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Preorder on the Set of Analytic Functions

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Abstract. In the paper, we study a preorder, which can be naturally defined on the set of all analytic functions of two complex variables with the help of superpositions of analytic functions of one variable.

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Introduction

The celebrated Hilbert 13th problem, even though it has been solved by A. N. Kolmogorov and V. I. Arnold, leaves around itself many interesting questions. Thus, the original problem about the representation of the solutions of a polynomial equation of seventh degree as a finite superposition of *continuous* functions of two variables could be looked from a different angle, if smooth, analytic or algebraic functions are being considered, instead of continuous functions. Such interpretation of the Hilbert question gave rise to plenty of research and deep results about the representation of functions with the help of superpositions of functions of lesser number of variables in different functional classes (see, e.g., [1–4]).

If we replace superpositions of continuous functions by superpositions of germs of analytic functions, then the beautiful and contensive theory, constructed in papers of V. K. Beloshapka, arises ([5,6]). This theory allows to introduce a natural preorder on the set of analytic functions of several variables (the definition was proposed by V. K. Beloshapka). We shall restrict ourselves to functions of two variables, although it is possible to study analogous structures also in the case of larger number of variables.

Let two analytic functions $f_1(x, y)$ and $f_2(x, y)$ be given. We say that f_1 is not more complex then f_2 ($f_1 \preccurlyeq f_2$), if f_2 is representable as a superposition of analytic functions of one variable and the function f_1 (see Definition 1 in Section 1). The minimal depth of the superposition is referred to as complexity (with respect to f_1) and is designated by $N(f_1)$. Preorder $\preccurlyeq$ brings to the set of analytic functions a new structure, which needs further comprehension.

In the paper, we give answers to the main questions arising here, which, as we see it, are essential for the description of the complete picture. Let us note the following three main points of our exposition:

1) Does there exist a function $f(x, y)$, which is simpler than $x + y$ (i.e., such that $f \preccurlyeq x + y$)? Here we suppose that the complexity $N_{x+y}(f)$ of the function f with respect to addition $x + y$

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is greater than one, because otherwise the question ceases to be interesting, as we will show in what follows.

2) Given functions f_1 and f_2. Is it possible that f_1 is expressible in terms of f_2 with depth $n > 1$, and f_2 is expressible in terms of f_1 with depth $m > 1$?

3) Given functions f_1 and f_2. Is it possible that f_1 is expressible in terms of f_2 with depth $n > 1$, and f_2 is not expressible in terms of f_1?

In all the cases the answer is affirmative. But if in 2) and 3) the answer is something we should expect, in 1) the answer is counterintuitive: we obtain that the simpler object $x + y$ is a superposition constructed with the help of the more complex object.

1. Definition of the preorder

The classes $Cl_n = Cl_n(x + y)$ of functions of finite analytical complexity are constructed inductively with the help of finite superpositions of analytic functions of one variable and addition $x + y$. In analogous way the classes $Cl_n(\varphi)$ can be constructed, if we take as a binary operation an arbitrary analytic function $\varphi = \varphi(x, y)$ of two variables instead of addition. I.e., a function belongs to the class $Cl_0(\varphi)$, if it is an analytic function of one variable, and belongs to the class $Cl_{n+1}(\varphi)$, if in a neighbourhood of some point its germ has the form $C(\varphi(A_n, B_n))$, where $A_n, B_n \in Cl_n(\varphi)$, and $C(t)$ is an analytic function of one variable. Here the expression $C(\varphi(A_n, B_n))$ means the following: for any analytic function in the superposition we take such germ that the whole expression is a well-defined germ. With the help of shifts of the form $(x \longrightarrow x + a,\ y \longrightarrow y + b,\ t \longrightarrow t + c)$, where $a, b, c \in \mathbb{C}$, we can move the germs of all functions to the origin. Therefore, it could be assumed that all germs are defined at the origin.

