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On the Solvability of Burgers-type Equation with Special Type of Non-linearity

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Abstract. A one-dimensional parabolic Burgers equation of special form with Cauchy data is considered in this paper. To prove the theorem on the solvability of this problem the method of weak approximation developed by Yu. Ya. Belov is used. The results of this paper enhance the results obtained in [2].

Keywords: inverse problem, parabolic equation, Burgers type equation, Cauchy problem, method of weak approximation.

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1. Problem statement

Let us choose r different points $\alpha_1, \alpha_2, \ldots, \alpha_r$ in the space E_1.

Consider the Cauchy problem

$$u_t = a(t)u_{xx} + b(t, x, u(t, x), \omega(t))u_x + f(t, x, u(t, x), \omega(t)), \tag{1}$$

$$u(0, x) = u_0(x). \tag{2}$$

In the strip $G_{[0,T]} = \{(t, x)|0 \leqslant t \leqslant T, x \in E_1\}$. Let us introduce the vector-function $\omega(t) = \Big(u(t, \alpha_j), \dfrac{\partial^k}{\partial x^k}u(t, \alpha_j)\Big)$, $k = 0, \ldots, p_1, j = 1, \ldots, r$. Components of this function are the traces (depending only on the variable t) of function $u(t, x)$ and all its derivatives with respect to x up to order p_1 inclusive. Choose and fix the constant $p \geqslant \max\{2, p_1\} \geqslant 2$.

Definition 1. Let us denote the set of functions $u(t, x)$ defined in $G_{[0,t^*]}$ belonging to the class

$$C^{1,p}_{t,x}(G_{[0,t^*]}) = \left\{u(t, x)| \frac{\partial u}{\partial t}, \frac{\partial^k u}{\partial x^k} \in C(G_{[0,t^*]}), k = 0, \ldots, p\right\},$$

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by $Z_x^p([0,t^*])$. Functions are bounded for $(t,x) \in G_{[0,t^*]}$ together with all derivatives satisfying inequalities

$$\sum_{k=0}^{p} \left| \frac{\partial^k u(t,x)}{\partial x^k} \right| \leqslant C. \tag{3}$$

Definition 2. Classical solution of problem (1), (2) in $G_{[0,t^*]}$ is a function $u(t,x) \in Z_x^p([0,t^*])$ that satisfies (1) and initial data (2) in $G_{[0,t^*]}$.

Here $0 < t^* \leqslant T$ is a fixed constant. Let us assume that the following conditions are satisfied.

Condition 1. Functions $b(t,x,u(t,x),\omega(t)), f(t,x,u(t,x),\omega(t))$ are real-valued continuous functions that are defined for any values of their arguments. For all $t^* \in (0,T]$ and for all $u(t,x) \in Z_x^{p+2}([0,t^*])$ these functions, as functions of the variables $(t,x) \in G_{[0,t^*]}$, are continuous and have continuous derivatives involved in (5) and (6). Function $a(t) \geqslant a_0 > 0$ is a continuous bounded function on the interval $[0,T]$. Function $u_0(x)$ has continuous derivatives satisfying inequalities

$$\sum_{k=0}^{p+2} \left| \frac{d^k u_0(x)}{dx^k} \right| \leqslant C. \tag{4}$$

Condition 2. Let us introduce the following notations

$$U_k(0) = \sup_{x \in E_1} \left| \frac{d^k}{dx^k} u_0(x) \right|, \quad k = 0,1,\ldots,p+2,$$

$$U_k(t) = \sup_{0<\xi\leqslant t} \sup_{x \in E_1} \left| \frac{\partial^k}{\partial x^k} u(\xi,x) \right|, \quad k = 0,1,\ldots,p+2,$$

$$U(t) = \sum_{k=0}^{p+2} U_k(t), \quad U(0) = \sum_{k=0}^{p+2} U_k(0).$$

Let us assume that for all $t^* \in (0,T]$, for all $t \in [0,t^*]$ and for any $u(t,x) \in Z_x^{p+2}([0,t^*])$ the following estimates hold

$$\sum_{k=0}^{p+2} \left| \frac{\partial^k}{\partial x^k} b(t,x,u(t,x),\omega(t)) \right| \leqslant P_{\gamma_1}(U(t)), \tag{5}$$

$$\sum_{k=0}^{p+2} \left| \frac{\partial^k}{\partial x^k} f(t,x,u(t,x),\omega(t)) \right| \leqslant P_{\gamma_2}(U(t)). \tag{6}$$

Here $\gamma_1, \gamma_2 \geqslant 0$ are some fixed integer constants and

$$P_\xi(y) = C(1 + |y| + |y|^2 + \ldots + |y|^\xi),$$

where $C \geqslant 1$ is a constant independent of function $u(t,x)$ and its derivatives.

