(n)}, z \rangle} \frac{dy}{y},$$

where

$$f(y^\eta) = \sum_{\alpha \in A} a_\alpha y_1^{\langle \mu^{(k)}, \alpha \rangle} y_2^{\langle \eta^{(2)}, \alpha \rangle} \dots y_n^{\langle \eta^{(n)}, \alpha \rangle}.$$

Remark that $\max_{\alpha \in A} \{\langle \mu^{(k)}, \alpha \rangle\} = \nu^{(k)}$, and this quantity is the degree of $f(y^\eta)$ over y_1. Next, we integrate over y_1 for the fixed value $y' = (y_2, \dots, y_n) \in \mathbb{R}_+^{n-1}$:

$$M[1/f(y^\eta)](z) = \det \eta \int_{\mathbb{R}_+^{n-1}} y_2^{\langle \eta^{(2)}, z \rangle} \dots y_n^{\langle \eta^{(n)}, z \rangle} \frac{dy'}{y'} \int_0^{+\infty} \frac{y_1^{\langle \mu^{(k)}, z \rangle}}{f(y^\eta)} \frac{dy_1}{y_1}.$$

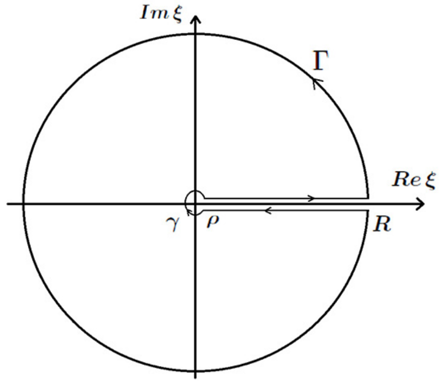

Fig. 3. The integration contour Γ

In order to calculate the inner integral over y_1, we introduce the complex variable ξ, setting $y_1 = Re\xi$. Traditionally, we consider the integral over the contour Γ (see Fig. 3) of the following type

$$\int\limits_{\Gamma} \frac{\xi^{\langle\mu^{(k)},z\rangle}}{f\left((\xi,y')^{\eta}\right)}\frac{d\xi}{\xi} = \left(1-e^{2\pi i(\langle\mu^{(k)},z\rangle-1)}\right)\int\limits_{\rho}^{R} \frac{\xi^{\langle\mu^{(k)},z\rangle-1}}{f\left((\xi,y')^{\eta}\right)}d\xi + \int\limits_{C_R} \frac{\xi^{\langle\mu^{(k)},z\rangle-1}}{f\left((\xi,y')^{\eta}\right)}d\xi + \int\limits_{C_\rho} \frac{\xi^{\langle\mu^{(k)},z\rangle-1}}{f\left((\xi,y')^{\eta}\right)}d\xi.$$

Since the degree of $f(y^{\eta})$ by y_1 equals $\nu^{(k)}$, then, by the residue total sum theorem, this integral vanishes if $\langle\mu^{(k)}, \mathrm{Re} z\rangle < \nu^{(k)}$. Passing to the limit as $\rho \to 0$, $R \to \infty$ and applying the residue theorem, we obtain

$$\int_0^{+\infty} \frac{y_1^{\langle\mu^{(k)},z\rangle}}{f(y^{\eta})}\frac{dy_1}{y_1} = \frac{2\pi i}{1-e^{2\pi i(\langle\mu^{(k)},z\rangle-1)}}\sum_j \frac{\left(\xi^j(y')\right)^{\langle\mu^{(k)},z\rangle-1}}{f'_{y_1}(\xi^j(y'),y')},$$

where $\xi^j(y')$ are roots of $f(y^{\eta})$. Thus we get

$$M[1/f(y^{\eta})] = \frac{2\pi i}{1-e^{2\pi i(\langle\mu^{(k)},z\rangle-1)}}\int\limits_{\mathbb{R}_+^{n-1}}\sum_j \frac{\left(\xi^j(y')\right)^{\langle\mu^{(k)},z\rangle-1}}{f'_{y_1}(\xi^j(y'),y')} y_2^{\langle\eta^{(2)},z\rangle}\dots y_n^{\langle\eta^{(n)},z\rangle}\frac{dy'}{y'} =$$

$$= \frac{2\pi i}{1-e^{2\pi i(\langle\mu^{(k)},z\rangle-1)}}\int_{V_k}\mathrm{Res}\left(\frac{x^{z-I}}{f(x)}dx\right).$$

