Journal of Siberian Federal University. Mathematics & Physics 2023, 16(5), 620–627

EDN: HOXOPX
510.643; 517.11

Criterion of Global Admissibility for Logic IPC

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Received 29.04.2023, received in revised form 16.06.2023, accepted 01.8.2023

Abstract. We describe globally admissible inference rules for logic IPC.

Keywords: modal logic, frame and model Kripke, admissible and globally admissible infere

Citation: V.V. Rimatskiy, Criterion of Global Admissibility for Logic IPC, J. Sib. Fed. Univ. Math. Phys., 2023, 16(5), 620–627. EDN: HOXOPX.

Introduction

Setting the basic rules of inference is fundamental to logic and its deductive system. The most general variant of possible inference rules is the admissible inference rules introduced by Lorenzen in 1955. An inference rule is admissible in a logic L if the set of theorems L is closed with respect to this rule. Directly from the definition it follows that the set of all admissible rules is the most general concept of rules compatible with the logic L that can be added to the logic without changing the set of theorems provable in L. For the majority of basic non-classical logics ($IPC; KC; K4; S4; S5; S4.3$, etc.), the problem of decidability with respect to the admissibility of inference rules (the Friedman problem) was solved by V. V. Rybakov in 1980s (see, for example, [1, 2]).

To the Kuznetsov's problem (1975) goes back to another way of describing all rules admissible in logic: the assignment of some (finite, explicit) set of admissible rules, from which all other rules admissible in logic will be derived as consequences, i.e. setting a (finite, explicit) basis. It turned out that most basic logics ($IPC; KC; K4; S4; Grz$, etc.) do not have a finite basis for admissible inference rules, i.e. the Kuznetsov problem for them was solved by V. V. Rybakov in 1980s in the negative (see [1]). For a wide class of logics (including most basic and some tabular logics), an explicit basis of admissible rules was obtained at the beginning 2000s (see [3–5]). The question arose about the further development of the theory of admissible rules.

In addition to using admissible inference rules to describe non-trivial semantic properties of non-classical logics (see [6, 7]), one can also propose the following approach. The next step in the study of admissible inference rules for non-classical logics was a globally admissible inference rule. The concept of a globally admissible inference rule was introduced in 2005 (see [8]). Globally admissible rules in the logic L are those inference rules that are admissible simultaneously in all (finitely approximable) extensions of this logic. Such rules develop and generalize the concept of an admissible inference rule.

As for admissible rules when studying a new individual logic or a whole class of logics, for globally admissible rules of inference, questions of resolvability arise (an algorithm for recognizing

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the global admissibility of a given rule), the existence of a basis for them (or an anti-basis from which all rules that are not globally admissible are derived), its finiteness and explicit description (the Kuznetsov problem). Ideally, both a certain criterion for the global admissibility of a rule in a given logic and an explicit description (construction) of an explicit (or finite) basis for such rules is desirable. Whether this problem has a solution is not yet clear.

To date the author is aware of relatively few results devoted to the study of globally admissible inference rules. In a short note [8] the reduction of global admissibility to tabular admissibility was proved: a rule is globally admissible in the logic L if and only if it is admissible in all tabular extensions of the logic L. The (recursive) basis of globally admissible rules in semi-reduced form for the logic IPC was described in [9]. In [10] an explicit (infinite) basis of inference rules that are globally admissible in modal pretabular logics $PT2$, $PT3$ was obtained. In [11] the conditions for global admissibility in the logic $S4$ were obtained, the (recursive) basis and anti-basis of such inference rules globally admissible in $S4$ were described.

The presented work is devoted to the study of globally admissible rules of intuitionistic logic IPC. A necessary and sufficient condition for global admissibility in the logic IPC is obtained.

1. Denotation, preliminary facts

We assume the reader to be aware of algebraic and Kripke semantics for superintuitionistic logics and also assume a certain initial knowledge concerning basic facts on inference rules and their admissibility (though we recall below briefly all necessary facts). As a good entry point to the subject in whole we would recommend Rybakov [1] for advanced technique concerning superintuitionistic logics and inference rules. According to modern trends by a *logic* we understand the set of all theorem provable in a given axiomatic system. In particular, a superintuitionistic logic λ is a set of theorems of a logical axiomatic system, where λ includes all theorems of the intuitionistic propositional calculus IPC. In definitions which follow we mean under a propositional logic an algebraic propositional logic (cf. [1]) though the reader can understand λ as a superintuitionistic logic (s.i.) which is enough for our purposes.