Theorem 1 (Existence)**.** *Let us assume that Conditions (1) and (2) are satisfied and $0 \leqslant \gamma_1 < \infty$, $0 \leqslant \gamma_2 < \infty$. Then there exists the constant $t^* \in (0,T]$ that depends on a_0 from Condition (1) and C from inequalities (5), (6) such that classical solution $u(t,x)$ of problem (1), (2) exists in the class $Z_x^p([0,t^*])$.*

The proof of the theorem for $0 \leqslant \gamma_2 \leqslant 1$ is given in [2]. The case $2 \leqslant \gamma_2 < \infty$ is considered in this paper.

Proof. To prove the existence of a solution of Cauchy problem (1), (2) *the weak approximation method* [1] is used. Let us consider an auxiliary split problem with time shift $\left(t-\dfrac{\tau}{3}\right)$ in unknown functions and non-linear terms

$$u^\tau_t(t,x) = 3a(t)u^\tau_{xx}(t,x), \quad n\tau < t \leqslant \left(n+\frac{1}{3}\right)\tau; \tag{7}$$

$$u^\tau_t(t,x) = 3b\Bigg(t-\frac{\tau}{3}, x, u^\tau\left(t-\frac{\tau}{3},x\right), \omega^\tau\left(t-\frac{\tau}{3}\right)\Bigg)u^\tau_x(t,x), \quad \left(n+\frac{1}{3}\right)\tau < t \leqslant \left(n+\frac{2}{3}\right)\tau; \tag{8}$$

$$u^\tau_t(t,x) = 3f\left(t-\frac{\tau}{3}, x, u^\tau\left(t-\frac{\tau}{3},x\right), \omega^\tau\left(t-\frac{\tau}{3}\right)\right), \quad \left(n+\frac{2}{3}\right)\tau < t \leqslant (n+1)\tau; \tag{9}$$

$$u^\tau(t,x)\big|_{t\leqslant 0} = u_0(x). \tag{10}$$

Let us prove a priori estimates that ensure the compactness of the family of solutions $u(t,x)$ of problem (7)–(10) in the class $C^{1,p}_{t,x}(G_{[0,t^*]})$ for some constant $0 < t^* \leqslant T$.

At the first fractional step $\left(0 < t \leqslant \dfrac{\tau}{3}\right)$ for $(n = 0)$ we apply the maximum principle to problem (7), (10) and obtain the estimate for function $u^\tau(t,x)$

$$|u^\tau(t,x)| \leqslant U_0(0), \quad 0 < t \leqslant \frac{\tau}{3}.$$

Differentiating problem (7), (10) k times with respect to x, we obtain similar estimates

$$\left|\frac{\partial^k}{\partial x^k}u^\tau(t,x)\right| \leqslant U_k(0), \quad 0 < t \leqslant \frac{\tau}{3}, \quad k = 1,\dots,p+2.$$

Summing up the obtained inequalities, we obtain the estimate

$$U^\tau(t) \leqslant U(0), \quad 0 < t \leqslant \frac{\tau}{3}. \tag{11}$$

At the second fractional step $\left(\dfrac{\tau}{3} < t \leqslant \dfrac{2\tau}{3}\right)$ we solve equation (8). Since function $b\left(t-\dfrac{\tau}{3}, x, u^\tau\left(t-\dfrac{\tau}{3},x\right), \omega^\tau\left(t-\dfrac{\tau}{3}\right)\right)$ is continuous and it is known from the previous fractional step solution of this equation exists ([3], item 2.6). Let us consider the characteristic equation for equation (8)

$$\frac{dx}{dt} = -3b\left(t-\frac{\tau}{3}, x, u^\tau\left(t-\frac{\tau}{3},x\right), \omega^\tau\left(t-\frac{\tau}{3}\right)\right).$$

Let us denote the characteristic function of the resulting characteristic equation by $\varphi(\xi,\zeta,\eta)$, that is, $x = \varphi(\xi,\zeta,\eta)$ is the integral curve passing through the point (ζ,η). Then the solution at the second fractional step has the form

$$u^\tau(t,x) = u^\tau\left(\frac{\tau}{3}, \varphi\left(\frac{\tau}{3},t,x\right)\right), \quad \frac{\tau}{3} < t \leqslant \frac{2\tau}{3}. \tag{12}$$

Therefore, the following estimate is true

$$U^\tau_0(t) \leqslant U^\tau\left(\frac{\tau}{3}\right) \leqslant U(0), \quad \frac{\tau}{3} < t \leqslant \frac{2\tau}{3}. \tag{13}$$