If one germ of an analytic function $G(x, y)$ is representable with the help of the superposition of the specified form, then almost all germs are also representable in that way (see [6]). Namely, if G admits a holomorphic branch in some polydisc $\Pi \in \mathbb{C}^2$, then the representation takes place at a generic point in Π (i.e., outside some proper analytic subset in Π). A function has complexity $n = N_\varphi$ with respect to the function φ, if it belongs to the set $Cl_n(\varphi) \setminus Cl_{n-1}(\varphi)$. Next, if a function does not belong to the set $Cl_n(\varphi)$ for any finite value n, then we assume $N_\varphi = \infty$. The complexity is also computed at a generic point. We denote the hierarchy of all functions of finite complexity with respect to φ by $\mathbf{Cl}(\varphi)$, i.e., $\mathbf{Cl}(\varphi) = \cup_{n=0}^{\infty} Cl_n(\varphi)$.

Let us recall that a preorder $\preccurlyeq$ on a set is a binary relation, satisfying the two following properties:

1) reflexivity ($f \preccurlyeq f \ \ \forall f$),

2) transitivity ($f_1 \preccurlyeq f_2,\ f_2 \preccurlyeq f_3 \Rightarrow f_1 \preccurlyeq f_3 \ \ \forall f_1, f_2, f_3$).

If in addition to this properties we have one more,

3) antisymmetry ($f_1 \preccurlyeq f_2,\ f_2 \preccurlyeq f_1 \Rightarrow f_1 = f_2 \ \ \forall f_1, f_2$),

then we say that on the set a partial order is defined.

With the help of the classes $Cl_n(\varphi)$ we can introduce a preorder on the set $\mathcal{A}(x, y)$ of analytic functions of two variables in the following way:

Definition 1. *$f_1(x, y) \preccurlyeq f_2(x, y)$ if $f_2 \in \mathbf{Cl}(f_1)$.*

Reflexivity and transitivity of the relation $\preccurlyeq$ follow from the definition.

2. Properties of the preorder

On the set of germs of analytic functions $f(x, y)$ acts the group of transformations

$$\mathcal{G} = \{x \longrightarrow \alpha(x),\, y \longrightarrow \beta(y),\, f \longrightarrow \gamma(f)\},$$

where α, β, γ are germs in the origin of holomorphic invertible functions of one variable, such that $\alpha(0) = \beta(0) = \gamma(0) = 0$. Here, as above, we assume that the germ of the function f is defined at the origin and vanishes there.

We call two germs $[f_1]_0$ and $[f_2]_0$ at the origin of the functions f_1 and f_2 gauge equivalent, if there exists an element $g \in \mathcal{G}$, such that $[f_1]_0 = g \circ [f_2]_0$. Next, two germs $[f_1]_a$ and $[f_2]_b$ at points $a = (a_1, a_2)$ and $b = (b_1, b_2)$ are gauge equivalent, if germs of the functions $\Big(f_1(x - a_1, y - b_1) - f_1(a_1, b_1)\Big)$ and $\Big(f_2(x - a_2, y - b_2) - f_2(a_2, b_2)\Big)$ at the origin are gauge equivalent. Finally, two analytic functions f_1 and f_2 are gauge equivalent, if they have gauge equivalent germs (notation: $f_1 \sim f_2$).

Let us note an important property of the group $\mathcal{G}$: it acts on the set of all analytic functions without changing their complexity (with respect to any function φ). I.e., $\mathcal{G} \circ \Big(Cl_n(\varphi) \setminus Cl_{n-1}(\varphi)\Big) = \Big(Cl_n(\varphi) \setminus Cl_{n-1}(\varphi)\Big)$ for all n and φ.

2.1. The preorder is not a partial order

Let us show that the introduced preorder is not a partial order, i.e., the property of antisymmetry fails.