Since we are dealing below with only superintuitionistic logics, a Kripke frame, or merely a frame, is a partially ordered set $\mathcal{F} := \langle \mathcal{F}, \leqslant \rangle$ (a poset for short). As usually, we understand by intuitionistic Kripke model a frame $\mathcal{F}$ with a certain valuation V of a set of propositional letters P, where V is upwards stable: $\forall p \in P$ and $\forall x, y \in \mathcal{F}(x \in V(p) \& (x \leqslant y) \Rightarrow y \in V(p))$. For any propositional formula α with variables from the domain of V and any $a \in \mathcal{F}$, $a \models_V \alpha$ is the denotation for α is valid (or true) at a in the model $\mathcal{M} := \langle F, \leqslant, V \rangle$ under the valuation V. If we need to distinguish in which namely model the truth takes place we write $(\mathcal{M}, a) \models_V \alpha$. Well known fact is that the intuitionistic truth of formulas is stable upwards ($\forall a, b \in \mathcal{F}$, if $a \models_V \alpha$ and $a \leqslant b$ then $b \models_V \alpha$) in Kripke models. For definition of open subframes and open submodels we refer to, for example, [1]. If $\mathcal{M}_1$ and $\mathcal{M}_2$ are frames or models $\mathcal{M}_1 \sqsubseteq \mathcal{M}_2$ is the abbreviation for $\mathcal{M}_1$ is open subframe, or respectively submodel, of $\mathcal{M}_2$. A useful fact concerning this is that, if $\mathcal{M}_1$ is an open submodel of $\mathcal{M}_2$, and $a \in \mathcal{M}_1$ then $(\mathcal{M}_1, a) \models_V \alpha) \Leftrightarrow (\mathcal{M}_2, a) \models_V \alpha$, i.e. the truth is the same at the open submodel as in the model itself.

For any subset $\mathcal{X}$ of a fame $\mathcal{F}$, $\mathcal{X}^{\leqslant} := \{a \mid \exists b \in \mathcal{X}(b \leqslant a)\}$, that is $\mathcal{X}^{\leqslant}$ is the upwards cone generated by $\mathcal{X}$, and $\mathcal{X}^{\leqslant +} := \{a \mid \exists b \in \mathcal{X}(b \leqslant a) \& \forall c \in \mathcal{X}(\neg(a \leqslant c))\}$. For any antichain $\mathcal{Y}$ of elements from $\mathcal{F}$, an element c from $\mathcal{F}$ is a **co-cover** for $\mathcal{Y}$ if and only if $c^{\leqslant +} = \bigcup_{c_1 \in \mathcal{Y}} (c_1^{\leqslant})$. A frame $\mathcal{F}$ is rooted, or sharp, if $\exists a \in \mathcal{F}$ such that $\forall b \in \mathcal{F}$, $a \leqslant b$, then we say a is the root of $\mathcal{F}$ and denote that element by $root(\mathcal{F})$. $S_m(\mathcal{F})$ denotes the set of all elements of $\mathcal{F}$ with depth not exceeding m, and $Sl_m(\mathcal{F})$ is the set of all elements of $\mathcal{F}$ with depth m, i.e. — m-slice of $\mathcal{F}$.

All preliminary information concerning inference rules and their admissibility can be found in [1]. We recall briefly below necessary definitions and facts. Let $\alpha_1, \ldots, \alpha_n, \beta$ be some formulas. We understand the figure r, where

$$r := \frac{\alpha_1, \ldots, \alpha_n}{\beta},$$

as the (structural) inference rule, which derives $s(\beta)$ from $s(\alpha_1), \ldots, s(\alpha_n)$ for every substitution s. We say r is derivable in a logic λ if there is a derivation β in λ from the set of assumptions $\{\alpha_1, \ldots, \alpha_n\}$. And r is called *admissible* in λ if, for every substitution s, $s(\beta) \in \lambda$ whenever $s(\alpha_1) \in \lambda$, ..., $s(\alpha_n) \in \lambda$. Clearly any derivable rule is admissible, but not conversely in general. Also immediately from the definition we see that the set of all rules admissible in a logic λ is the *greatest* class of inference rules by which we can extend axiomatic system of the logic λ preserving theorems of λ (which is a particular interest for studying these rules). Derivable rules may replace some fragments of fixed length in derivations, thereby shortening them linearly. Admissible rules, which are not derivable, in principle may reduce derivations even more drastically.