Let us differentiate equation (8) with respect to x and introduce the following notations

$$\begin{aligned}
&u_x^\tau(t,x) = z^\tau(t,x),\\
&b_0^\tau(t,x) = 3b\left(t-\frac{\tau}{3}, x, u^\tau\left(t-\frac{\tau}{3}, x\right), \omega^\tau\left(t-\frac{\tau}{3}\right)\right),\\
&b_1^\tau(t,x) = 3\frac{\partial}{\partial x}b\left(t-\frac{\tau}{3}, x, u^\tau\left(t-\frac{\tau}{3}, x\right), \omega^\tau\left(t-\frac{\tau}{3}\right)\right).
\end{aligned}$$

Using new notations, the differentiated equation is written in the form

$$z_t^\tau = b_0^\tau(t,x) z_x^\tau + b_1^\tau(t,x) z^\tau.$$

The solution of this equation can be written in the parametric form ([3], p. 4.3)

$$z^\tau(t,x) = e^{F_0^\tau(t,\frac{\tau}{3},\eta)} z^\tau\left(\frac{\tau}{3},\eta\right), \quad x = \varphi^\tau\left(t,\frac{\tau}{3},\eta\right),$$

where

$$F_0^\tau = F_0^\tau(t,\zeta,\eta) = -\int_\zeta^t b_1^\tau(\xi,\zeta,\eta)\,d\xi$$

and $x = \varphi^\tau(\xi,\zeta,\eta)$ is the characteristic function of the equation

$$\frac{dx}{dt} = -b_0^\tau(t,x) = -3b\left(t-\frac{\tau}{3}, x, u^\tau\left(t-\frac{\tau}{3}, x\right), \omega^\tau\left(t-\frac{\tau}{3}\right)\right).$$

Therefore, the following estimate is true

$$|u_x^\tau(t,x)| = |z^\tau(t,x)| \leqslant U_1^\tau\left(\frac{\tau}{3}\right) e^{P_{\gamma_1}(U^\tau(t-\frac{\tau}{3}))\tau} \leqslant U_1^\tau\left(\frac{\tau}{3}\right) e^{P_{\gamma_1}(U(0))\tau}.$$

Now we take from the left and right parts of the resulting inequality sup for $x \in E_1$ and obtain

$$U_1^\tau(t) \leqslant U_1^\tau\left(\frac{\tau}{3}\right) e^{P_{\gamma_1}(U(0))\tau}, \quad \frac{\tau}{3} < t \leqslant \frac{2\tau}{3}. \tag{14}$$

Next, we differentiate equation (8) twice with respect to x and introduce the following notations

$$\begin{aligned}
&u_{xx}^\tau(t,x) = v^\tau(t,x),\\
&c_0^\tau(t,x) = 3b\left(t-\frac{\tau}{3}, x, u^\tau\left(t-\frac{\tau}{3}, x\right), \omega^\tau\left(t-\frac{\tau}{3}\right)\right),\\
&c_1^\tau(t,x) = 6\frac{\partial}{\partial x}b\left(t-\frac{\tau}{3}, x, u^\tau\left(t-\frac{\tau}{3}, x\right), \omega^\tau\left(t-\frac{\tau}{3}\right)\right),\\
&c_2^\tau(t,x) = 3\frac{\partial^2}{\partial x^2}b\left(t-\frac{\tau}{3}, x, u^\tau\left(t-\frac{\tau}{3}, x\right), \omega^\tau\left(t-\frac{\tau}{3}\right)\right).
\end{aligned}$$

Using new notations, the equation is written in the form

$$v_t^\tau = c_0^\tau(t,x) v_x^\tau + c_1^\tau(t,x) v^\tau + c_2^\tau(t,x) z^\tau(t,x).$$

The solution of this equation can be written in the parametric form ([3], p. 4.3)

$$\begin{aligned}
&v^\tau = e^{G_0^\tau(t,\frac{\tau}{3},\eta)}\Big(v^\tau\left(\frac{\tau}{3},\eta\right) + \int_{\frac{\tau}{3}}^t c_2^\tau\left(\xi,\varphi\left(\xi,\frac{\tau}{3},\eta\right)\right) z^\tau\left(\xi,\varphi\left(\xi,\frac{\tau}{3},\eta\right)\right) e^{G_0^\tau(\xi,\frac{\tau}{3},\eta)}d\xi\Big),\\
&x = \varphi^\tau(\xi,\zeta,\eta),
\end{aligned}$$

where

$$G_0^\tau = G_0^\tau(t,\zeta,\eta) = -\int_\zeta^t c_1^\tau(\xi,\zeta,\eta)d\xi.$$