Consider two arbitrary different functions f_1 and f_2 of complexity one with respect to addition. The action of $\mathcal{G}$ on such functions is transitive, i.e., there exists an element $g \in \mathcal{G}$, such that $f_1 = g \circ f_2$. This means that the inequalities $f_1 \preccurlyeq f_2$ and $f_2 \preccurlyeq f_1$ hold, although $f_1 \neq f_2$. Thus, if we consider the preorder on the set of all functions, then it is easy to construct the counterexample to the antisymmetry condition. But it seems natural to consider the quotient set of all functions with respect to the action of the group $\mathcal{G}$ and ask the question about antisymmetry of the introduced preorder for resulting equivalence classes.

On the set $\Big(\mathcal{A}(x, y)/\sim\Big)$ of classes of gauge equivalence the preorder $\lesssim$ can be defined as follows: $[f_1] \lesssim [f_2]$, if for representatives f_1 and f_2 of classes $[f_1]$ and $[f_2]$ of gauge equivalence we have $f_1 \preccurlyeq f_2$. This definition does not depend on the choice of representatives of the classes.

Let $\varphi_1 = \varphi_1(x, y) = (x - 1)y + \dfrac{y^2}{1 + y}$, $\varphi_2 = \varphi_2(x, y) = x^2 + xy$.

Theorem 2. *1) $\varphi_1 \preccurlyeq x + y$, and $N_{\varphi_1}(x + y) = 2$.*
2) $\varphi_2 \preccurlyeq x + y$, and $N_{\varphi_2}(x + y) = 2$.

Proof. We have:

$$((x - 1)y + \frac{y^2}{1 + y})y + \frac{y^2}{1 + y} = xy^2,$$

whence

$$\ln\Big(((e^x - 1)\sqrt{e^y} + \frac{e^y}{1 + \sqrt{e^y}})\sqrt{e^y} + \frac{e^y}{1 + \sqrt{e^y}}\Big) = x + y,$$

i.e., $x + y = \ln\Big(\varphi_1(\varphi_1(e^x, \sqrt{e^y}) + 1, \sqrt{e^y})\Big) \in Cl_2(\varphi_1)$.

We also have $x + y \notin Cl_1(\varphi_1)$, because otherwise φ_1 would have complexity one with respect to addition, but it has complexity two.

Next, $x^3y = -\varphi_2(x^2, -\varphi_2(x, y))$ and $x^3y \sim x+y$ holds, i.e., $x+y \in Cl_2(\varphi_2)$. And complexity $N_{\varphi_2}(x + y)$ also equals two, because the complexity $N_{x+y}(\varphi_2)$ equals two.

The proof of Theorem 2 is complete. $\square$

Thus, we have

$$x + y \preccurlyeq (x - 1)y + \frac{y^2}{1 + y} \preccurlyeq x + y, \qquad x + y \preccurlyeq x^2 + xy \preccurlyeq x + y.$$

And also the functions $x+y$ and $\varphi_1 = (x-1)y + \dfrac{y^2}{1 + y}$ have different complexity with respect to the hierarchy **Cl**, hence, they belong to the different classes $[x+y]$ and $[\varphi_1]$ of gauge equivalence. I.e., the relation $\lesssim$ is also not a partial order. The same is true also for the pair of functions $x + y$ and φ_2 : the classes $[x + y]$ and $[\varphi_2]$ are different.

2.2. Increase of dimension of the stabilizer of a function

The functions φ_1 and φ_2 demonstrate one more interesting effect. Namely, every function f have the stabilizer $St(f)$ with respect to the action of the group $\mathcal{G}$, which consists of all elements $g \in \mathcal{G}$, such that $g \circ f = f$. For the dimension of the stabilizer there exist only four possibilities (see [7]):

1) infinity — in this case the function depends on only one variable,
2) three — in this case $f \sim x + y$,
3) one — in this case $f \sim r(x + y) - x$ for an analytic function $r(t)$,
4) zero in other cases.