Algebraic description of admissible rules came from Polish Logical School. A rule r is admissible in λ iff the quasi-identity $q(r) := \alpha_1 = \top \& \ldots \& \alpha_n = \top \Rightarrow \beta = \top$ is valid in the free algebra of countable rank $\mathcal{F}_\lambda(\omega)$ from the variety $Var(\lambda)$ of all algebras on which all theorems of λ are valid (cf. [1] for details).

Given a propositional logic λ, a rule $r := \alpha_1, \ldots, \alpha_n/\beta$ is a *consequence* of a family of rules F in λ (denotation $F \vdash_\lambda r$) if there is a derivation of β from $\alpha_1, \ldots, \alpha_n$ as assumptions in the axiomatic system of λ extended by adding F as new rules.

A set of inference rules S admissible in a propositional logic λ is a *bases* for all rules admissible in λ if, for any admissible rule r, $S \vdash_\lambda r$, i.e. r is a consequence of S in λ.

It is clear now why we are interested to describe a bases for admissible rules of IPC in precise — doing that we will have exhaustive collection of rules compatible with derivability in IPC, all others will be their consequences.

The admissibility of inference rules in IPC can be described through their validness in certain special n-characterizing Kripke models. Description of these models $Ch_{IPC}(n)$ and criteria for recognizing admissibility in IPC by means of them are given, for instance, in [1]. Since we will strongly occupy these techniques in our paper, we recall briefly the construction of $Ch_{IPC}(n)$ and the semantic criterion for recognizing admissibility.

Given a set $P_n := \{p_1, \ldots, p_n\}$ of propositional letters, we construct the first slice $S_1(Ch_{IPC}(n)$ as follows. It consists of the collection of all elements with all possible valuations V of letters from P_n which does not have doubling - elements with the same valuation. Recall that, for any element a of a Kripke model $\mathcal{M}$ with a valuation V, $V(a)$ is the set of all propositional letters which are valid under V at a. Assuming $S_m(Ch_{IPC}(m))$ to be constructed, we put in $Sl_{m+1}(Ch_\lambda(m))$ the elements as follows. We take arbitrary antichain $\mathcal{Y}$ of elements from $S_m(Ch_{IPC}(m))$ having at least one element of depth m and put in $Sl_{m+1}(Ch_{IPC}(n))$ all elements c from $S_1(Ch_{IPC}(n))$, assuming any c to be immediate predecessor for all elements from $\mathcal{Y}$, such that

(i) $V(c) \subseteq \bigcap_{a \in \mathcal{Y}} V(a)$; and

(ii) if $\mathcal{Y} := \{a\}$ then $V(c) \subset V(a)$.

Iterating this procedure we get as the result the model $Ch_{IPC}(n)$. Recall a model $\mathcal{M}$ is n-characterizing for a logic λ if, for any formula α, which is built up out of letters from P_n, $\alpha \in \lambda$ iff $\mathcal{M} \models \alpha$. We need the following facts.

Theorem 1.1 ([1], Theorem 3.3.11). *The model $Ch_{IPC}(n)$ is n-characterizing for the logic IPC.*

For a given frame $\mathcal{F}$, a given valuation V and a given inference rule $r := \alpha_1, \dots, \alpha_n/\beta$, we say r is valid at $\mathcal{F}$ under V, and write $\mathcal{F} \models_V r$, if as soon as $\forall x \in \mathcal{F}$ and $\forall i$ $(x \models_V \alpha_i)$ holds, we have $\forall x \in \mathcal{F}(x \models_V \beta)$. A rule r is valid at an intuitionistic frame $\mathcal{F}$ if r is valid at $\mathcal{F}$ under any intuitionistic valuation, we write then $\mathcal{F} \models r$.

