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Elasto-plastic Twisting of a Two-layer Rod Weakened by Holes

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Abstract. Under study is the elasto-plastic twisting of a multiply-connected two-layer prismatic rod under the influence of a couple of forces with a given moment. It is assumed that the rod consists of two layers. Either layer has its own elastic properties, but the plastic characteristics of both layers are the same. The contact boundary of the layers is located alongside Axis ox. The lateral boundary of the rod is free from stresses; at the interface, continuous are movements and stresses. Stress components at a point are calculated with the help of contour integrals obtained from the conservation laws, calculated on the lateral boundary and the boundaries of the holes. At those points of the rod where yield stress is achieved — plastic state is present, at the rest of them — elastic. This allows building the boundary between the plastic and elastic areas.

Keywords: elastic-plastic torsion, multi-layer materials, conservation laws.

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Introduction

In the article we continue to use the conservation laws for solving the boundary value problems of the equations of mechanics of a solid body being deformed. For more than 50 years the equations of elasticity and plasticity are studied with the help of symmetries [1, 2]. This allowed building a series of exact solutions and studying some qualitative properties of these equations. There were attempts to solve the boundary value problems [2] with the help of symmetries, but here good results did not manage to be achieved, which is explained by the local nature of the symmetries. The found conservation laws turned out to be more suitable for solving the boundary value problems of mechanics equations [3–6]. For the first time, the conservation laws

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were used for solving the boundary value problems for two-dimensional equations of plasticity [3–11], and with their use it turned out well to solve the basic boundary value problems. This is explained by the fact that the symmetries by their nature are local, unlike the conservation laws – global in and of itself. Further, the conservation laws were used for solving the elasto-plastic problems on the twisting of rods and bending of cantilevers, and also solving the elasto-plastic problems for plates with finite dimensions, weakened by holes [12–18]. In the present work it is demonstrated that the conservation laws can be successfully used also for solving the boundary value problems for multi-layer materials.

1. Problem setting

Let us consider a rectilinear rod, a cross section of which is shown in Fig. 1.

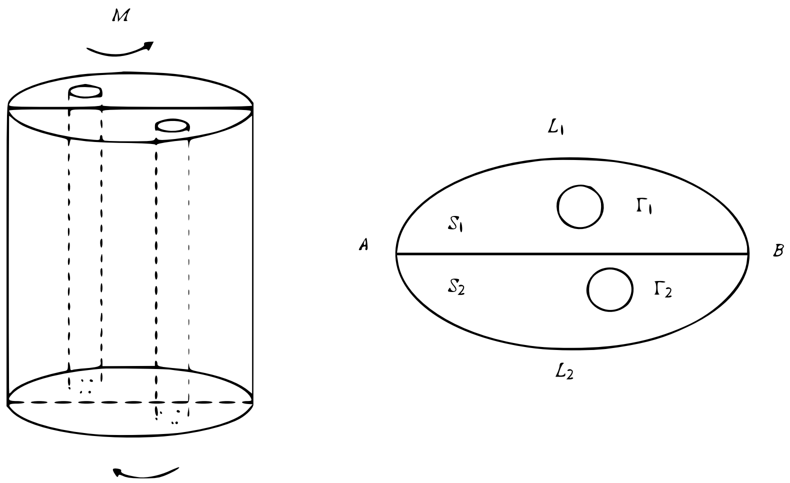

Fig. 1. Multi-linked rod

Assume S_1 and S_2 are the areas, occupied by elasto-plastic isotropic materials that have their yield stress at pure shift identical and equal to k and Lame elastic constants are different and equal to λ_1, μ_1 and λ_2, μ_2 respectively. There are two holes limited by the contours Γ_1 and Γ_2 Assume that the boundary line of the materials is rectilinear. We will choose the axis of coordinate alongside the boundary line. It is assumed, as usual, that the lateral surface of the rod and the side walls of the holes are free from stresses, and the rod is being twisted by a couple of forces with the moment

$$M = \iint (y\sigma_{13} - x\sigma_{23})dxdy$$

In this case, the equations describing stress state in area S_i, $i = 1, 2$ are written as

$$F_1 = \partial_x\sigma_{13} + \partial_y\sigma_{23} = 0, \quad F_2 = \partial_y\sigma_{13} - \partial_x\sigma_{23} + \mu_i\omega = 0, \quad \mu_i\omega = K_i, \tag{1}$$

where σ_{13}, σ_{23} are stress components, ω is twist angle, it is assumed to be constant. On the lateral surface of the rod and the holes these conditions comply

$$\sigma_{13}n_1 + \sigma_{23}n_2 = 0, \quad \sigma_{13}^2 + \sigma_{23}^2 = k^2, \tag{2}$$

and they mean that the lateral surface and the walls of the holes are free from stresses and are in plastic state.