For the functions φ_1 and φ_2 the dimensions of the stabilizers are the following: $\dim St(\varphi_2)=1$, $\dim St(\varphi_1) = 0$ (it is easy to check by substituting φ_1 in the published at the web page vkb.strogino.ru criterion of one-dimensionality of a stabilizer, see section "Другое", item "Maple-приложение к статье 'Об аналитических функциях двух переменных с одномерным стабилизатором'"). But on the other hand, the function $x + y$, having the three-dimensional stabilizer, belongs simultaneously to the classes $Cl_2(\varphi_1)$ and $Cl_2(\varphi_2)$. I.e., the more complicated function (with respect to φ_1 and φ_2) has higher dimension of the stabilizer. Such effect is absent in the standard hierarchy **Cl** with respect to addition: the simplest function $x+y$ has the largest stabilizer.

2.3. Partial order on the equivalence classes

Using the preorder $\preccurlyeq$, we can divide the set of analytic functions of two variables into equivalence classes: $f_1(x, y)$ is equivalent to $f_2(x, y)$, if $f_1(x, y) \preccurlyeq f_2(x, y)$ and $f_2(x, y) \preccurlyeq f_1(x, y)$ (notation: $f_1 \approx f_2$). Note that the orbits of the action of the group $\mathcal{G}$ fall into the same equivalence class. In accordance with the general set-theoretic construction the preorder relation $\preccurlyeq$ on the set $\Big(\mathcal{A}(x, y)\big/\approx\Big)$ of equivalence classes become a partial order relation.

2.4. Existence of nonequivalent functions inside one hierarchy

It follows from Theorem 2 that some functions of different complexity are not equivalent with respect to the relation $\approx$. The question arise: can all the functions from the hierarchy $\mathbf{Cl}(\varphi)$ fall into the same equivalence class for all φ? We will give an example of a function $\Phi = \Phi(x,y)$, such that among the functions from the hierarchy $\mathbf{Cl}(\Phi)$ there exist nonequivalent ones. For this purpose we will write down the function Φ of infinite complexity with respect to the standard hierarchy $\mathbf{Cl}$, such that the function $x+y$ has complexity two with respect to the hierarchy $\mathbf{Cl}(\Phi)$.

We need some auxiliary constructions. Let $F(x,y)$ be a function, satisfying the following three conditions:

$$\begin{aligned} &1)\ \ F(x,y) = F(y,x),\\ &2)\ \ F(x,y) = F(-x,y),\\ &3)\ \ \text{the function } y=\chi(x,w) \text{ given by the implicit equation } F(x,y)=w\\ &\quad\ \ \text{has infinite complexity.}\end{aligned} \tag{1}$$

It can be shown that functions $F(x,y)$ with these properties exists, and there are many functions of the same sort (see below), but we will give explicit examples of functions $F(x,y)$ with the specified properties.

To begin with, there exists an entire function ψ, such that the implicit equation

$$\psi(y) + xy = w \tag{2}$$

defines the function $\Psi(x,w)$ of infinite complexity (see [8]). At the same time, there are plenty of such functions ψ (further we will only need that there are more than two of them). By the implicit function theorem, $\Psi(x,w)$ is defined everywhere outside a proper analytic subset of the space $\mathbb{C}^2$. Note that equation (2) for all possible ψ defines the general solution of the Hopf equation $\dfrac{\partial y}{\partial x} = y\dfrac{\partial y}{\partial w}$.

Then the function $F(x,y) = \psi(y^2) + x^2y^2 + \psi(x^2)$ satisfies all three conditions given above. It is clear that conditions 1) and 2) are satisfied. Let us check condition 3).

If $\psi(y^2) + x^2y^2 + \psi(x^2) = w$, then $y = \chi(x,w) = \sqrt{\Psi(x, w-\psi(x^2))}$. Since $\Psi(x,w)$ has infinite complexity, then $\chi(x,w)$ also has infinite complexity, because the change of coordinates $\{x \longrightarrow x,\ w \longrightarrow w - \psi(x^2)\}$ does not change complexity, and also extracting a root does not change complexity.