Theorem 1.2 ([1], Theorem 3.5.8, Lemma 3.4.2). *For any inference rule r, r is admissible in IPC iff r is valid in the frame of $Ch_{IPC}(n)$ under any intuitionistic valuation for any given n.*

We say that the inference rule r *is globally admissible in the logic* λ_0 if r is admissible in any finitely approximate (tabular) logic λ, extending logic λ_0 and denote $r \in TAd(\lambda_0)$. The set of globally admissible rules $\mathcal{B}$ is called *the basis of the globally admissible over logic* λ_0, if any globally admissible rule r is consequence from $\mathcal{B}$ in any tabular logic λ that extends the logic λ_0, i.e. $\forall r \in TAd(\lambda_0)\ \forall \lambda \supseteq \lambda_0\ \mathcal{B} \vdash_\lambda r$.

The main result of [8] was the reduction of the global admissibility of the rule in logic $S4(Int)$ to the admissibility in all tabular extensions of this logic:

Theorem 1.3 (Th.3, [8]). *The inference rule r is admissible in all finitely approximated logics, expanding $S4(Int)$ $\iff$ r is admissuble in all tabular logics (including those generated by root $S4(Int)$-frames), expanding $S4(Int)$.*

We also need for our research a certain reduction of any intuitionistic inference rule to the most simple form as it is possible. These forms — semi-reduced forms — are defined below as in [12] or [9]. Given an inference rule $r := \alpha_1, \dots, \alpha_n/\beta$. To transform r into semi-reduced form $sr(r)$ we are doing the following steps. First, we take the rule $r_1 := \alpha_1 \wedge \dots \wedge \alpha_n/\beta$, and then the rule

$$r_2 := \frac{\alpha_1 \wedge \cdots \wedge \alpha_n \wedge (\beta \equiv x_0)}{x_0},$$

where x_0 is a new variable having no occurrences in r_1. Evidently that all rules r, r_1 and r_2 are equivalent with respect to admissibility in any superintuitionistic logic and with respect to validness at any pseudo-boolean algebra, and at any Kripke frame as well, i.e. they are equivalent in any semantic sense. Our rule r_2 has the form α/x_0, and now we will disclose (or, maybe better to say, to decompose) the premise of r_2 introducing new variables with the aim to make formulas from the premise possibly most simple. What we are doing first, we introduce the rule

$$r_3 := \frac{x, x \equiv \alpha}{x_0},$$

where x is a new variable not occurring in r_2; evidently r_3 is equivalent to r_2 in any sense mentioned above. We call the variable x standing alone at the premise of r_3 *the main variable of the premise* and denote it by $mv(r_3)$.

Assume we already have a rule r_4 equivalent to r in the form

$$r_4 := \frac{x, x_1 \equiv \gamma_1, \dots, x_k \equiv \gamma_k}{x_0},$$

where any variable x_i does not have occurrences in the formula γ_i. Take first formula γ_i from the premise which is not a simple term, i.e. (i) $\gamma_i = \delta_1 \circ \delta_2$, where $\circ \in \{\vee, \wedge, \rightarrow\}$ and δ_1 or δ_2 is not a variable, or (ii) $\gamma_i = \neg\delta$, where δ is not a variable. In the case (i) we take the rule

$$r_5 := \frac{x, \{x_j \equiv \gamma_j \mid 1 \leqslant j < i, i < j \leqslant k\}, y_1 \equiv \delta_1, y_2 \equiv \delta_2, x_i \equiv y_1 \circ y_2}{x_0},$$

where y_1 and y_2 are new variables having no occurrences in r_4. For the case (ii) we put

$$r_5 := \frac{x, x_1 \equiv \gamma_1, \ldots, x_{i-1} \equiv \gamma_{i-1}, x_{i+1} \equiv \gamma_{i+1}, \ldots, x_k \equiv \gamma_k, y \equiv \delta, x_i \equiv \neg y}{x_0},$$

where y is a new variable having no occurrences in r_4. Continuing the described procedure till up all formulas γ_i to become simple terms we get the rule $sr(r)$ which we call *semi-reduced form* of the rule r. Again $mv(sr(r))$ is the main variable of $sr(r)$ — the variable x having occurrence in the premise of $sr(r)$ as the formula x.