The factor before the integral can be rewritten as follows

$$\frac{2\pi i}{1-e^{2\pi i(\langle\mu^{(k)},z\rangle-1)}} = \frac{\Gamma(-\langle\mu^{(k)},z\rangle)\Gamma(1+\langle\mu^{(k)},z\rangle)}{e^{i\pi(\langle\mu^{(k)},z\rangle)}}.$$

Thus, we obtain the first assertion of Theorem 1. The second one follows from the fact that the integral over V_k vanishes if $\langle\mu^{(k)}, \mathrm{Re} z\rangle = \langle\mu^{(k)}, u\rangle < \nu^{(k)}$.

4. Examples

I. Consider the quasi-elliptic polynomial $f(x) = 2 - x_1 + x_1^2 - x_2 + x_2^2 - x_1x_2$ and the Mellin transform

$$M[1/f](z) = \int_{\mathbb{R}^2_+} \frac{x_1^{z_1} x_2^{z_2}}{2 - x_1 + x_1^2 - x_2 + x_2^2 - x_1 x_2} \frac{dx}{x}. \tag{6}$$

The Newton polytope of the polynomial $f(x)$ is given by three inequalities:

$$\mathcal{N}_f = \{-z_1 \leqslant 0\} \cap \{-z_2 \leqslant 0\} \cap \{z_1 + z_2 \leqslant 2\}.$$

It has outward normals $\mu^{(1)} = (-1, 0)$, $\mu^{(2)} = (0, -1)$, $\mu^{(3)} = (1, 1)$.

We consider the monomial change of variables

$$x_1 = t^{-1}, \qquad x_2 = \tau^{-1},$$

associated with the vector $\mu^{(1)} = (-1, 0)$. As a result, the integral (6) will take the form:

$$M[1/f](z) = \int_{\mathbb{R}^2_+} \frac{t^{1-z_1} \tau^{1-z_2}}{2t^2\tau^2 - t\tau^2 + \tau^2 - t^2\tau + t^2 - t\tau} dt \wedge d\tau. \tag{7}$$

The denominator of the integrand in (7) has roots

$$t^{(1)} = \frac{\tau^2 + \tau + i\tau\sqrt{7\tau^2 - 6\tau + 3}}{2(2\tau^2 - \tau + 1)} \text{ and } t^{(2)} = \frac{\tau^2 + \tau - i\tau\sqrt{7\tau^2 - 6\tau + 3}}{2(2\tau^2 - \tau + 1)}.$$

According to Theorem 1, the Mellin transform $M[1/f](z)$ admits the representation

$$M_1(z) = \Gamma(z_1)\Gamma(1 - z_1) e^{i\pi z_1} \int_{V_1} \operatorname{Res} \omega,$$

where

$$V_1 = \{x_1 = 1/t^{(1)},\, x_2 = 1/\tau,\, \tau \in [0; +\infty]\} \cup \{x_1 = 1/t^{(2)},\, x_2 = 1/\tau,\, \tau \in [0; +\infty]\}.$$

Since the hypersurface $V = \{f(x) = 0\}$ is smooth, the principal value $v.p.$ in the representation is omitted. Thus,

$$M_1(z) = \Gamma(z_1)\Gamma(1 - z_1) e^{i\pi z_1} 2^{z_1 - 1} \times$$
$$\times \int_0^\infty \frac{\left(\tau^2 + \tau + i\tau\sqrt{7\tau^2 - 6\tau + 3}\right)^{1-z_1} - \left(\tau^2 + \tau - i\tau\sqrt{7\tau^2 - 6\tau + 3}\right)^{1-z_1}}{\tau^{z_2} \left(2\tau^2 - \tau + 1\right)^{1-z_1} i\sqrt{7\tau^2 - 6\tau + 3}} d\tau. \tag{8}$$