From (2) we obtain

$$\sigma_{13} = kn_1, \quad \sigma_{23} = -kn_2. \tag{3}$$

Also we assume that on the boundary line stress components are continuous, this means absence of stress interruption for this given rod alongside .

2. Conservation laws

The conservation law we will search in the form of

$$A_x + B_y = \rho_1 F_1 + \rho_2 F_2, \tag{4}$$

where ρ_1, ρ_2 are some functions, simultaneously not identically zero, the letter subscripts stand for derivatives with respect to the corresponding variables.

Note. More detailed information on the conservation laws, their calculating and usage can be found in the literature referenced above.

Assume

$$A = \alpha^1 u + \alpha^2 v + \alpha^3, \quad B = \beta^1 u + \beta^2 v + \beta^3, \tag{5}$$

where for convenience it was set $\sigma_{13} = u$, $\sigma_{23} = v$, $\alpha^1, \alpha^2, \alpha^3, \beta^1, \beta^2, \beta^3$ are assumed to be functions only x, y.

Inserting (5) into (4) we obtain

$$\alpha^1 = \beta^2, \quad \alpha^2 = -\beta^1, \quad \alpha^1_x - \alpha^2_y = 0, \quad \alpha^1_y + \alpha^2_x = 0, \quad \alpha^3_x + \beta^3_y = -\alpha^2 K_i. \tag{6}$$

Assume

$$\alpha_x^{1(i)} - \alpha_y^{2(i)} = 0, \quad \alpha_y^{1(i)} + \alpha_x^{2(i)} = 0, \quad \alpha_x^{3(i0} + \beta_y^{3(i0} = -\alpha^2 K_i, \quad i = 1, 2. \tag{7}$$

Here the index i in brackets corresponds to the area S_i.

Let us assume that at the point x_0, y_0 the subintegral functions have a singularity, and this point is located within a circle with the radius $\varepsilon : (x - x_0)^2 + (y - y_0)^2 = \varepsilon^2$, then from (4) we obtain (see Fig. 2)

$$\iint_S (A_x + B_y)dxdy = \iint_{S_1} ({A^1}_x + {B^1}_y)dxdy + \iint_{S_2} ({A^2}_x + {B^2}_y)dxdy =$$
$$= -\int_\varepsilon A^1 dy - B^1 dx + \int_{L_1} A^1 dy - B^1 dx + \int_{L_2} A^2 dy - B^2 dx + \int\limits_{\Gamma_1} A^1 dy - B^1 dx +$$
$$+ \int_{\Gamma_2} A^2 dy - B^2 dx + \int_{AB} A^1 dy - B^1 dx + \int_{BA} A^2 dy - B^2 dx = 0.$$

We have alongside

$$\int_{AB} A^1 dy - B^1 dx + \int_{BA} A^2 dy - B^2 dx =$$
$$= \int_{AB} (\alpha^{1(1)} u + \alpha^{2(1)} v + \alpha^{3(1)})dy - (-\alpha^{2(1)} u + \alpha^{1(1)} v + \beta^{3(1)})dx +$$
$$+ \int_{BA} (\alpha^{1(2)} u + \alpha^{2(2)} v + \alpha^{3(2)})dy - (-\alpha^{2(2)} u + \alpha^{1(2)} v + \beta^{3(2)})dx = 0$$

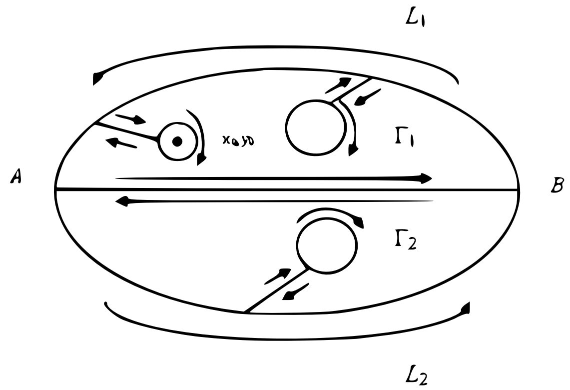

Fig. 2. Contour bypass scheme when computing the integral

Since alongside $dy = 0$, then we assume $\beta^{3(i)} = 0$, $\alpha_x^{3(i)} = \alpha^{2(i)} K_i$, therefore $\alpha^{1(1)} = \alpha^{1(2)}$, $\alpha^{2(1)} = \alpha^{2(2)}$.