Let $\Phi(x,w) = \chi(x+w,w)+w$. The function $\Phi(x,w)$ also has infinite complexity.

Theorem 3. $x+w \in Cl_2(\Phi)$.

Proof. Since $\Phi(x,w)$ has infinite complexity with respect to $w+x$, then $w+x \notin Cl_1(\Phi)$. Let us show that

$$\Phi(-\Phi(x,w),w) = x+2w. \tag{3}$$

As before in the definition of the complexity classes, we understand equation (3) as equality of the function $x+2w$ to the correctly chosen composition of germs of the function Φ. Here we consider this equality only in those points, in which it is possible to choose germs so that the composition is well-defined (these are generic points). I.e., we will show that equality (3) holds at a generic point.

Let $\Phi(-\Phi(x,w),w) = \chi\Big(-\Big(\chi(x+w,w)+w\Big)+w,w\Big)+w = u$. Then

$$\chi\Big(-\chi(x+w,w),w\Big) = u-w,$$

which is equivalent to

$$w = F\Big(-\chi(x+w,w), u-w\Big).$$

Using properties 2) and 1), we obtain

$$w = F\Big(\chi(x+w,w), u-w\Big) = F\Big(u-w, \chi(x+w,w)\Big),$$

whence

$$\chi(x+w,w) = \chi\Big(u-w,w\Big).$$

Therefore we can suppose $u-w = x+w$, whence $u = x+2w$. Following back the chain of equalities obtained in the proof of the theorem, we obtain that the function $u = x+2w$ satisfies the equality $\Phi(-\Phi(x,w),w) = u$. After the change $\Big\{x \longrightarrow x,\ w \longrightarrow \dfrac{w}{2}\Big\}$ we get what we need.

The proof of Theorem 3 is complete. □

Corollary 4. *Any function $f(x,y)$ of finite complexity with respect to addition belongs to the hierarchy* $\mathbf{Cl}(\Phi)$*, but in the same time* $\Phi \notin \mathbf{Cl}(f)$*.*

2.5. Pairs of functions of different finite relative complexity

Thus, we have constructed the function Φ, such that $N_\Phi(x+y) = 2$, but $N_{x+y}(\Phi) = \infty$ for all natural n. The given above construction can be slightly modified to obtain examples of functions Φ_n, such that $N_{\Phi_n}(x+y) = 2$, but $N_{x+y}(\Phi_n) = \nu(n)$ for some natural $\nu(n) \in [n-2, n+2]$. For this purpose we need to construct a function $F_n(x,y)$ satisfying the following three conditions:

1) $F_n(x,y) = F_n(y,x)$,

2) $F_n(x,y) = F_n(-x,y)$,

3) the function $y = \chi_n(x,w)$, given by the implicit equation $F_n(x,y) = w$, has complexity not less than $n-1$ and not greater than $n+1$.

To construct the functions $F_n(x,y)$ with the given properties it is necessary to choose, as above, an entire function ψ_n, such that the implicit equation

$$\psi_n(y) + xy = w$$

defines the function $\Psi_n(x,w)$ of complexity n (see [8]). Note that there are also plenty of such functions ψ_n.

Then the function $F_n(x,y) = \psi_n(y^2) + x^2y^2 + \psi_n(x^2)$ satisfies all the three conditions. It is clear that conditions 1) and 2) are satisfied. Let us check condition 3).

If $\psi_n(y^2) + x^2y^2 + \psi_n(x^2) = w$, then $y = \chi_n(x,w) = \sqrt{\Psi_n(x, w-\psi_n(x^2))}$. Since $\Psi_n(x,w)$ has complexity n, then the complexity of the function $\chi_n(x,w)$ differs from n by no more than one, because the coordinate change $\{x \longrightarrow x,\ w \longrightarrow w - \psi_n(x^2)\}$ can change complexity by not more than one, and extracting a root does not change complexity.