Now let us define the domain of convergence of the integral in (8). At the origin, its convergence is ensured by the condition

$$u_1 + u_2 < 2,$$

where $u_1 = \mathrm{Re} z_1$, $u_2 = \mathrm{Re} z_2$. Next, we study the convergence in the neighborhood of the infinity using the substitution $\tau = 1/\lambda$. As a result, we obtain the integral

$$\int_0^\infty \frac{\left(1 + \lambda + i\sqrt{7 - 6\lambda + 3\lambda^2}\right)^{1-z_1} - \left(1 + \lambda - i\sqrt{7 - 6\lambda + 3\lambda^2}\right)^{1-z_1}}{\lambda^{1-z_2} \left(2 - \lambda + \lambda^2\right)^{1-z_1} i\sqrt{7 - 6\lambda + 3\lambda^2}} d\lambda.$$

The convergence of this integral at the origin is ensured by the condition $u_2 > 0$. Thus, the integral on the right side of (8) converges in the tube domain with the base $U_{[1]} = \left\{u \in \mathbb{R}^2 : u_1 + u_2 < 2, u_2 > 0\right\}$.

Next, consider the monomial change of coordinates

$$x_1 = \tau, \qquad x_2 = t^{-1},$$

associated with the vector $\mu^{(2)} = (0,-1)$. As a result of this change of variables, the Mellin transform is expressed by the integral

$$M[1/f] = \int_{\mathbb{R}^2_+} \frac{t^{1-z_2}\tau^{z_1-1}}{2t^2 - \tau t^2 + \tau^2 t^2 - t + 1 - t\tau} dt \wedge d\tau,$$

in which the denominator of the integrand has roots:

$$t^{(1)} = \frac{\tau + 1 + i\sqrt{3\tau^2 - 6\tau + 7}}{2(\tau^2 - \tau + 2)}, \qquad t^{(2)} = \frac{\tau + 1 - i\sqrt{3\tau^2 - 6\tau + 7}}{2(\tau^2 - \tau + 2)}.$$

According to Theorem 1, the Mellin transform $M[1/f](z)$ admits the representation

$$M_2(z) = \Gamma(z_2)\Gamma(1 - z_2)e^{i\pi z_2} \int_{V_2} \operatorname{Res}\omega,$$

where

$$V_2 = \{x_1 = \tau,\ x_2 = 1/t^{(1)},\ \tau \in [0;+\infty]\} \cup \{x_1 = \tau,\ x_2 = 1/t^{(2)},\ \tau \in [0;+\infty]\}.$$

Thus, we obtain the formula

$$M_2(z) = \Gamma(z_2)\Gamma(1 - z_2)e^{i\pi z_2} 2^{z_2-1} \times$$
$$\times \int_0^\infty \frac{\left(\tau + 1 + i\sqrt{3\tau^2 - 6\tau + 7}\right)^{1-z_2} - \left(\tau + 1 - i\sqrt{3\tau^2 - 6\tau + 7}\right)^{1-z_2}}{\tau^{1-z_1}\left(\tau^2 - \tau + 2\right)^{1-z_2} i\sqrt{3\tau^2 - 6\tau + 7}} d\tau. \quad (9)$$

The integral on the right side of (9) converges in a tube domain with the base $U_{[2]} = \left\{u \in \mathbb{R}^2 : u_1 > 0, u_1 + u_2 < 2\right\}$.