Lemma 1.1 ([12])**.** *The rules $sr(r)$ and r are equipotent w.r.t. admissibility in any superintuitionistic logic and w.r.t. semantic validness at any pseudo-boolean algebra and at any Kripke frame.*

Lemma 1.2 ([12])**.** *Any rule r is a consequence of the rule $sr(r)$ in any superintuitionistic logic λ, i.e. $sr(r) \vdash_\lambda r$.*

2. Main result

Next, we consider the intuitionistic rules of inference in a semi-reduced form. The following two statements are quite obvious.

Lemma 2.1. *If the rule r is valid on any finite $IPC(S4)$-model M for any valuation, then it is globally admissible in $S4(Int)$.*

Proof. Indeed, the n-characteristic model of an arbitrary tabular logic over $Int(S4)$ is finite. Consequently, for arbitrary formulaic valuation, the rule will be true on all n-characteristic models, i.e. admissible in all tabular logics over $IPC(S4)$. □

Lemma 2.2. *If the rule r is globally admissible in $IPC(S4)$, then this rule is true on the intuitionistic (modal) model $\mathcal{E} = \langle \{e\}, R, S\rangle$ generated by the only reflexive element e for any valuation S.*

Proof by contradiction. Indeed, consider the tabular logic $\lambda_e = \lambda(e)$ generated by the singleton cluster e. For any n, the frame of the n-characteristic model of this logic is a direct union of one-element clusters (a finite number of copies of the element e). Under valuation S, the rule r is refuted on this n-characteristic model (due to the finiteness of the model, the valuation is formulaic). Hence we conclude that the rule r is not admissible in tabular logic $\lambda_e = \lambda(e)$, which contradicts the original assumption. □

Let's define the root frame $\mathcal{T} := \langle \{a, b, c\}, \leqslant\rangle = c^{\leqslant}$, where $c \leqslant a$, $c \leqslant b$ and elements $\{a, b\}$ forms an antichain (i.e. the element c is a co-cover of the antichain $\{a, b\}$ and the root of the given frame).

Lemma 2.3. *If the rule $sr(r)$ in the semi-reduced form is not globally admissible in the logic IPC, then this rule is refuted on some finite model, and in particular, on the frame $\mathcal{T}$ under some valuation.*

Proof. Let us assume that the rule $sr(r)$ is not globally admissible in the logic IPC, i.e. is not admissible in some tabular superintuitionistic logic L. So the rule $sr(r)$ is refuted on some n-characteristic model $Ch_L(n)$ (for some suitable n) with valuation V:

$$Ch_L(n) \models_V \alpha, \forall \alpha \in Pr(sr(r)),\ \exists y \in Ch_L(n)\ :\ y \not\models_V x_0. \tag{1}$$

Consider the model $\mathcal{F} := \langle y^R, V\rangle$ which is a finite open submodel of the model $Ch_L(n)$. By the property of an open submodel we have

$$\forall \alpha \in Pr(sr(r))\, \forall e \in \mathcal{F}\ e \models_V \alpha, \quad y \not\models_V x_0. \tag{2}$$

This proves the first part of the assertion: the rule $sr(r)$ is refuted on some finite model.

Let us now show that this rule is also refuted on the frame $\mathcal{T}$ for some valuation. Let us define the valuation of the variables of the rule $sr(r)$ on $\mathcal{T}$ as follows:

$$c \models_W p \iff y \models_V p; \quad b \models_W p \iff \exists e \in y^< \ e \models_V p,$$
$$a \models_W p \iff \forall e \in y^< \ e \models_V p.$$

Note that $W(c) = V(y),\ W(b) = \bigcup_{e \in y^<} V(e),\ W(a) = \bigcap_{e \in y^<} V(e)$. In particular, $c \not\models_W x_0$. Let us show that the premise $Pr(sr(r))$ is true on $\mathcal{T}$ for the given valuation.