Note that estimate for function $z^\tau(t,x)$ is already available. Therefore, one can evaluate function $v^\tau(t,x)$

$$|u_{xx}^\tau(t,x)| = |v^\tau(t,x)| \leqslant e^{2\tau P_{\gamma_1}(U(0))}\Big(U_2^\tau\left(\frac{\tau}{3}\right) + 3P_{\gamma_1}(U(0))e^{2\tau P_{\gamma_1}(U(0))}\int_{\frac{\tau}{3}}^t U_1^\tau(\xi)d\xi\Big) \leqslant$$
$$\leqslant e^{C\tau P_{\gamma_1}(U(0))}\Big(U_2^\tau\left(\frac{\tau}{3}\right) + C\tau P_{\gamma_1}(U(0))U_1^\tau\left(\frac{\tau}{3}\right)e^{\tau P_{\gamma_1}(U(0))}\Big) \leqslant$$
$$\leqslant e^{C\tau P_{\gamma_1}(U(0))}\Big(U_2^\tau\left(\frac{\tau}{3}\right) + C\tau P_{\gamma_1}(U(0))U_1^\tau\left(\frac{\tau}{3}\right)\Big) \leqslant$$
$$\leqslant e^{C\tau P_{\gamma_1}(U(0))}\Big(U_2^\tau\left(\frac{\tau}{3}\right) + U_1^\tau\left(\frac{\tau}{3}\right)\Big)(1 + C\tau P_{\gamma_1}(U(0))) \leqslant e^{C\tau P_{\gamma_1}(U(0))}\Big(U_2^\tau\left(\frac{\tau}{3}\right) + U_1^\tau\left(\frac{\tau}{3}\right)\Big).$$

Now we take from the left and right parts of the resulting inequality sup for $x \in E_1$ and obtain

$$U_2^\tau(t) \leqslant e^{C\tau P_{\gamma_1}(U(0))}\Big(U_2^\tau\left(\frac{\tau}{3}\right) + U_1^\tau\left(\frac{\tau}{3}\right)\Big), \quad \frac{\tau}{3} < t \leqslant \frac{2\tau}{3}. \tag{15}$$

Next, we differentiate equation (8) $k = 3, \dots p+2$ times with respect to x. Using the Leibniz formula for the k-th derivative of the product of two functions, we obtain the equation in general form

$$\frac{\partial^k}{\partial x^k}u_t^\tau = g_0^\tau\frac{\partial^k}{\partial x^k}u_x^\tau + g_1^\tau\frac{\partial^k}{\partial x^k}u^\tau + \sum_{j=2}^k g_j^\tau\frac{\partial^{k-j+1}}{\partial x^{k-j+1}}u^\tau,$$

where

$$g_0^\tau = 3b\left(t-\frac{\tau}{3}, x, u^\tau\left(t-\frac{\tau}{3},x\right), \omega^\tau\left(t-\frac{\tau}{3}\right)\right),$$
$$g_j^\tau = 3C_k^j\frac{\partial^j}{\partial x^j}b\left(t-\frac{\tau}{3}, x, u^\tau\left(t-\frac{\tau}{3},x\right), \omega^\tau\left(t-\frac{\tau}{3}\right)\right).$$

Writing the solution in explicit form, we obtain the following estimate

$$\left|\frac{\partial^k}{\partial x^k}u^\tau(t,x)\right| \leqslant e^{C\tau P_{\gamma_1}(U(0))}\left(U_k^\tau\left(\frac{\tau}{3}\right) + CP_{\gamma_1}(U(0))\int_{\frac{\tau}{3}}^t\sum_{j=1}^{k-1}U_j^\tau(\xi)d\xi\right) \leqslant$$
$$\leqslant e^{C\tau P_{\gamma_1}(U(0))}\left(U_k^\tau\left(\frac{\tau}{3}\right) + C\tau P_{\gamma_1}(U(0))e^{C\tau P_{\gamma_1}(U(0))}\sum_{j=1}^{k-1}U_j^\tau\left(\frac{\tau}{3}\right)\right) \leqslant$$
$$\leqslant e^{C\tau P_{\gamma_1}(U(0))}\left(\sum_{j=1}^k U_j^\tau\left(\frac{\tau}{3}\right)\right)(1 + C\tau P_{\gamma_1}(U(0))) \leqslant$$
$$\leqslant e^{C\tau P_{\gamma_1}(U(0))}\left(\sum_{j=1}^k U_j^\tau\left(\frac{\tau}{3}\right)\right), \quad k = 3,\dots,p+2, \quad \frac{\tau}{3} < t \leqslant \frac{2\tau}{3}.$$

Now we take from the left and right parts of the resulting inequality sup for $x \in E_1$ and obtain

$$U_k^\tau(t) \leqslant e^{C\tau P_{\gamma_1}(U(0))}\left(\sum_{j=1}^k U_j^\tau\left(\frac{\tau}{3}\right)\right), \quad \frac{\tau}{3} < t \leqslant \frac{2\tau}{3}. \tag{16}$$