Finally, consider the third normal $\mu^{(3)} = (1,1)$ and the corresponding monomial mapping:

$$x_1 = t, \qquad x_2 = \tau t.$$

The Mellin transform takes the form:

$$M[1/f](z) = \int_{\mathbb{R}^2_+} \frac{t^{z_1+z_2-1}\tau^{z_2-1}}{t^2\left(\tau^2 - \tau + 1\right) + t\left(-1 - \tau\right) + 2} dt \wedge d\tau.$$

The denominator of the integrand in the resulting integral has two roots:

$$t^{(1)} = \frac{\tau + 1 + i\sqrt{7\tau^2 - 10\tau + 7}}{2(\tau^2 - \tau + 1)}, \qquad t^{(2)} = \frac{\tau + 1 - i\sqrt{7\tau^2 - 10\tau + 7}}{2(\tau^2 - \tau + 1)}.$$

According to Theorem 1, the Mellin transform $M[1/f](z)$ admits the representation

$$M_3(z) = \Gamma(-z_1 - z_2)\Gamma(z_1 + z_2 + 1)e^{-i\pi(z_1+z_2)} \int_{V_3} \operatorname{Res}\omega,$$

where

$$V_3 = \{x_1 = t^{(1)},\ x_2 = \tau t^{(1)},\ \tau \in [0;+\infty]\} \cup \{x_1 = t^{(2)},\ x_2 = \tau t^{(2)},\ \tau \in [0;+\infty]\}.$$

Calculating the residue, we get the representation

$$M_3(z) = \Gamma(-z_1 - z_2)\Gamma(z_1 + z_2 + 1)e^{-i\pi(z_1+z_2)} 2^{1-z_1-z_2} \times$$
$$\times \int_0^\infty \frac{\left(\tau + 1 + i\sqrt{7\tau^2 - 10\tau + 7}\right)^{z_1+z_2-1} - \left(\tau + 1 - i\sqrt{7\tau^2 - 10\tau + 7}\right)^{z_1+z_2-1}}{\tau^{1-z_2}\left(\tau^2 - \tau + 1\right)^{z_1+z_2-1} i\sqrt{7\tau^2 - 10\tau + 7}} d\tau. \quad (10)$$

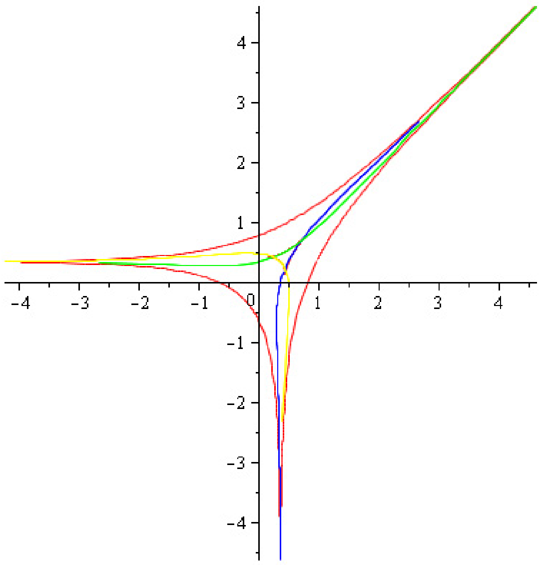

Fig. 4. Contour of the amoeba for $f(x)=2-x_1+x_1^2-x_2+x_2^2-x_1x_2$ and $Log(V_1), Log(V_2), Log(V_3)$

The integral in (10) converges in a tube domain with the base $U_{[3]} = \mathbb{R}^2_+$.

In Fig. 4, the logarithmic projections of the sets V_1, V_2, and V_3 are shown in blue, green, and yellow, respectively. The contour of the amoeba is highlighted in red.

II. Consider a quasi-elliptic polynomial $f(x) = 5 + x_1 + x_2 + x_1x_2$ and the Mellin transform:

$$M[1/f](z) = \int_{\mathbb{R}^2_+} \frac{x_1^{z_1} x_2^{z_2}}{5 + x_1 + x_2 + x_1x_2} \frac{dx}{x}. \tag{11}$$

The Newton polytope of the polynomial $f(x)$ is given by the inequalities

$$\mathcal{N}_f = \{-z_1 \leqslant 0\} \cap \{-z_2 \leqslant 0\} \cap \{z_1 \leqslant 1\} \cap \{z_2 \leqslant 1\}$$

and therefore has outward normals $\mu^{(1)} = (-1, 0)$, $\mu^{(2)} = (0, -1)$, $\mu^{(3)} = (1, 0)$, $\mu^{(4)} = (0, 1)$.