Proposition 1. *$\forall \alpha \in Pr(sr(r))$ $c \models_W \alpha$ holds in model $\langle \mathcal{T}, W\rangle$ by force of $y \models_V \alpha$ in the model $\mathcal{F}$.*

Proof. (1) Consider first the case when $\alpha = (x \equiv x_1 \vee x_2) \in Pr(sr(r))$ or $\alpha = (x \equiv x_1 \wedge x_2) \in Pr(sr(r))$.

Let $\alpha = (x \equiv x_1 \wedge x_2) \in Pr(sr(r))$ and $y \models_V (x \to x_1 \wedge x_2)$ (**i**) & $y \models_V (x_1 \wedge x_2 \to x)$ (**ii**) holds. If $c \models_W x$, then $y \models_V x$. In force of (**i**) we get $\forall u\ y \leqslant u \Longrightarrow u \models_V x_1 \wedge x_2$, in particular $y \models_V x_1 \wedge x_2$. Hence $y \models_V x_1$ & $y \models_V x_2 \Longrightarrow c \models_W x_1$ & $c \models_V x_2 \Longrightarrow c \models_W x_1 \wedge x_2$.

If $c \models_W x_1 \wedge x_2$ holds then $c \models_W x_1$ & $c \models_V x_2 \implies y \models_V x_1$ & $y \models_V x_2 \implies y \models_V x_1 \wedge x_2$. In force of truth of (**ii**) we have $y \models_V x$. From where should $c \models_W x$. In this way, we proove $c \models_W \alpha$.

The case $\alpha = (x \equiv x_1 \vee x_2) \in Pr(sr(r))$ proves analogously. Let $\alpha = (x \equiv x_1 \vee x_2) \in Pr(sr(r))$ and $y \models_V (x \to x_1 \vee x_2)$ (**i**) & $y \models_V (x_1 \vee x_2 \to x)$ (**ii**). If $c \models_W x$, then $y \models_V x$. In force of (**i**) we have $\forall u\ y \leqslant u \implies u \models_V x_1 \vee x_2$, in particular $y \models_V x_1 \vee x_2$. From this we get $y \models_V x_1\ \vee\ y \models_V x_2 \Longrightarrow c \models_W x_1\ \vee\ c \models_V x_2 \Longrightarrow c \models_W x_1 \vee x_2$.

If $c \models_W x_1 \vee x_2 \implies c \models_W x_1\ \vee\ c \models_V x_2 \implies y \models_V x_1\ \vee\ y \models_V x_2 \implies y \models_V x_1 \vee x_2$. In force of (**ii**) we have $y \models_V x$. From where it follows $c \models_W x$. So, we get $c \models_W \alpha$.

(2) Let now $\alpha = (x \equiv \neg x_1) \in Pr(sr(r))$ and $y \models_V (x \to \neg x_1)$ (**i**) & $y \models_V (\neg x_1 \to x)$ (**ii**). If $c \models_W x$, then $y \models_V x$. In force of truth $\alpha \in Pr(sr(r))$ in the model $\mathcal{F}$, by (**i**) we conclude $y \models_V \neg x_1$, i.e. $\forall u (y \leqslant u \Longrightarrow u \not\models_V x_1)$. From this we have $x_1 \notin V(y)$ & $x_1 \notin \bigcap_{e \in y^<} V(e)$ & $x_1 \notin \bigcup_{e \in y^<} V(e)$. This entails a, b, $c \not\models_W x_1$ Therefore, $c \models_W \neg x_1$, what did we need to show.

Let now $c \models_W \neg x_1$ holds, therefore, by definition of W we have $x_1 \notin V(y)$ & $x_1 \notin \bigcup_{e \in y^<} V(e)$ & $x_1 \notin \bigcap_{e \in y^<} V(e)$. Hence, for all $e \in y^{\leqslant}$ we get $e \not\models_V x_1$ in the model $\mathcal{F}$, i.e. $y \models_V \neg x_1$.

From the force of truth $\alpha \in Pr(sr(r))$ in the model $\mathcal{F}$ by (**ii**) we conclude $y \models_V x$ in model $\mathcal{F}$, what does it entail $c \models_W x$. In this way, is true $\mathcal{T} \models_W \alpha$.