Combining inequalities (13), (14), (15) and (16), we obtain estimates for the solution at the second fractional step

$$U^\tau(t) \leqslant U^\tau\left(\frac{\tau}{3}\right) e^{C\tau P_{\gamma_1}(U(0))}, \quad \frac{\tau}{3} < t \leqslant \frac{2\tau}{3}. \tag{17}$$

At the third fractional step $\left(\frac{2\tau}{3} < t \leqslant \tau\right)$ we integrate equation (9) with respect to variable t

$$u^\tau(t,x) = u^\tau\left(\frac{2\tau}{3}, x\right) + 3\int_{\frac{2\tau}{3}}^{t} f\left(\eta - \frac{\tau}{3}, x, u^\tau\left(\eta - \frac{\tau}{3}, x\right), \omega^\tau\left(\eta - \frac{\tau}{3}\right)\right) d\eta.$$

Condition (2) implies that

$$U_0^\tau(t) \leqslant U_0^\tau\left(\frac{2\tau}{3}\right) + C\tau P_{\gamma_2}\left(U^\tau\left(\frac{2\tau}{3}\right)\right).$$

Differentiating equation (9) k times with respect to $x, k = 1, \ldots, p+2$ and using condition (2), we obtain

$$U_k^\tau(t) \leqslant U_k^\tau\left(\frac{2\tau}{3}\right) + C\tau P_{\gamma_2}\left(U^\tau\left(\frac{2\tau}{3}\right)\right).$$

Combining the obtained inequalities, we have

$$\begin{aligned} U^\tau(t) &\leqslant U^\tau\left(\frac{2\tau}{3}\right) + C\tau P_{\gamma_2}\left(U^\tau\left(\frac{2\tau}{3}\right)\right) \leqslant 1 + U^\tau\left(\frac{2\tau}{3}\right) + C\tau\left(1 + U^\tau\left(\frac{2\tau}{3}\right)\right)^{\gamma_2} - 1 \leqslant \\ &\leqslant \left(1 + U^\tau\left(\frac{2\tau}{3}\right)\right)\left(1 + C\tau\left(1 + U^\tau\left(\frac{2\tau}{3}\right)\right)^{\gamma_2 - 1}\right) - 1 \leqslant \\ &\leqslant \left(1 + U^\tau\left(\frac{2\tau}{3}\right)\right) e^{C\tau\left(1 + U^\tau\left(\frac{2\tau}{3}\right)\right)^{\gamma_2 - 1}} - 1. \end{aligned} \tag{18}$$

Using estimates (11), (17), (18), the following inequality holds at the zero time step $t \in [0, \tau]$

$$\begin{aligned} U^\tau(t) &\leqslant \left(1 + U(0) e^{C\tau P_{\gamma_1}(U(0))}\right) e^{C\tau\left[1 + U(0) e^{C\tau P_{\gamma_1}(U(0))}\right]^{\gamma_2 - 1}} - 1 \leqslant \\ &\leqslant (1 + U(0)) e^{C\tau P_{\gamma_1}(U(0)) + C\tau(1+U(0))^{\gamma_2 - 1} e^{(\gamma_2 - 1)C\tau P_{\gamma_1}(U(0))}} - 1 \leqslant \\ &\leqslant (1 + U(0)) e^{C\tau\left[P_{\gamma_1}(U(0)) + (1+U(0))^{\gamma_2 - 1}\right] e^{(\gamma_2 - 1)C\tau P_{\gamma_1}(U(0))}} - 1. \end{aligned}$$

Let us choose $\gamma_3 = \max\{\gamma_1; \gamma_2 - 1\}$ then

$$U^\tau(t) \leqslant (1 + U(0)) e^{C\tau P_{\gamma_3}(1+U(0)) e^{\gamma_3 C\tau P_{\gamma_3}(U(0))}} - 1.$$

Let τ be such that inequality

$$e^{\gamma_3 C\tau P_{\gamma_3}(U(0))} \leqslant 2$$

is satisfied then

$$U^\tau(t) \leqslant (1 + U(0)) e^{2C\tau P_{\gamma_3}(1+U(0))} - 1, \quad t \in [0, \tau].$$

Using the same line of reasoning, at the first time step $(\tau < t \leqslant 2\tau)$, we obtain the estimate

$$\begin{aligned} U^\tau(t) &\leqslant (1 + U^\tau(\tau)) e^{2C\tau P_{\gamma_3}(1+U^\tau(\tau))} - 1 \leqslant \\ &\leqslant \left(1 + (1 + U(0)) e^{2C\tau P_{\gamma_3}(1+U(0))} - 1\right) e^{2C\tau P_{\gamma_3}\left[1 + (1+U(0)) e^{2C\tau P_{\gamma_3}(1+U(0))} - 1\right]} - 1 \leqslant \\ &\leqslant (1 + U(0)) e^{2C\tau P_{\gamma_3}(1+U(0)) + 2C\tau P_{\gamma_3}(1+U(0)) e^{2C\tau\gamma_3 P_{\gamma_3}(1+U(0))}} \leqslant \\ &\leqslant (1 + U(0)) e^{2C\tau P_{\gamma_3}(1+U(0))\left[1 + e^{2C\tau\gamma_3 P_{\gamma_3}(1+U(0))}\right]} - 1. \end{aligned}$$