Let $\Phi_n(x,w) = \chi_n(x+w,w) + w$. Complexity $\nu(n)$ of the function $\Phi_n(x,w)$ differs from the complexity of the function χ_n by no more than two, therefore $n-3 \leqslant \nu(n) \leqslant n+3$.

2.6. Metric and continuous dependence on function

For functions f and g, which are holomorphic in U, the metric $\rho_U(f,g)$ is introduced in a standard way with the help of the enumerable sequence of seminorms in the Frechet space.

In the same time, it is clear that there is a continuous dependence of the classes $Cl_n(\varphi)$ on the function φ. Namely, let a function $f = f(B, S, \varphi)(x, y) \in Cl_n(\varphi)$ be given. Here $B = (B_1(t), \ldots, B_N(t))$ for $N = (2^{n+1} - 1)$ is the set of functions of one variable, which enter into the superposition of the function $f(B, S, \varphi)(x, y)$, and S is a scheme of the composition, i.e., a way of arrangement of the functions from the set B inside the superposition (see [6] for more information on schemes). Consider a function $\tilde{f} = f(B, S, \tilde{\varphi})(x, y) \in Cl_n(\tilde{\varphi})$, which differs from f only by replacing φ by $\tilde{\varphi}$. And let the functions φ and $\tilde{\varphi}$ have holomorphic elements $\Big(\varphi(U), U\Big)$ and $\Big(\tilde{\varphi}(U), U\Big)$ in a common domain U. Then $\forall\ \varepsilon > 0\ \exists\ \delta > 0$, such that for $\rho_U(\varphi(U), \tilde{\varphi}(U)) < \delta$ the functions $f, \tilde{f}$ have holomorphic elements $\Big(f_V, V\Big)$ and $\Big(\tilde{f}_V, V\Big)$ in a common domain $V = V(\varepsilon)$, for which $\rho_V(f_V, \tilde{f}_V) < \varepsilon$ holds.

In particular, from the property of continuity follows that the condition that a function does not belong to the class $Cl_n(\varphi)$ is an open condition. Namely, we have the following

Proposition 5. *Let $f \notin Cl_n(\varphi)$. Then there exists a number $\delta > 0$, such that for $\rho_U(\varphi(U), \tilde{\varphi}(U)) < \delta$ we have $f \notin Cl_n(\tilde{\varphi})$.*

Proof. Indeed, assume the contrary, i.e., let for all $\delta > 0$ there exist a function $\varphi_\delta(U)$, such that $\rho_U(\varphi(U), \varphi_\delta(U)) < \delta$, and $f \in Cl_n(\varphi_\delta)$ holds. But the sequence of functions φ_δ converges to φ in the metric for $\delta \longrightarrow 0$, whence by the property of continuity we obtain $f \in Cl_n(\varphi)$, which contradicts the condition. □

2.6. Almost all hierarchies contain nonequivalent functions

Choose a function $\tilde{\Phi}$, such that the following condition hold:

1) there exist a domain U, such that elements $\Phi(U)$ and $\tilde{\Phi}(U)$ of the functions Φ and $\tilde{\Phi}$ are defined in it,
2) $\rho(\Phi(U), \tilde{\Phi}(U)) < \varepsilon$,
3) $\tilde{\Phi} \notin \mathbf{Cl}(\Phi)$.

Condition 3) can be attained, because the set $\mathbf{Cl}(\Phi)$ is equal to the countable union of nowhere dense sets $Cl_n(\Phi)$. Hence, $\mathbf{Cl}(\Phi)$ can not include an entire neighbourhood of the function $\Phi(U)$.

For small enough ε we obtain that $\tilde{\Phi}(-\tilde{\Phi}(x, w), w) \notin \mathbf{Cl}(\tilde{\Phi})$, because $\Phi(-\Phi(x, w), w) \notin \mathbf{Cl}(\Phi)$ and the condition that a function does not belong to the set $\mathbf{Cl}(\Phi)$ is an open condition by continuous dependence of the differential criteria $J_n(\varphi)$ on φ. This means that the set $\mathbf{Cl}(\tilde{\Phi})/\approx$ also contain more than one equivalence class. It is also clear that almost all (in the sense of the metric ρ_U) functions $\tilde{\Phi}$ possess this property.