(3) Consider now the case $\alpha = (x \equiv x_1 \to x_2) \in Pr(sr(r))$ and $y \models_V (x \to (x_1 \to x_2))$ (**i**) & $y \models_V ((x_1 \to x_2) \to x)$ (**ii**). And let performed $c \models_W x$, what entails by definition of

W $y \models_V x$ in model $\mathcal{F}$. By force of α in the model $\mathcal{F}$ and by (**i**) we conclude $y \models_V x_1 \to x_2$. If $c \models_W x_1$, then $y \models_V x_1$. From this and $y \models_V x_1 \to x_2$ we get $y \models_V x_2$ in the model $\mathcal{F}$. Again, according to the definition of the valuation of W we have $c \models_W x_2$. So, we prove $c \models_W x_1 \implies c \models_W x_2 \implies \forall e \in \mathcal{T}(c \leqslant e \Longrightarrow e \models_W x_2)$, i.e. $c \models_W x_1 \to x_2$ in the model $\mathcal{T}$ holds.

Let $c \models_W x_1 \to x_2$ in the model $\mathcal{T}$ holds. And let $c \not\models_W x$ is true, hence $y \not\models_V x$ in the model $\mathcal{F}$. In force of α in the model $\mathcal{F}$ and by (**ii**) we conclude $y \not\models_V x_1 \to x_2$ in the model $\mathcal{F}$, what does it entail $\exists u \ : \ y \leqslant u \ \& \ y \models_V x_1;\ u \not\models_V x_2$. It means it's running $c \models_V x_1,\ x_2 \notin \bigcap_{e \in y^<} V(e)$. From this we receive $c \models_W x_1,\ c \leqslant a \ \& \ a \not\models_W x_2$, which contradicts the original assumption $c \models_W x_1 \to x_2$. Hence it is true $c \models_W \alpha$ in the model $\mathcal{T}$. □

Lemma 2.4. *If the rule $sr(r)$ is refuted on the frame $\mathcal{T}$ for some valuation W, then $sr(r)$ is not globally admissible in the logic IPC.*

Proof. Let for some valuation W the rule $sr(r)$ is refuted on $\mathcal{T}$:

$$\forall \alpha \in Pr(sr(r))\, \forall e \in \mathcal{T}\ e \models_V \alpha, \quad \exists z \in \mathcal{T} z \not\models_V x_0. \tag{3}$$

We define a tabular logic $\mathcal{L} := L(\mathcal{T})$ and show that the rule $sr(r)$ is not admissible in it. Recall that, by constructing the n-characteristic model, the frame $Ch_n(\mathcal{L})$ of this model has the following structure: the first slice consists of 2^n reflexive elements, the second slice consists of co-covers of antichains of at most 2 elements of the first slice.

Note also: (1) frame $\mathcal{T}$ is an open subframe of frame $Ch_n(\mathcal{L})$ for some n; (2) there is a p-morphism f of the frame $Ch_n(\mathcal{L})$ onto the subframe $\mathcal{T}' \sqsubseteq Ch_n(\mathcal{L})$ isomorphic to the frame $\mathcal{T}$. Transferring the valuation from $\mathcal{T}$ to the frame $Ch_n(\mathcal{L})$ with the help of f, we obtain a p-morphism of models $f \ : \ \langle Ch_n(\mathcal{L}), f^{-1}(V)\rangle \to \langle \mathcal{T}, V\rangle$, which preserves the truth of the formulas. Therefore, the rule $sr(r)$ is refuted on the n-characteristic model $Ch_n(\mathcal{L})$ under the valuation $f^{-1}(V)$. And that means it will not be admissible in tabular logic $\mathcal{L}$, i.e. is not globally admissible in the logic IPC. □

From the proved theorems it follows:

Theorem 2.1. *The rule $sr(r)$ in semi-reduced form is not globally admissible in the logic IPC $\iff$ this rule is refuted on some finite model, in particular, is refuted on the frame $\mathcal{T}$ for some valuation.*

The research was financially supported by the Russian Scientific Foundation (Project no. 23-21-00213).

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Критерий глобальной допустимости в логике IPC

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Аннотация. В работе исследуется глобальная допустимость правил вывода в интуиционистской пропозициональной логике IPC.

Ключевые слова: модальная логика, фрейм и модель Крипке, допустимое правило вывода, глобально допустимые правила вывода.