Let τ be such that inequality
$$e^{2C\tau\gamma_3 P_{\gamma_3}(1+U(0))} \leqslant 2$$
is true, then
$$U^\tau(t) \leqslant (1+U(0))e^{6C\tau P_{\gamma_3}(1+U(0))} - 1, \quad t \in [0, 2\tau].$$

At the second time step ($2\tau < t \leqslant 3\tau$) we obtain the estimate
$$\begin{aligned} U^\tau(t) &\leqslant (1+U^\tau(2\tau))e^{2C\tau P_{\gamma_3}(1+U^\tau(2\tau))} - 1 \leqslant \\ &\leqslant \left(1+(1+U(0))e^{6C\tau P_{\gamma_3}(1+U(0))} - 1\right) e^{2C\tau P_{\gamma_3}[1+(1+U(0))e^{6C\tau P_{\gamma_3}(1+U(0))}-1]} - 1 \leqslant \\ &\leqslant (1+U(0))e^{6C\tau P_{\gamma_3}(1+U(0))+2C\tau P_{\gamma_3}(1+U(0))e^{6C\tau\gamma_3 P_{\gamma_3}(1+U(0))}} \leqslant \\ &\leqslant (1+U(0))e^{2C\tau P_{\gamma_3}(1+U(0))\left[3+e^{6C\tau\gamma_3 P_{\gamma_3}(1+U(0))}\right]} - 1. \end{aligned}$$

Let τ be such that inequality
$$e^{6C\tau\gamma_3 P_{\gamma_3}(1+U(0))} \leqslant 2$$
is satisfied, then
$$U^\tau(t) \leqslant (1+U(0))e^{10C\tau P_{\gamma_3}(1+U(0))} - 1, \quad t \in [0, 3\tau].$$

Continuing given above argument, at the i-th time step we obtain the estimate
$$U^\tau(t) \leqslant (1+U(0))e^{(4i+2)C\tau P_{\gamma_3}(1+U(0))} - 1, \quad t \in [0, i\tau].$$

Let t^* ($0 < t^* \leqslant T$) be such that
$$e^{t^* C\gamma_3 P_{\gamma_3}(1+U(0))} \leqslant 2.$$

Then for all $i \geqslant 0$ such that $(4i+2)\tau \leqslant t^*$ the following estimate holds
$$U^\tau(t) \leqslant (1+U(0))e^{(4i+2)C\tau P_{\gamma_3}(1+U(0))} - 1 \leqslant (1+U(0))e^{t^* CP_{\gamma_3}(1+U(0))} - 1.$$

Since t^*, C, γ_3 and $U(0)$ depend on the input data but do not depend on τ we obtain
$$\left|\frac{\partial^k}{\partial x^k} u^\tau(t,x)\right| \leqslant U^\tau(t) \leqslant (1+U(0))e^{t^* CP_{\gamma_3}(1+U(0))} - 1 = K, \quad t \in [0, t^*]. \tag{19}$$

This implies that function $u^\tau(t,x)$ and its derivatives with respect to x are bounded uniformly in terms of variable τ up to order $p+2$ inclusive in the strip $G_{[0,t^*]}$.

By virtue of equations (7)–(9) it also follows that derivatives are bounded uniformly in terms of variable τ
$$\frac{\partial}{\partial t}\frac{\partial^k u^\tau}{\partial x^k}, \quad k = 0, \dots, p. \tag{20}$$

Taking into account the boundedness of derivatives $\dfrac{\partial}{\partial x}\dfrac{\partial^k u^\tau}{\partial x^k}$, $k=1,\dots,p$, it guarantees equicontinuity in $G^N_{[0,t^*]} = \{(t,x) | 0 \leqslant t \leqslant t^*, |x| \leqslant N\}$ of sets of functions $\left\{\dfrac{\partial^k u^\tau}{\partial x^k}\right\}$, $k = 0, \dots p$ for any fixed constant N.