2.7. Intersecting hierarchies, which do not coincide

Suppose that two hierarchies $\mathbf{Cl}(\varphi_1)$ and $\mathbf{Cl}(\varphi_2)$, based on different functions φ_1 and φ_2, intersect, i.e., there exists a function f, such that $f \in \mathbf{Cl}(\varphi_1)$ and $f \in \mathbf{Cl}(\varphi_2)$, and also f depends on both variables x and y. This takes place, for example, if $\varphi_2 \in \mathbf{Cl}(\varphi_1)$ or $\varphi_1 \in \mathbf{Cl}(\varphi_2)$ holds. In this case one hierarchy is nested within one another. It is natural to ask: is it the only reason for intersection? We will show in what follows that the answer is negative – intersecting hierarchies are not necessary nested within one another.

Consider the set $\mathcal{F}$ of functions $F(x, y)$, satisfying properties 1) and 2) in (1), which are holomorphic in a domain U and for which the nonequality $\rho_U(\Phi(U), F) < \varepsilon$ for some $\varepsilon > 0$

holds. The set $\mathcal{F}$ defines the set $\mathcal{Y}$ of functions $y = \chi(x, w)$. The set $\mathcal{Y}$ is a complete metric space, which can be represented as the countable union $\mathcal{Y} = \cup_{1 \leqslant n \leqslant \infty} \mathcal{Y}_n$ of the sets $\mathcal{Y}_n$, consisting of functions of complexity n. Moreover, the sets $\mathcal{Y}_n$ for $n < \infty$ are nowhere dense. By the Baire theorem we obtain from here that the set $\mathcal{Y}_\infty$ can not be represented as an at most countable union of nowhere dense sets. Note that $\mathcal{Y}_\infty$ coincide with the set of all functions $y = \chi(x, w)$, given by property 3) in (1).

The set $Cl_n(\Phi)$ is nowhere dense for all fixed n. Hence, there exists a function $\Phi_1 \in \mathcal{Y}$ (and there are plenty of such functions), such that $x + y \in Cl_2(\Phi_1)$, but $\Phi_1 \notin Cl_n(\Phi)\, \forall n$. We also can obtain the condition $\Phi(U) \notin Cl_n(\Phi_1)\, \forall n$ in an analogous way.

This means that the hierarchies $Cl_n(\Phi)$ and $Cl_n(\Phi_1)$ intersect (in the function $x + y$ and some others), but are not nested within one another. It is clear that the intersection contains the set $\mathbf{Cl}(x + y)$.

In conclusion, let us formulate a question, which was left out of our considerations.

Question 6. Fix a function φ. Does there exist the minimal element in the hierarchy $\mathbf{Cl}(\varphi)$?

Here we mean the following. As was shown, there exists the function Φ, in terms of which we can express the function $x + y$, such that the hierarchy $\mathbf{Cl}(\Phi)$ contains the hierarchy $\mathbf{Cl}(x + y)$, and moreover, this inclusion is strict. The question arise: is it possible to find an infinite sequence of functions Ψ_j, $j = 1, 2, 3, \ldots$, such that the strict inclusion $\mathbf{Cl}(\Psi_1) \subset \mathbf{Cl}(\Psi_2) \subset \mathbf{Cl}(\Psi_3) \subset \ldots$ holds?

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Отношение предпорядка на множестве аналитических функций

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Аннотация. В работе изучено отношение предпорядка, которое естественно определяется на множестве аналитических функций двух комплексных переменных с помощью суперпозиций аналитических функций одной переменной.

Ключевые слова: аналитическая сложность, суперпозиции аналитических функций.