We consider the monomial change of variables

$$x_1 = t^{-1}, \qquad x_2 = \tau^{-1},$$

associated with the vector $\mu^{(1)} = (-1, 0)$. The Mellin transform after the change of variables will take the form:

$$M[1/f](z) = \int_{\mathbb{R}^2_+} \frac{t^{-z_1} \tau^{-z_2}}{5t\tau + t + \tau + 1} dt \wedge d\tau.$$

According to Theorem 1, the Mellin transform $M[1/f](z)$ admits the representation

$$M_1(z) = \Gamma(z_1)\Gamma(1 - z_1) e^{i\pi z_1} v.p. \int_{V_1} \operatorname{Res} \omega,$$

where

$$V_1 = \left\{ x_1 = -\frac{5\tau + 1}{\tau + 1}, x_2 = \tau^{-1}, \tau \in [0; +\infty] \right\}.$$

Calculating the residue, we get the following result:

$$M_1(z) = \Gamma(z_1)\Gamma(1 - z_1) e^{i\pi z_1} \int_0^{\infty} (-1)^{-z_1} \frac{\tau^{-z_2} (\tau + 1)^{-z_1}}{(5\tau + 1)^{1 - z_1}} d\tau. \tag{12}$$

The integral in (12) converges in the domain $\{(z_1, z_2) \in \mathbb{C}^2 : 0 < \mathrm{Re} z_2 < 1\}\}$.

Next, consider the monomial change of coordinates

$$x_1 = \tau, \qquad x_2 = t^{-1},$$

associated with the vector $\mu^{(2)} = (0, -1)$. As a result of this change of variables, the Mellin transform is expressed by the integral

$$M[1/f](z) = \int_{\mathbb{R}^2_+} \frac{\tau^{z_1-1} t^{-z_2}}{5t + \tau t + \tau + 1} dt \wedge d\tau.$$

According to Theorem 1, the Mellin transform $M[1/f](z)$ admits the representation

$$M_2(z) = \Gamma(z_2)\Gamma(1 - z_2)e^{i\pi z_2} v.p. \int_{V_2} \mathrm{Res}\, \omega,$$

where

$$V_2 = \left\{ x_1 = \tau, x_2 = -\frac{\tau + 5}{\tau + 1}, \tau \in [0; +\infty] \right\}.$$

Calculating the residue, we obtain the result:

$$M_2(z) = \Gamma(z_2)\Gamma(1 - z_2)e^{i\pi z_2} \int_0^\infty (-1)^{-z_2} \frac{\tau^{z_1-1}(\tau + 1)^{-z_2}}{(5 + \tau)^{1-z_2}} d\tau. \tag{13}$$

The integral in (13) converges in the domain $\{(z_1, z_2) \in \mathbb{C}^2 : 0 < \mathrm{Re} z_1 < 1\}\}$.

Further we consider the vector $\mu^{(3)} = (1, 0)$ and do the substitution

$$x_1 = t, \qquad x_2 = \tau.$$

As a result, the Mellin transform takes the form

$$M[1/f](z) = \int_{\mathbb{R}^2_+} \frac{t^{z_1-1} \tau^{z_2-1}}{5 + \tau + t + t\tau} dt \wedge d\tau.$$

According to Theorem 1, the Mellin transform $M[1/f](z)$ admits the representation

$$M_3(z) = \Gamma(-z_1)\Gamma(1 + z_1)e^{-i\pi z_1} v.p. \int_{V_3} \mathrm{Res}\, \omega,$$

where

$$V_3 = \left\{ x_1 = \frac{-\tau - 5}{\tau + 1}, x_2 = \tau, \tau \in [0; +\infty] \right\}.$$

Calculating the residue, we get the representation

$$M_3(z) = \Gamma(-z_1)\Gamma(1 + z_1)e^{-i\pi z_1} \int_0^\infty (-1)^{z_1-1} \frac{\tau^{z_2-1}(\tau + 5)^{z_1-1}}{(1 + \tau)^{z_1}} d\tau. \tag{14}$$

The integral in (14) converges in the domain $\{(z_1, z_2) \in \mathbb{C}^2 : 0 < \mathrm{Re} z_2 < 1\}\}$.