By virtue of the Arzela theorem some subsequence $u^{\tau_k}(t,x)$ of the sequence $u^\tau(t,x)$ of solutions of split problem (7)–(10) converges together with derivatives with respect to x up to order p inclusive to function $u(t,x) \in C^{1,p}_{t,x}\left(G_{[0,t^*]}\right)$. By virtue of the convergence theorem for the weak approximation method [1], $u(t,x)$ is a solution of original problem (1), (2). Moreover, for $(t,x) \in G_{[0,t^*]}$ the following estimate is satisfied
$$\sum_{k=0}^{p}\left|\frac{\partial^k u(t,x)}{\partial x^k}\right| \leqslant C.$$

Thus, $u(t,x) \in Z^p_x([0,t^*])$. The theorem is proved. □

2. Example

Let us consider an example of application of Theorem (1) to the proof of the solvability of one inverse coefficient problem for a parabolic type equation.

Let us consider the Cauchy problem

$$u_t(t,x) = a^2 u_{xx}(t,x) + (u(t,x) + \lambda_1(t))u_x(t,x) + \lambda_2(t) f(t,x), \tag{21}$$

$$u(0,x) = u_0(x). \tag{22}$$

that is posed in the domain $G_{[0,T]} = \{(t,x) \,|\, 0 \leqslant t \leqslant T, x \in E_1\}$. Functions $\lambda_1(t), \lambda_2(t)$ are to be determined simultaneously with solution $u(t,x)$ of problem (21), (22) satisfying redefinition conditions

$$u(t,\alpha) = \varphi_1(t), \tag{23}$$

$$u_x(t,\alpha) = \varphi_2(t) \tag{24}$$

and conditions of agreement

$$u(0,\alpha) = \varphi_1(0), \tag{25}$$

$$u_x(0,\alpha) = \varphi_2(0). \tag{26}$$

Regarding functions $\varphi_1(t), \varphi_2(t), u_0(x), f(t,x)$, we assume that they are sufficiently smooth, and they have all continuous derivatives that satisfy the following inequality for all $(t,x) \in G_{[0,T]}$

$$|\varphi_1(t)| + |\varphi_1'(t)| + |\varphi_2(t)| + |\varphi_2'(t)| + \left|\frac{d^k}{dx^k} u_0(x)\right| + \left|\frac{\partial^k}{\partial x^k} f(t,x)\right| \leqslant C, \quad k = 0, \ldots, 5. \tag{27}$$

Let us also assume that for all $t \in [0,T]$ the following inequality is satisfied

$$\varphi_2(t) f_x(t,\alpha) - f(t,\alpha) \frac{\partial^2}{\partial x^2} u_0(\alpha) \geqslant \delta > 0. \tag{28}$$

The original problem is reduced to the auxiliary direct problem

$$\begin{aligned} u_t = a^2 u_{xx} + \Bigg[u &+ \frac{(\psi_1(t) - a^2 u_{xx}(t,\alpha)) f_x(t,\alpha)}{\varphi_2 f_x(t,\alpha) - u_{xx}(t,\alpha) f(t,\alpha)} - \\ &- \frac{(\psi_2(t) - a^2 u_{xxx}(t,\alpha) - \varphi_1 u_{xx}(t,\alpha)) f(t,\alpha)}{\varphi_2 f_x(t,\alpha) - u_{xx}(t,\alpha) f(t,\alpha)} \Bigg] u_x + \\ &+ \Bigg[\frac{(\psi_2(t) - a^2 u_{xxx}(t,\alpha) - \varphi_1 u_{xx}(t,\alpha)) \varphi_2}{\varphi_2 f_x(t,\alpha) - u_{xx}(t,\alpha) f(t,\alpha)} - \\ &- \frac{(\psi_1(t) - a^2 u_{xx}(t,\alpha)) u_{xx}(t,\alpha)}{\varphi_2 f_x(t,\alpha) - u_{xx}(t,\alpha) f(t,\alpha)} \Bigg] f(t,x), \end{aligned} \tag{29}$$

$$u(0,x) = u_0(x), \tag{30}$$

where

$$\psi_1(t) = \varphi_1'(t) - \varphi_1(t)\varphi_2(t), \qquad \psi_2(t) = \varphi_2'(t) - \varphi_2^2(t).$$

In order to guarantee that denominator of expression (29) does not vanish we introduce *the cut-off function* $S_\delta(y)$. It is differentiable as many times as needed and has the following properties

$$S_\delta(y) \geqslant \frac{\delta}{3} > 0, \qquad \forall y \in E_1, \tag{31}$$

$$S_\delta(y) = \begin{cases} y, & y \geqslant \dfrac{\delta}{2}; \\ \dfrac{\delta}{3}, & y \leqslant \dfrac{\delta}{3}. \end{cases} \tag{32}$$