As a result we obtain

$$\int_{\varepsilon} A^1 dy - B^1 dx = \int_{L_1} A^1 dy - B^1 dx + \int_{\Gamma_1} A^1 dy - B^1 dx + \int_{L_2} A^2 dy - B^2 dx + \\ + \int_{\Gamma_2} A^2 dy - B^2 dx. \tag{8}$$

Let us use formula (8) to find the functions u, v at the point x_0, y_0.
For this, we will consider the solution of equations (7) in the form of

$$\alpha^1 = \frac{x - x_0}{(x - x_0)^2 + (y - y_0)^2}, \ \alpha^2 = -\frac{y - y_0}{(x - x_0)^2 + (y - y_0)^2}, \ \alpha^3 = \omega \mu_1 \operatorname{arctg} \frac{x - x_0}{y - y_0}. \tag{9}$$

Inserting (9) into (8) we obtain

$$\int_{\varepsilon} A^1 dy - B^1 dx = \int_{\varepsilon} (\alpha^1 u + \alpha^2 v + \alpha^3) dy - (-\alpha^2 u + \alpha^1 v) dx =$$

$$= \int_{\varepsilon} \left(\frac{x - x_0}{(x - x_0)^2 + (y - y_0)^2} u - \frac{y - y_0}{(x - x_0)^2 + (y - y_0)^2} v + \varpi \mu_1 \operatorname{arctg} \frac{x - x_0}{y - y_0} \right) dy -$$

$$- \left(\frac{y - y_0}{(x - x_0)^2 + (y - y_0)^2} u \right) dx + \int_{\varepsilon} \frac{x - x_0}{(x - x_0)^2 + (y - y_0)^2} v) dx.$$

Assume $x - x_0 = \varepsilon \cos \phi, \ y - y_0 = \varepsilon \sin \phi$, then we obtain

$$\int_{\varepsilon} A^1 dy - B^1 dx = \int_0^{2\pi} [(u \cos \phi + v \sin \phi) \cos \phi + (u \sin \phi + v \cos \phi) \sin \phi] d\phi = \\ = \int_0^{2\pi} u d\phi = 2\pi \, u(x_0, y_0).$$

In the last equation, being used are the mean-value theorem and the passage to the limit $\varepsilon \to 0$.

As a result from formula (8) it follows

$$
\begin{aligned}
&2\pi\,\sigma_{13}(x_0,y_0)=\int_{L_1}\left(\frac{x-x_0}{(x-x_0)^2+(y-y_0)^2}kn_1+\frac{y-y_0}{(x-x_0)^2+(y-y_0)^2}kn_2+\right.\\
&+\omega\mu_1\,\mathrm{arctg}\,\frac{x=x_0}{y-y_0}\Bigg)dy-\left(\frac{y-y_0}{(x-x_0)^2+(y-y_0)^2}kn_1-\frac{x-x_0}{(x-x_0)^2+(y-y_0)^2}kn_2\right)dx+\\
&+\int_{\Gamma_1}\left(\frac{x-x_0}{(x-x_0)^2+(y-y_0)^2}kn_1+\frac{y-y_0}{(x-x_0)^2+(y-y_0)^2}kn_2+\omega\mu_1\,\mathrm{arctg}\,\frac{x=x_0}{y-y_0}\right)dy-\\
&-\left(\frac{y-y_0}{(x-x_0)^2+(y-y_0)^2}kn_1-\frac{x-x_0}{(x-x_0)^2+(y-y_0)^2}kn_2\right)dx+\\
&+\int_{L_2}\left(\frac{x-x_0}{(x-x_0)^2+(y-y_0)^2}kn_1+\frac{y-y_0}{(x-x_0)^2+(y-y_0)^2}kn_2+\omega\mu_2\,\mathrm{arctg}\,\frac{x=x_0}{y-y_0}\right)dy-\\
&-\left(\frac{y-y_0}{(x-x_0)^2+(y-y_0)^2}kn_1-\frac{x-x_0}{(x-x_0)^2+(y-y_0)^2}kn_2\right)dx+\\
&+\int_{\Gamma_2}\left(\frac{x-x_0}{(x-x_0)^2+(y-y_0)^2}kn_1+\frac{y-y_0}{(x-x_0)^2+(y-y_0)^2}kn_2+\omega\mu_2\,\mathrm{arctg}\,\frac{x=x_0}{y-y_0}\right)dy-\\
&-\left(\frac{y-y_0}{(x-x_0)^2+(y-y_0)^2}kn_1-\frac{x-x_0}{(x-x_0)^2+(y-y_0)^2}kn_2\right)dx.
\end{aligned}
\tag{10}
$$