Finally, we consider the normal $\mu^{(4)} = (0, 1)$ and the corresponding monomial mapping:

$$x_1 = \tau^{-1}, \qquad x_2 = t.$$

The Mellin transform after the change of variables is as follows:

$$M[1/f] = \int_{\mathbb{R}^2_+} \frac{\tau^{-z_1} t^{z_2-1}}{5\tau + 1 + t + t\tau} dt \wedge d\tau.$$

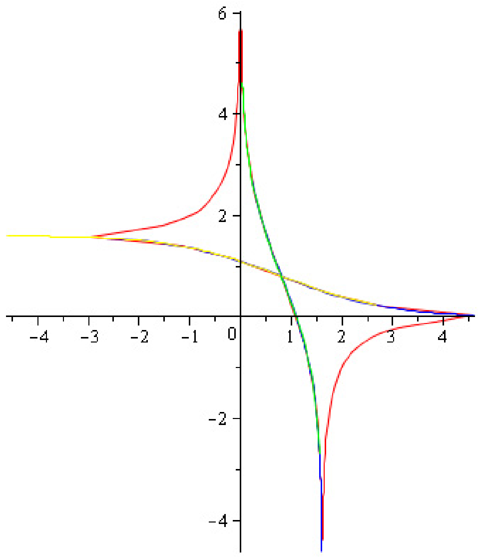

Fig. 5. Contour of the amoeba of $V = \{5 + x_1 + x_2 + x_1 x_2 = 0\}$ and logarithmic projections of V_k

According to Theorem 1, the Mellin transform $M[1/f](z)$ admits the representation

$$M_4(z) = \Gamma(-z_2)\Gamma(1+z_2)e^{-i\pi z_2} v.p. \int_{V_4} \operatorname{Res} \omega,$$

where

$$V_4 = \left\{ x_1 = \tau^{-1}, x_2 = \frac{-5\tau - 1}{\tau + 1}, \tau \in [0; +\infty] \right\}.$$

Calculating the residue, we get the representation

$$M_4(z) = \Gamma(-z_2)\Gamma(1+z_2)e^{-i\pi z_2} \int_0^\infty (-1)^{z_2 - 1} \frac{\tau^{-z_1}(5\tau + 1)^{z_2 - 1}}{(1+\tau)^{z_2}} d\tau. \tag{15}$$

The integral in (15) converges in the domain $\{(z_1, z_2) \in \mathbb{C}^2 : 0 < \mathrm{Re} z_1 < 1\}$. The contour of the amoeba of $V = \{5 + x_1 + x_2 + x_1 x_2 = 0\}$ is shown in Fig. 5. The sets $\mathrm{Log}(V_1)$ and $\mathrm{Log}(V_3)$ coincide with the green part of the contour, and the sets $\mathrm{Log}(V_2)$ and $\mathrm{Log}(V_4)$ coincide with the yellow one.

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Преобразование Меллина для рациональных функций с квазиэллиптическими знаменателями

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Аннотация. В статье рассматриваются вычетные представления n-мерных преобразований Меллина для рациональных функций с квазиэллиптическим знаменателем. Эти представления задаются интегралами по $(n-1)$-мерным относительным циклам. Количество представлений (или циклов) равно числу граней многогранника Ньютона знаменателя рациональной функции.

Ключевые слова: многомерное преобразование Меллина, квазиэллиптический полином, форма-вычет Лере, амёба.