Let us substitute the cut-off function into the denominator of fractional expressions

$$\begin{aligned} u_t = a^2 u_{xx} + \Bigg[u + \frac{(\psi_1(t) - a^2 u_{xx}(t,\alpha)) f_x(t,\alpha)}{S_\delta(\varphi_2 f_x(t,\alpha) - u_{xx}(t,\alpha) f(t,\alpha))} - \\ - \frac{(\psi_2(t) - a^2 u_{xxx}(t,\alpha) - \varphi_1 u_{xx}(t,\alpha)) f(t,\alpha)}{S_\delta(\varphi_2 f_x(t,\alpha) - u_{xx}(t,\alpha) f(t,\alpha))} \Bigg] u_x + \\ + \Bigg[\frac{(\psi_2(t) - a^2 u_{xxx}(t,\alpha) - \varphi_1 u_{xx}(t,\alpha)) \varphi_2}{S_\delta(\varphi_2 f_x(t,\alpha) - u_{xx}(t,\alpha) f(t,\alpha))} - \\ - \frac{(\psi_1(t) - a^2 u_{xx}(t,\alpha)) u_{xx}(t,\alpha)}{S_\delta(\varphi_2 f_x(t,\alpha) - u_{xx}(t,\alpha) f(t,\alpha))} \Bigg] f(t,x), \end{aligned} \tag{33}$$

$$u(0,x) = u_0(x). \tag{34}$$

The resulting direct problem (33), (34) is a problem of form (1), (2). Let us check the conditions of Theorem (1) for $p = 3$,

$$b = u + \frac{(\psi_1(t) - a^2 u_{xx}(t,\alpha)) f_x(t,\alpha) - (\psi_2(t) - a^2 u_{xxx}(t,\alpha) - \varphi_1 u_{xx}(t,\alpha)) f(t,\alpha)}{S_\delta(\varphi_2 f_x(t,\alpha) - u_{xx}(t,\alpha) f(t,\alpha))},$$

$$f = \frac{(\psi_2(t) - a^2 u_{xxx}(t,\alpha) - \varphi_1 u_{xx}(t,\alpha)) \varphi_2 - (\psi_1(t) - a^2 u_{xx}(t,\alpha)) u_{xx}(t,\alpha)}{S_\delta(\varphi_2 f_x(t,\alpha) - u_{xx}(t,\alpha) f(t,\alpha))}.$$

Condition (1) is satisfied due to assumption (27), and condition (2) becomes

$$\sum_{k=0}^{5} \left| \frac{\partial^k}{\partial x^k} b(t,x,u(t,x),\omega(t)) \right| \leqslant P_1(U(t)),$$

$$\sum_{k=0}^{5} \left| \frac{\partial^k}{\partial x^k} f(t,x,u(t,x),\omega(t)) \right| \leqslant P_2(U(t)).$$

Thus, all conditions of Theorem (1) are satisfied for $p = 3, \gamma_1 = 1, \gamma_2 = 2$. Therefore, there exists a constant $t^* : 0 < t^* \leqslant T$ depending on the constants that constrain the input data such that classical solution $u(t,x)$ of problem (33), (34) exists in the class $Z_x^3(G_{[0,t^*]})$.

Note that at this point, the existence of a solution of direct problem is proved but not the existence of a solution of inverse problem. After that, we need to remove the cut-off function from the denominators of fractional expressions. In order to guarantee that conditions imposed on cut-off function (31)–(32) are satisfied it is necessary to use inequality (28).

Then, using the agreement conditions and redefinition conditions, one can show that solution of the inverse problem also exists, and function $u(t,x)$ which is the solution of direct problem (33), (34) is also the solution of inverse problem (21)–(22). Parameters $\lambda_1(t), \lambda_2(t)$ are defined as follows

$$\lambda_1(t) = \frac{(\psi_1(t) - a^2 u_{xx}(t,\alpha)) f_x(t,\alpha) - (\psi_2(t) - a^2 u_{xxx}(t,\alpha) - \varphi_1 u_{xx}(t,\alpha)) f(t,\alpha)}{\varphi_2 f_x(t,\alpha) - u_{xx}(t,\alpha) f(t,\alpha)},$$

$$\lambda_2(t) = \frac{(\psi_2(t) - a^2 u_{xxx}(t,\alpha) - \varphi_1 u_{xx}(t,\alpha)) \varphi_2 - (\psi_1(t) - a^2 u_{xx}(t,\alpha)) u_{xx}(t,\alpha)}{\varphi_2 f_x(t,\alpha) - u_{xx}(t,\alpha) f(t,\alpha)}.$$

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О разрешимости уравнения типа Бюргерса с нелинейностью специального вида

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Аннотация. В данной работе рассматривается одномерное параболическое уравнение Бюргерса специального вида с данными Коши. При доказательстве теоремы о разрешимости этой задачи используется метод слабой аппроксимации, разработанный Ю. Я. Беловым. Результаты, полученные в данной работе, усиливают результаты, полученные в [2].

Ключевые слова: обратная задача, параболическое уравнение, уравнение типа Бюргерса, задача Коши, метод слабой аппроксимации.