Let us consider the solution of equations (7) in the form of

$$
\begin{gathered}
\alpha^1=\frac{y-y_0}{(x-x_0)^2+(y-y_0)^2},\quad \alpha^2=\frac{x-x_0}{(x-x_0)^2+(y-y_0)^2},\\
\alpha^3=\frac{1}{2}\omega\mu_2\ln((x-x_0)^2+(y-y_0)^2).
\end{gathered}
\tag{11}
$$

Inserting (11) into (8) we obtain

$$
\begin{aligned}
&2\pi\,\sigma_{23}(x_0,y_0)=\int\limits_{L_1}\left(\frac{y-y_0}{(x-x_0)^2+(y-y_0)^2}kn_1-\frac{x-x_0}{(x-x_0)^2+(y-y_0)^2}kn_2+\right.\\
&+\frac{1}{2}\omega\mu_2\ln\Big((x-x_0)^2+(y-y_0)^2\Big)dy-\left(-\frac{x-x_0}{(x-x_0)^2+(y-y_0)^2}kn_1+\frac{y-y_0}{(x-x_0)^2+(y-y_0)^2}kn_2\right)\Bigg)dx+\\
&+\int\limits_{\Gamma_1}\left(\frac{y-y_0}{(x-x_0)^2+(y-y_0)^2}kn_1-\frac{x-x_0}{(x-x_0)^2+(y-y_0)^2}kn_2+\frac{1}{2}\omega\mu_2\ln((x-x_0)^2+(y-y_0)^2)dy-\right.\\
&\qquad-\left(-\frac{x-x_0}{(x-x_0)^2+(y-y_0)^2}kn_1+\frac{y-y_0}{(x-x_0)^2+(y-y_0)^2}kn_2\right)\Bigg)dx+\\
&+\int\limits_{L_2}\left(-\frac{y-y_0}{(x-x_0)^2+(y-y_0)^2}kn_1+\frac{x-x_0}{(x-x_0)^2+(y-y_0)^2}kn_2+\frac{1}{2}\omega\mu_2\ln((x-x_0)^2+(y-y_0)^2)dy-\right.\\
&\qquad-\left(-\frac{x-x_0}{(x-x_0)^2+(y-y_0)^2}kn_1+\frac{y-y_0}{(x-x_0)^2+(y-y_0)^2}kn_2\right)\Bigg)dx+\\
&+\int\limits_{\Gamma_2}\left(-\frac{y-y_0}{(x-x_0)^2+(y-y_0)^2}kn_1+\frac{x-x_0}{(x-x_0)^2+(y-y_0)^2}kn_2+\frac{1}{2}\omega\mu_2\ln((x-x_0)^2+(y-y_0)^2)dy-\right.\\
&\qquad-\left(-\frac{x-x_0}{(x-x_0)^2+(y-y_0)^2}kn_1+\frac{y-y_0}{(x-x_0)^2+(y-y_0)^2}kn_2\right)\Bigg)dx.
\end{aligned}
\tag{12}
$$

Conclusion

Formulas (10), (12) allow calculating stress components' values at all points of a cross section. Further, at each point x_0, y_0 to be checked is the condition of plasticity $\sigma_{13}^2 + \sigma_{23}^2 = k^2$. Those points where $\sigma_{13}^2 + \sigma_{23}^2 < k^2$. belong to the elastic zone, and the rest of the points belong to the plastic zone. In this way, the described procedure allows separating the plastic and elastic zones and building the elasto-plastic boundary which beforehand was unknown and was to be determined.

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Упруго-пластическое кручение двухслойного стержня, ослабленного отверстиями

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Аннотация. Изучается упруго-пластическое кручение многосвязного двухслойного призматического стержня под действием пары сил с заданным моментом. Предполагается, что стержень состоит из двух слоев. Каждый слой обладает своими упругими свойствами, но пластические характеристики у обоих слоев одинаковые. Граница контакта слоев расположена вдоль оси ох. Боковая граница стержня свободна от напряжений, на границе раздела непрерывны перемещения и напряжения. Компоненты тензора напряжений в точке вычисляются с помощью контурных интегралов, полученных из законов сохранения, вычисленных по боковой границе и границам отверстий. В тех точках стержня, где достигается предел текучести, реализуется пластическое состояние, в остальных — упругое. Это позволяет построить границу между пластической и упругой областями.

Ключевые слова: упруго-пластическое кручение, многослойные материалы, законы сохранения.